ON THE EXTENSION OF WEAK ARMENDARIZ RINGS RELATIVE TO A MONOID

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ABSTRACT

For a monoid M, we introduce the concept of weak 3-M-Armendariz rings, which are a common generalization of 3-M-Armendariz rings and weak M-Armendariz rings, and investigates its properties. Moreover, this paper proves that a ring R is a weak 3-M-Armendariz if and only if for any n, the n-by-n upper triangular matrix ring $T_n(R)$ over R is a weak 3-M-Armendariz. If the ideal I is a reduced and R/I is a weak 3-M-Armendariz, then R is a weak 3-M-Armendariz, where M is strictly totally ordered monoid. Also we show that if a ring R satisfy condition (P) and a weak 3-M-Armendariz, then R is a weak 3-$(M \times N)$-Armendariz, where N is a unique product monoid.

Keywords: *unique product monoid, 3-Armendariz ring, 3-M-Armendariz ring, weak 3-M-Armendariz ring.*
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1. INTRODUCTION

Throughout this paper, R and M denote an associative ring, not necessary with identity and a monoid, respectively. Given a ring R, the polynomial ring over R is denoted by $R[x]$. We denote by $T_n(R)$ the n-by-n upper triangular matrix ring over R. The study of Armendariz rings was initiated by Armendariz [5] and Rege and Chhawchharia [9]. A ring R is called Armendariz if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$, for each i, j. (The converse is always true.) Some properties of Armendariz rings have been studied in Rege and Chhawchharia [9], Anderson and Camillo [3], Kim and Lee [10], Hong et al. [2], and Lee and Wong [12]. Suiyi [15] introduced the notion of 3-Armendariz rings. A ring R is called a 3-Armendariz ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m, h(x) = c_0 + c_1x + \cdots + c_rx^r \in R[x],$ satisfy $f(x)g(x)h(x) = 0,$ then $a_ib_jc_k = 0,$ for each $i, j,$ and $k.$ Zhongkui [7], studied a generalization of Armendariz rings, which are called M-Armendariz rings, where M is a monoid. A ring R is called M-Armendariz if whenever elements $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m \in R[M],$ satisfy $\alpha\beta = 0$, then $a_ib_j = 0$ for each $i, j,$ where $g_i, h_j \in M.$ A ring R is called reduced if it has no nonzero nilpotent elements. Reduced rings are Armendariz by Armendariz [5, Lemma 1.1] and subrings of a Armendariz ring are also Armendariz. A ring R is called abelian if every idempotent is central. Armendariz ring are abelian by Kim and Lee [10]. Subrings of M-Armendariz ring are also M-Armendariz by Zhongkui [7]. Subrings of a 3-Armendariz ring are also 3-Armendariz by Suiyi [15]. Liu and Zhao [8]. Introduced the notion of weak Armendariz. A ring R is called weak Armendariz if whenever polynomials $f(x) = a_0 + \cdots + a_nx^n, g(x) = b_0 + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0,$ then $a_ib_j \in nil(R)$ for each $i, j.$ They showed that for a semicommutative ideal I such that R/I is weak Armendariz, then R is weak Armendariz, and R is weak Armendariz if and only if for any $n,$ the $n \times n$ upper triangular matrix ring over R is weak Armendariz. Wu Hui-feng [14]. Introduced the notion of weak 3-Armendariz. A ring R is called a weak 3-Armendariz if whenever polynomials $f(x) = a_0 + \cdots + a_nx^n, g(x) = b_0 + \cdots + b_mx^m, h(x) = c_0 + \cdots + c_rx^r \in R[x]$ satisfy $f(x)g(x)h(x) = 0,$ then $a_ib_jc_k \in nil(R)$ for each $i, j, k.$ Elshokry and et al. [1]. Introduced the notion of 3-M-Armendariz, where

M is the a monoid. A ring R is called 3-M-Armendariz if whenever elements $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m, \gamma = c_1l_1 + \cdots + c_rl_r \in R[M]$, satisfy $\alpha\beta\gamma = 0$, then $a_ib_jc_k = 0$ for each i, j, k where $g_i, h_j, l_k \in M$. In [16]. A ring R is called weak M-Armendariz if whenever elements $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m \in R[M]$, satisfy $\alpha\beta = 0$, then $a_ib_j \in nil(R)$ for each i, j. Clearly, M-Armendariz rings are weak M-Armendariz.

Recall that a monoid M is called a u.p.-monoid (unique product monoid) if for any two non-empty finite subsets $A, B \subseteq M$ there exists an element $g \in M$ uniquely presented in the form ab where $a \in A$ and $b \in B$. The class of u.p.-monoids is quite large and important (see Birkenmeier and Park [6], Passman [4]). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid M has non unity element of finite order.

Motivated by results in Elshokry and et al. [1], Suiyi [15], Zhongkui [7], Rege and Chhawchharia [9], Zhongkui and Zhao [8], Kim and Lee [10], Wu Hui-feng [14] and Cuiping and Jianlong [16], we will investigate a common generalization of M-Armendariz and 3-M-Armendariz rings, which we called weak 3-M-Armendariz rings.

