APPLICATION OF THE BPOMS FOR NUMERICAL SOLUTION OF THE ISOTHERMAL GAS SPHERES EQUATIONS

Behrooz Basirat [1,*] & Mohammad Amin Shahdadi[2]

[1,2] Department of Mathematics, Birjand Branch, Islamic Azad University, Birjand, Iran

ABSTRACT

In this article we present a matrix method for solving isothermal gas spheres equations (IGSEs) with initial conditions by Bernstein polynomials operational matrices (BPOMs) on interval [a,b]. This method transforms (IGSEs) and the given conditions into matrix equation which corresponds to a system of linear algebraic equations.

Keywords: *Lane-Emden type equations, Isothermal gas spheres equations, Singular initial value problems, Bernstein polynomials operational matrices*

1. INTRODUCTION

The study of singular initial value problems (IVPs) modeled by second-order nonlinear ordinary differential equations have attracted many mathematicians and physicists. One of the equations in this category is the following Lane-Emden type equations:

$$u''(x)+\frac{L}{x}u'(x)+f\,(x,u(x))=g(x), \quad x\geq 0, \quad \mathrm{L}\geq 1, \tag{1}$$

subject to initial conditions

$$u(0)=\alpha \text{ and } u'(0)=\beta, \tag{2}$$

where the prime denotes differentiation with respect to x, and α, β and L are constants and f is a real valued continuous function.

Lane-Emden type equation, first published by Jonathan Homer Lane in 1870 [1], and further explored in detail by Emden [2]. Lane-Emden type equation models many phenomena in mathematical physics and astrophysics. It is categorized as singular nonlinear initial value problem. This equation describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. The polytropic theory of stars essentially follows out of thermodynamic considerations, that deals with the issue of energy transport through the transfer of material between different levels of the star. Also this equation describes the equilibrium density distribution in self gravitating sphere of polytrophic isothermal gas and has a singularity at the origin.

Many problems in mathematical physics and astrophysics can be modeled by the singular IVPs of Lane-Emden type equation such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere and theory of thermionic currents [3-5].

In recent years, many powerful methods have been presented for solving of Lane-Emden type equations. For instance, the homotopy perturbation method [6-8], the Legendre wavelets [9], the variational iteration method [10-11], the B-spline method [12], the Adomian decomposition method [13], the Bessel collocation method [14], the Pade series method [15], the rational Legendre pseudospectral method [16], the nonperturbative approximate method [17], the Hermite functions collocation method [18], and the variational approach method [19].

The Lane-Emden equation of the first kind, used in the field of astrophysics, is actually the Poisson's equation describing the gravitational potential of a self-gravitating spherically symmetric polytropic fluid (Emden, 1970; Lane, 1870), and as such is often used as crude model of stellar structure. The Lane-Emden equation of the first kind governing a polytropic gas sphere reads

$$u''(x)+\frac{2}{x}u'(x)+u^{m}(x)=0, \tag{3}$$

subject to initial conditions

$$u(0)=1 \text{ and } u'(0)=0. \tag{4}$$

Here, the power-law index m ranges through the natural numbers ($m=0,\ 1,\ 2,\ ...$). For $m \geq 2,$ equation