2. Weak 3-Armendariz rings relative to a monoid

For a monid M, e will always stand for the identity of M. If R is a ring, then $R[M]$ denotes the monoid ring over R.

Definition 2.1 *Let M be a monoid. A ring R is called a weak 3-M-Armendariz ring if whenever elements $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m$ and $\gamma = c_1l_1 + \cdots + c_rl_r \in R[M]$, satisfy $\alpha\beta\gamma = 0$, then $a_ib_jc_k \in nil(R)$ for each i, j and k, where $a_i, b_j, c_k \in R$ and $g_i, h_j, l_k \in M$.*

Proposition 2.2 *Every subring of a weak 3-M-Armendariz rings is a weak 3-M-Armendariz.*

Proof. It is obvious.

We introduce the following notation (see [15]).

Condition (P): For all $a, b, c \in R$, if $(abc)^2 = 0$, then $abc = 0$.

Lemma 2.3 *[13, Proposition 1]. If R is a reduced ring, then R satisfies the condition (P), but the converse is not true.*

Lemma 2.4 *[6, Lemma 1.1]. Assume M is a $u.p$.-monoid. Then M is cancellative (i.e., for $g, h, x \in M$, if $gx = hx$ or $xg = xh$, then $g = h$).*

Theorem 2.5 *Let M be a $u.p.$-monoid and $nil(R)$ an ideal of R. Then R is weak 3-M-Armendariz.*

Proof. Let $\alpha = \sum_{i=1}^{n} a_ig_i, \beta = \sum_{j=1}^{m} b_jh_j$ and $\gamma = \sum_{k=1}^{r} c_kl_k$ in $R[M]$, satisfy $\alpha\beta\gamma = 0$. Since $nil(R)$ is an ideal of R, the ring $\overline{R} = R/nil(R)$ is reduced. By Lemma 2.3, $\overline{R} = R/nil(R)$ satisfies the condition (P) and so 3-M-Armendariz, by [1, Theorem 2.6]. Also, $\alpha\beta\gamma = 0$ implies that $\overline{\alpha}\overline{\beta}\overline{\gamma} = \overline{0}$. So $\overline{a}_i \bullet \overline{b}_j\overline{c}_k = \overline{0}$, for each i, j and k, since $\overline{R}$ is 3-M-Armendariz. Thus $a_ib_jc_k \in nil(R)$, for each i, j and k, and the result follows.

Proposition 2.6 *Let M be a $u.p.$-monoid and R a reduced ring. Then R is weak 3-M-Armendariz.*

Proof. Since R is reduced, hence $nil(R) = 0$ is an ideal of R. Thus, the result follows from Theorem 2.5.

Corollary 2.7 *Let M be a $u.p.$-monoid and R a ring satisfying the condition (P). Then R is a weak 3-M-Armendariz.*

Proof. Let M be a $u.p.$-monoid and R a ring satisfying the condition (P). Then by [1, Theorem 2.6], R is 3-M-Armendariz ring. Thus, R is a weak 3-M-Armendariz.

Let $(M,\leq)$ be an ordered monoid. If for any $g,g',h\in M, g<g'$ implies that $gh<g'h$ and $hg<hg'$, then $(M,\leq)$ is called a strictly ordered monoid.

Corollary 2.8 *Let M be a strictly totally ordered monoid and R a ring satisfying the condition (P). Then R is a weak 3-M-Armendariz.*

Corollary 2.9 *If a ring R satisfies condition (P), then R is a weak 3-$\mathbb{Z}$-Armendariz, that is, for any $\alpha = a_{-m}x^{-m} + a_{-(m-1)}x^{-(m-1)} + \cdots + a_p x^p, \beta = b_{-n}x^{-n} + b_{-(n-1)}x^{-(n-1)} + \cdots + b_q x^q$ and $\gamma = c_{-t}x^{-t} + c_{-(t-1)}x^{-(t-1)} + \cdots + c_s x^s \in R[x,x^{-1}]$, if $\alpha\beta\gamma = 0$, then $a_i b_j c_k \in nil(R)$ for $-m \leq i \leq p, -n \leq j \leq q$ and $-t \leq k \leq s$.*

In [1, Theorem 2.9], it was shown that if I is a reduced ideal of R such that R/I is 3-M-Armendariz, then R is 3-M-Armendariz. Here we have the following result for a weak 3-M-Armendariz property.

Theorem 2.10 *Let M be a strictly totally ordered monoid and I an ideal of R. If I is reduced and R/I is a weak 3-M-Armendariz, then R is a weak 3-M-Armendariz.*

Proof. Let $\alpha,\beta,\gamma \in R[M]$ be such that $\alpha\beta\gamma = 0$. We write $\alpha = a_1g_1 + \cdots + a_n g_n, \beta = b_1h_1 + \cdots + b_m h_m$ and $\gamma = c_1l_1 + \cdots + c_r l_r \in R[M]$, with

$$g_1 < g_2 < \cdots < g_n, h_1 < h_2 < \cdots < h_m, l_1 < l_2 < \cdots < l_r.$$

We will use transfinite induction on the strictly totally ordered set $(M,\leq)$ to show that $a_i b_j c_k \in \text{nil}(R)$, for any i, j and k. Note that in $(R/I)[M], (\bar{a}_1 g_1 + \bar{a}_2 g_2 + \cdots + \bar{a}_n g_n)(\bar{b}_1 h_1 + \bar{b}_2 h_2 + \cdots + \bar{b}_m h_m)(\bar{c}_1 l_1 + \bar{c}_2 l_2 + \cdots + \bar{c}_r l_r) = 0.$ Thus we have $a_i b_j c_k \in I$ for all i, j and k with $1\leq i\leq n, 1\leq j\leq m$ and $1\leq k\leq r$, since R/I is a weak 3-M-Armendariz.

If there exist $1\leq i\leq n, 1\leq j\leq m$ and $1\leq k\leq r$, such that $g_i h_j l_k = g_1 h_1 l_1$, then $g_1 \leq g_i, h_1\leq h_j$ and $l_1\leq l_k$. If $g_1 < g_i$, then $g_1h_1l_1 < g_ih_1l_1 \leq g_ih_jl_k = g_1h_1l_1$, a contradiction, thus $g_1 = g_i$. Similarly, $h_1 = h_j$ and $l_1 = l_k$. Hence $a_1b_1c_1 = 0 \in \text{nil}(R)$. Now suppose that $w\in M$ is such that for any g_i, h_j and l_k, if $g_ih_jl_k < w$, then $a_ib_jc_k = 0$. We will show that $a_ib_jc_k \in \text{nil}(R)$, for any g_i, h_j and l_k, with $g_ih_jl_k = w$. Set $X = \{(g_i, h_j, l_k) \mid g_ih_jl_k = w\}$. Then X is a finite set. We write X as $\{(g_{i_t}, h_{j_t}, l_{k_t}) \mid t = 1,2,\cdots,u\}$ such that

$$g_{i_1} < g_{i_2} < \cdots < g_{i_u}.$$

We claim that

$$h_{j_u} l_{k_u} < \cdots < h_{j_2} l_{k_2} < h_{j_1} l_{k_1}.$$

In fact, if $h_{j_1} l_{k_1} < h_{j_2} l_{k_2}$, then

$$w = g_{i_1} h_{j_1} l_{k_1} < g_{i_1} h_{j_2} l_{k_2} < g_{i_2} h_{j_2} l_{k_2} = w,$$

a contradiction. If $h_{j_1}l_{k_1} = h_{j_2}l_{k_2}$, then from $g_{i_1}h_{j_1}l_{k_1} = w = g_{i_2}h_{j_2}l_{k_2}$ it follows that $g_{i_1} = g_{i_2}$, a contradiction again. Thus, $h_{j_2}l_{k_2} < h_{j_1}l_{k_1}$. Similarly we have the claim. For any $t \geq 2, g_{i_1}h_{j_t}l_{k_t} < g_{i_t}h_{j_t}l_{k_t} = w,$ and thus, by induction hypothesis, we have $a_{i_1}b_{j_t}c_{k_t} \in$ nil(R). Then we have $b_{j_t}c_{k_t}Ia_{i_1} = 0,$ since $b_{j_t}c_{k_t}Ia_{i_1} \subseteq I, (b_{j_t}c_{k_t}Ia_{i_1})^2 = 0,$ and I is reduced. Thus for any $t \geq 2$, $(a_{i_t}b_{j_t}c_{k_t})(a_{i_1}b_{j_1}c_{k_1})^2 = (a_{i_t}b_{j_t}c_{k_t})(a_{i_1}b_{j_1}c_{k_1})(a_{i_1}b_{j_1}c_{k_1}) \in (a_{i_t}b_{j_t}c_{k_t})I(a_{i_1}b_{j_1}c_{k_1}) = a_{i_t}(b_{j_t}c_{k_t}Ia_{i_1})b_{j_1}c_{k_1} = 0.$ Which implies that $(a_{i_t}b_{j_t}c_{k_t})(a_{i_1}b_{j_1}c_{k_1})^2 = 0.$ Now, from

$$\sum_{(g_i,h_j,l_k)\in X}(a_ib_jc_k) = \sum_{t=1}^{u}a_{i_t}b_{j_t}c_{k_t} = 0,$$

it follows that

$$(\sum_{t=1}^{u}a_{i_t}b_{j_t}c_{k_t})(a_{i_1}b_{j_1}c_{k_1})^2 = (a_{i_1}b_{j_1}c_{k_1})^3 = 0.$$

Since $a_{i_1}b_{j_1}c_{k_1} \in I$ and I is reduced, we have $a_{i_1}b_{j_1}c_{k_1} \in$ nil(R). Thus, $\Sigma_{t=2}^{u}a_{i_t}b_{j_t}c_{k_t} = 0.$ Multiplying $(a_{i_2}b_{j_2}c_{k_2})^2$ on $\Sigma_{t=2}^{u}a_{i_t}b_{j_t}c_{k_t} = 0,$ from the right-hand side, we obtain $a_{i_2}b_{j_2}c_{k_2} \in$ nil(R), by the same way as the above. Continuing this process, we can prove $a_{i_t}b_{j_t}c_{k_t} \in$ nil(R), for $t = 1,2,\cdots,u.$ Thus, $a_ib_jc_k \in$ nil(R) for any i, j and k with $g_ih_jl_k = w.$ Therefore, by transfinite induction, $a_ib_jc_k \in nil(R)$ for any i, j and $k.$ Thus R is a weak 3-M-Armendariz.

Recall that a monoid M is called torsion-free if the following property holds: if $g, h \in M$ and $k \geq 1$ are such that $g^k = h^k$, then $g = h.$

Corollary 2.11 *Let M be a commutative, cancellative, and torsion-free monoid. If one of the following conditions holds, then R is a weak 3-M-Armendariz.*

1. R satisfies condition (P).

2. R/I is a weak 3-M-Armendariz for some ideal I of R, and I is reduced.

Proof. If M is commutative, cancellative, and torsion-free, then, by Ribenboim [11], there exists a compatible strict total order $\leq$ on $M.$ Now the results follow from Corollary 2.8 and Theorem 2.10.

Proposition 2.12 *Suppose that R is a weak 3-M-Armendariz, $n \geq 3.$ If $\alpha_1, \alpha_2, \cdots, \alpha_n \in R[M]$ are such that, $\alpha_1\alpha_2\cdots\alpha_n = 0,$ then $a_1a_2\cdots a_n \in$ nil$(R),$ where a_i is a coefficient of α_i.*

Proof. It follows easily from the definition.

In [16], Cuiping and Jianlong showed that a ring R is weak M-Armendariz if and only if $T_n(R)$ is weak M-Armendariz. For weak 3-M-Armendariz, in the following we will give more results:

Theorem 2.13 *Let M be a monoid with $|M| \geq 2.$ Then R is a weak 3-M-Armendariz if and only if, for any n, $T_n(R)$ is a weak 3-M-Armendariz.*

Proof. We note that any subring of weak 3- M -Armendariz rings is weak 3- M -Armendariz. Thus if $T_n(R)$ is a weak 3- M -Armendariz ring, then R is a weak 3- M -Armendariz ring. Conversely, Let $\alpha, \beta, \gamma \in R[M]$, we write $\alpha = A_1 g_1 + \cdots + A_n g_n, \beta = B_1 h_1 + \cdots + B_m h_m$ and $\gamma = C_1 l_1 + \cdots + C_r l_r \in R[M]$ be elements of $T_n(R)[M]$. Assume that $\alpha\beta\gamma = 0$. It is easy to see that there exists an isomorphism of rings $T_n(R)[M] \to T_n(R[M])$ define by:

$$\sum_{i=1}^{p} \begin{pmatrix} a_{11}^i & a_{12}^i & a_{13}^i & \cdots & a_{1n}^i \\ 0 & a_{22}^i & a_{23}^i & \cdots & a_{2n}^i \\ 0 & 0 & a_{33}^i & \cdots & a_{3n}^i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^i \end{pmatrix} g_i \mapsto$$

$$\begin{pmatrix} \sum_{i=1}^{p} a_{11}^i g_i & \sum_{i=1}^{p} a_{12}^i g_i & \sum_{i=1}^{p} a_{13}^i g_i & \cdots & \sum_{i=1}^{p} a_{1n}^i g_i \\ 0 & \sum_{i=1}^{p} a_{22}^i g_i & \sum_{i=1}^{p} a_{23}^i g_i & \cdots & \sum_{i=1}^{p} a_{2n}^i g_i \\ 0 & 0 & \sum_{i=1}^{p} a_{33}^i g_i & \cdots & \sum_{i=1}^{p} a_{3n}^i g_i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{i=1}^{p} a_{nn}^i g_i \end{pmatrix}.$$

Assume that

$$A_i = \begin{pmatrix} a_{11}^i & a_{12}^i & a_{13}^i & \cdots & a_{1n}^i \\ 0 & a_{22}^i & a_{23}^i & \cdots & a_{2n}^i \\ 0 & 0 & a_{33}^i & \cdots & a_{3n}^i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^i \end{pmatrix}, B_j = \begin{pmatrix} b_{11}^j & b_{12}^j & b_{13}^j & \cdots & b_{1n}^j \\ 0 & b_{22}^j & b_{23}^j & \cdots & b_{2n}^j \\ 0 & 0 & b_{33}^j & \cdots & b_{3n}^j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_{nn}^j \end{pmatrix}$$

and

$$C_k = \begin{pmatrix} c_{11}^k & c_{12}^k & c_{13}^k & \cdots & c_{1n}^k \\ 0 & c_{22}^k & c_{23}^k & \cdots & c_{2n}^k \\ 0 & 0 & c_{33}^k & \cdots & c_{3n}^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{nn}^k \end{pmatrix}.$$

Then

$$\begin{pmatrix} \sum_{i=1}^{p} a_{11}^i g_i & \sum_{i=1}^{p} a_{12}^i g_i & \sum_{i=1}^{p} a_{13}^i g_i & \cdots & \sum_{i=1}^{p} a_{1n}^i g_i \\ 0 & \sum_{i=1}^{p} a_{22}^i g_i & \sum_{i=1}^{p} a_{23}^i g_i & \cdots & \sum_{i=1}^{p} a_{2n}^i g_i \\ 0 & 0 & \sum_{i=1}^{p} a_{33}^i g_i & \cdots & \sum_{i=1}^{p} a_{3n}^i g_i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{i=1}^{p} a_{nn}^i g_i \end{pmatrix}$$

$$\times \begin{pmatrix} \sum_{j=1}^{q} b_{11}^j h_j & \sum_{j=1}^{q} b_{12}^j h_j & \sum_{j=1}^{q} b_{13}^j h_j & \cdots & \sum_{j=1}^{q} b_{1n}^j h_j \\ 0 & \sum_{j=1}^{q} b_{22}^j h_j & \sum_{j=1}^{q} b_{23}^j h_j & \cdots & \sum_{j=1}^{q} b_{2n}^j h_j \\ 0 & 0 & \sum_{j=1}^{q} b_{33}^j h_j & \cdots & \sum_{j=1}^{q} b_{3n}^j h_j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{j=1}^{q} b_{nn}^j h_j \end{pmatrix}$$

$$\times \begin{pmatrix} \sum_{k=1}^{d} c_{11}^k l_k & \sum_{k=1}^{d} c_{12}^k l_k & \sum_{k=1}^{d} c_{13}^k l_k & \cdots & \sum_{k=1}^{d} c_{1n}^k l_k \\ 0 & \sum_{k=1}^{d} c_{22}^k l_k & \sum_{k=1}^{d} c_{23}^k l_k & \cdots & \sum_{k=1}^{d} c_{2n}^k l_k \\ 0 & 0 & \sum_{k=1}^{d} c_{33}^k l_k & \cdots & \sum_{k=1}^{d} c_{3n}^k l_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sum_{k=1}^{d} c_{nn}^k l_k \end{pmatrix} = 0.$$

It follows that

$$(\sum_{i=1}^{p} a_{ss}^i g_i)(\sum_{j=1}^{q} b_{ss}^j h_j)(\sum_{k=1}^{d} c_{ss}^k l_k) = 0, s = 1,2,\cdots,n$$

Since R is weak 3-M-Armendariz, there exists $m_{ijks} \in \mathbb{N}$ such that $(a_{ss}^i b_{ss}^j c_{ss}^k)^{m_{ijks}} = 0$ for any s and any i, j, k. Let $m_{ijk} = max\{m_{ijk1}, m_{ijk2}, \cdots, m_{ijkn}\}$. Then

$$(A_iB_jC_k)^{m_{ijk}} = \begin{pmatrix} (a_{11}^i b_{11}^j c_{11}^k)^{m_{ijk}} & * & \cdots & * \\ 0 & (a_{22}^i b_{22}^j c_{22}^k)^{m_{ijk}} & \cdots & * \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \cdots & (a_{nn}^i b_{nn}^j c_{nn}^k)^{m_{ijk}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus $((A_iB_jC_k)^{m_{ijk}})^n = 0$ and so $A_iB_jC_k \in nil(T_n(R))$. This shows that $T_n(R)$ is a weak 3-M-Armendariz ring.

Corollary 2.14 *Let M be a monoid. If a ring R is 3-M-Armendariz ring, then, for any n, $T_n(R)$ is a weak 3-M-Armendariz ring.*

Proposition 2.15 *Let M be a monoid. A ring R is a weak 3-M-Armendariz if and only if the trivial extension $T(R,R)$ is a weak 3-M-Armendariz ring.*

Proof. It follows from Theorem 2.13.

From Theorem 2.13, one may suspect that if R is a weak 3-M-Armendariz then every n-by-n full matrix ring $M_n(R)$ over R is a weak 3-M-Armendariz, where $n \geq 2$. But the following example erases the possibility.

Example 2.16 *Let R be a ring and let $S = M_2(F)$. Let*

$$\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g, \beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad g$$

and

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} g$$

be elements in $S[M]$, where $e \neq g \in M$. Then $\alpha\beta\gamma = 0$. But

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

is not nilpotent. Thus S is not weak 3-M-Armendariz.

Clearly, every 3-M-Armendariz ring is a weak 3-M-Armendariz, but the converse is not true by the following example.

Example 2.17 *Let R be a weak 3-M-Armendariz ring. Then the ring*

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

is not 3-M -Armendariz by [1, Example 2.14], when $n \geq 4$, but R_n is a weak 3-M -Armendariz ring by Theorem 2.13, since any subring of a weak 3-M -Armendariz rings is a weak 3-M -Armendariz.

Proposition 2.18 *The class of weak 3-M -Armendariz rings is closed under finite direct products.*

Proof. Let $R = \prod_{s\in\rho} R_s$ be the finite direct product of R_s where $\rho = \{1,2,\cdots,p\}$, R_s is weak 3-M -Armendariz ring. Suppose $\alpha\beta\gamma = 0$ for some elements $\alpha = a_1 g_1 + \cdots + a_n g_n, \beta = b_1 h_1 + \cdots + b_m h_m$ and $\gamma = c_1 l_1 + \cdots + c_r l_r \in R[M]$, where $a_i = (a_{i1}, a_{i2}, \cdots, a_{ip}), b_j = (b_{j1}, b_{j2}, \cdots, b_{jp}), c_k = (c_{k1}, c_{k2}, \cdots, c_{kp}),$ are elements of the product ring R. Set $\alpha_s = \Sigma_{i=0}^{n} a_{is} g_i$, $\beta_s = \Sigma_{j=0}^{m} b_{js} h_j$ and $\gamma_s = \Sigma_{k=0}^{r} c_{ks} l_k \in R[M].$ Since $\alpha\beta\gamma = 0$ then $\Sigma_{i+j+k=t} a_i b_j c_k = 0, 0 \leq t \leq i+j+k,$ so $\Sigma_{i+j+k=t}(a_{i1} b_{j1} c_{k1}, \cdots, a_{ip} b_{jp} c_{kp}) = 0,$ and so $\Sigma_{i+j+k=t}(a_{is} b_{js} c_{ks}) = 0, 1 \leq s \leq p.$ Thus $\alpha_s \beta_s \gamma_s = 0$ in $R_s[M], 1 \leq s \leq p.$ Since R_s is weak 3-M-Armendariz rings, we have $a_{is} b_{js} c_{ks} \in nil(R_s).$ Now, for each $i, j, k,$ there exist positive integers m_{ijks} such that $(a_{is} b_{js} c_{ks})^{m_{ijks}} = 0$, in the ring $R_s, 1 \leq s \leq p.$ If we take $m_{ijk} = max\{m_{ijks} : 1 \leq s \leq p\},$ then it is clear that $(a_{is} b_{js} c_{ks})^{m_{ijk}} = 0.$ Therefore $a_i b_j c_k \in nil(R).$ This means that R is a weak 3-M -Armendariz.

Recall that an element u of a ring R is right regular if $ur = 0$ implies $r = 0$ for $r \in R.$ Similarly, left regular elements can be defined. An element is regular if it is both left and right regular (and hence not a zero divisor).

Theorem 2.19 *Let R be a ring and Δ be a multiplicative monoid in R consisting of central regular elements. Then R is a weak 3-M -Armendariz if and only if $\Delta^{-1}R$ is also weak 3-M -Armendariz.*

Proof. Let R be a weak 3-M -Armendariz ring, and $S = \Delta^{-1}R.$ Put $\alpha\beta\gamma = 0,$ where $\alpha = \sum_{i=1}^{n} a_i g_i, \beta = \sum_{j=1}^{m} b_j h_j$ and $\gamma = \sum_{k=1}^{r} c_k l_k \in S[M].$ We may assume that $a_i = \varepsilon_i u^{-1}, b_j = \eta_j v^{-1}$ and $c_k = \mu_k w^{-1}$ with ε_i, η_j, μ_k are in R for all i, j and $k,$ and $u, v, w \in \Delta.$ We will show that $a_i b_j c_k \in nil(S[M]).$ Now we have

$$\begin{aligned}0 &= \alpha\beta\gamma \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} a_i b_j c_k g_i h_j l_k \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} \varepsilon_i \eta_j \mu_k u^{-1} v^{-1} w^{-1} g_i h_j l_k \\ &= (\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} \varepsilon_i \eta_j \mu_k g_i h_j l_k)(uvw)^{-1}.\end{aligned}$$

Hence

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r} \varepsilon_i \eta_j \mu_k g_i h_j l_k = 0$$

in $R[M]$. Since R is a weak 3-M-Armendariz, $\varepsilon_i \eta_j \mu_k \in nil(R)$, for all i, j and k and so $a_i b_j c_k = \varepsilon_i u^{-1} \eta_j v^{-1} \mu_k w^{-1} = \varepsilon_i \eta_j \mu_k (uvw)^{-1} \in nil(S[M])$, for all i, j, k. Thus, S is a weak 3-M-Armendariz. The converse follows from Proposition 2.2.

Proposition 2.20 *Let M be a monoid, and R be a ring and e an idempotent of R. If e is central in R, then the following statements are equivalent:*

1. R is a weak 3-M-Armendariz;
2. eR and $(1-e)R$ are weak 3-M-Armendariz.

Proof. $(1)\Rightarrow(2)$ is obvious since eR and $(1-e)R$ are subrings of R. $(2)\Rightarrow(1)$ Let $\alpha = \sum_{i=1}^{n} a_i g_i, \beta = \sum_{j=1}^{m} b_j h_j$ and $\gamma = \sum_{k=1}^{r} c_k l_k \in R[M]$ be such that $\alpha\beta\gamma = 0$. Let $\alpha_1 = e\alpha, \alpha_2 = (1-e)\alpha, \beta_1 = e\beta, \beta_2 = (1-e)\beta$ and $\gamma_1 = e\gamma, \gamma_2 = (1-e)\gamma$. Then $\alpha_1\beta_1\gamma_1 = 0$ and $\alpha_2\beta_2\gamma_2 = 0$. Since eR and $(1-e)R$ are weak 3-M-Armendariz, there exist m_{ijk} and n_{ijk} such that $(ea_i eb_j ec_k)^{m_{ijk}} = (e(a_i b_j c_k))^{m_{ijk}} \in nil(eR)$ and $((1-e)a_i(1-e)b_j(1-e)c_k)^{n_{ijk}} = ((1-e)(a_i b_j c_k))^{n_{ijk}} \in nil((1-e)R)$. Thus, $e(a_i b_j c_k)^{m_{ijk}} = 0$ and $(1-e)(a_i b_j c_k)^{n_{ijk}} = 0$. Let $s_{ijk} = max\{m_{ijk}, n_{ijk}\}$. Then $e(a_i b_j c_k)^{s_{ijk}} = 0$ and $(1-e)(a_i b_j c_k)^{s_{ijk}} = 0$. Thus, $(a_i b_j c_k)^{s_{ijk}} \in nil(R)$. This means that R is weak 3-M-Armendariz.

Proposition 2.21 *Let M be a monoid, and R be a ring and I an ideal of R such that R/I is a weak 3-M-Armendariz. If $I \subseteq nil(R)$, then R is weak 3-M-Armendariz.*

Proof. Let $\alpha = \sum_{i=1}^{n} a_i g_i, \beta = \sum_{j=1}^{m} b_j h_j$ and $\gamma = \sum_{k=1}^{r} c_k l_k \in R[M]$ be such that $\alpha\beta\gamma = 0$. Then $(\sum_{i=1}^{n} \bar{a}_i g_i)(\sum_{j=1}^{m} \bar{b}_j h_j)(\sum_{k=1}^{r} \bar{c}_k l_k) = 0$. Since R/I is a weak 3-M-Armendariz, we have that $(\bar{a}_i \bar{b}_j \bar{c}_k) \in nil(R/I)$. Hence $a_i b_j c_k \in I$. Since $I \subseteq nil(R)$ then $a_i b_j c_k \in nil(R)$. This means that R is a weak 3-M-Armendariz.

In [7, proposition 2.1], it was shown that if R is a reduced and M-Armendariz ring, then $R[M]$ is N-Armendariz, where M is a monoid and N a $u.p.$-monoid. Elshokry and et al. [1, proposition 3.1], it was shown that if R satisfies the condition (P) and 3-M-Armendariz, then $R[M]$ is 3-N-Armendariz, where N is a

$u.p.$-monoid. For a weak 3-M -Armendariz, we have the following results.

Proposition 2.22 *Let M be a monoid and* N *a* $u.p$ *.-monoid. If* R *is a semicommutative ring which is also weak 3-*M *-Armendariz, then* $R[M]$ *is a weak 3-*N *-Armendariz.*

Proof. Suppose that $\alpha = a_1g_1 + \cdots + a_ng_n, \beta = b_1h_1 + \cdots + b_mh_m$ and $\gamma = c_1l_1 + \cdots + c_kl_k \in R[M],$ such that $(\alpha\beta\gamma)^2 = 0.$ Then $(a_ib_jc_k)^2 = 0$ for all i, j and k, since R satisfies condition (P) and a weak 3-M -Armendariz. Thus, $a_ib_jc_k \in nil(R)$ for all i, j and $k.$ Hence $\alpha\beta\gamma = 0.$ This shows that $R[M]$ satisfies condition (P). Now the result follows from Theorem 2.5.

Lemma 2.23 *[16, Lemma 3] Let* R *be a semicommutative ring and* M *a monoid. If* $a_1, \cdots, a_n \in nil(R),$ *then* $a_1g_1 + \cdots + a_ng_n \in nil(R[M])$

Proposition 2.24 *Let* M *be a monoid and* N *a* $u.p$ *.-monoid. If* R *a semicommutative ring which is also weak 3-*M *-Armendariz, then* $R[N]$ *is a weak 3-*M *-Armendariz.*

Proof. It is easy to see that there exists an isomorphism of rings $R[N][M] \cong R[M][N]$ defined by

$$\sum_p(\sum_i a_{ip}n_i)m_p \to \sum_i(\sum_p a_{ip}m_p)n_i.$$

Now suppose that $\alpha_i, \beta_j, \gamma_k \in R[N]$ are such that $(\sum_i \alpha_i g_i)(\sum_j \beta_j h_j)(\sum_k \gamma_k l_k) = 0,$ where $g_i, h_j, l_k \in M.$ We will show that $\alpha_i\beta_j\gamma_k \in nil(R[N])$ for all i, j and $k.$ Assume that $\alpha_i = \sum_p a_{ip}n_p, \beta_j = \sum_q b_{jq}n'_q$ and $\gamma_k = \sum_s c_{ks}n''_s,$ where $n_p, n'_q, n''_s \in N$ for all p, q and $s.$ Then

$$(\sum_i(\sum_p a_{ip}n_p)g_i)(\sum_j(\sum_q b_{jq}n'_q)h_j)(\sum_k(\sum_s c_{ks}n''_s)l_k) = 0.$$

Thus, in $R[M][N]$ we have

$$(\sum_p(\sum_i a_{ip}g_i)n_p)(\sum_q(\sum_j b_{jq}h_j)n'_q)(\sum_s(\sum_k c_{ks}l_k)n''_s) = 0.$$

By Proposition 2.22, $R[M]$ is a weak 3-N -Armendariz, $(\sum_i a_{ip}g_i)(\sum_j b_{jq}h_j)(\sum_k c_{ks}l_k) = 0$ for all p, q and $s.$ Since R is a weak 3-M -Armendariz, $a_{ip}b_{jq}c_{ks} \in nil(R)$ for all $i, j, k, p, q, s.$ Hence $\alpha_i\beta_j\gamma_k = 0$. By Lemma 2.23, this means that $R[N]$ is a weak 3-M -Armendariz.

Theorem 2.25 *Let* M *be a monoid and* N *a* $u.p.$*-monoid. If* R *a semicommutative ring which is also a weak 3-*M *-Armendariz, then* R *is a weak 3-*$(M \times N)$*-Armendariz.*

Proof. Suppose that $\sum_{i=1}^{s} a_i(m_i, n_i)$ is in $R[M \times N].$ Without loss of generality, we assume that $\{n_1, n_2, \cdots, n_s\} = \{n_1, n_2, \cdots, n_t\}$ with $n_i \neq n_j$ when $1 \leq i \neq j \leq t.$ For any $1 \leq p \leq t,$ denote $A_p = \{i \mid 1 \leq i \leq s, n_i = n_p\}$. Then $\sum_{p=1}^{t}\sum_{i \in A_p}(a_im_i)n_p \in R[M][N].$ Note that $m_i \neq m_{i'}$ for any $i, i' \in A_p$ with $i \neq i'.$ Now it is easy to see that there exists an isomorphism of rings $R[M \times N] \to R[M][N]$ define by

$$\sum_{i=1}^{s} a_i(m_i, n_i) \to \sum_{p=1}^{t} \sum_{i \in A_p} (a_i m_i) n_p.$$

Suppose that

$$(\sum_{i=1}^{s} a_i(m_i, n_i))(\sum_{j=1}^{s'} b_j(m'_j, n'_j))(\sum_{k=1}^{s''} c_k(m''_k, n''_k)) = 0$$

in $R[M \times N]$. Then from the above isomorphism it follows that

$$(\sum_{p=1}^{t} (\sum_{i \in A_p} a_i m_i) n_p)(\sum_{q=1}^{t'} (\sum_{j \in B_q} b_j m'_j) n'_q)(\sum_{r=1}^{t''} (\sum_{k \in C_r} c_k m''_k) n''_r) = 0.$$

By Proposition 2.22, $R[M]$ is a weak 3- N -Armendariz. Thus we have

$$(\sum_{i \in A_p} a_i m_i)(\sum_{j \in B_q} b_j m'_j)(\sum_{k \in C_r} c_k m''_k) \in nil(R[M])$$

for any p, q and r. Since R is a weak 3- M -Armendariz, $a_i b_j c_k \in nil(R)$ for any $i \in A_p$, $j \in B_q$ and any $k \in C_r$. Thus, $a_i b_j c_k \in nil(R)$ for all i, j, k. This means that R is a weak 3- $(M \times N)$ -Armendariz.

Let $M_i, i \in I$, be monoids. Denote $\coprod_{i \in I} M_i = \{(g_i)_{i \in I} \mid$ there exist only finite i's such that $g_i \neq e_i$, the identity of $M_i\}$. Then $\coprod_{i \in I} M_i$ is a monoid with the operation $(g_i)_{i \in I} (g'_i)_{i \in I} = (g_i g'_i)_{i \in I}$.

Corollary 2.26 *Let $M_i, i \in I$ be $u.p$.-monoids and R a semicommutative ring. If R is a weak 3- M_{i_0} -Armendariz for some $i_0 \in I$, then R is a weak 3- $\coprod_{i \in I} M_i$ -Armendariz.*

Proof. Let $\alpha = \Sigma_i a_i g_i, \beta = \Sigma_j b_j h_j, \gamma = \Sigma_k c_k l_k \in R[\coprod_{i \in I} M_i]$ such that $\alpha\beta\gamma \in nil(R[\coprod_{i \in I} M_i])$. Then $\alpha, \beta, \gamma \in R[M_1 \times M_2 \times \cdots \times M_n]$, for some finite subset $\{M_1, M_2, \cdots, M_n\} \subseteq \{M_i \mid i \in I\}$. Thus $\alpha, \beta, \gamma \in R[M_{i_0} \times M_1 \times M_2 \times \cdots \times M_n]$. The ring R, by Theorem 2.25 and by induction, is a weak 3- $(M_{i_0} \times M_1 \times M_2 \times \cdots \times M_n)$ -Armendariz, so $a_i b_j c_k \in nil(R)$ for all i, j and k. Hence R is a weak 3- $\coprod_{i \in I} M_i$ -Armendariz.

Corollary 2.27 *Let M be a monoid and R a semicommutative ring. If R is a weak 3- M -Armendariz, then $R[x]$ and $R[x, x^{-1}]$ are weak 3- M -Armendariz.*

Proof. Note that $R[x] \cong R[\mathbb{N} \cup \{0\}]$ and $R[x, x^{-1}] \cong R[\mathbb{Z}]$.

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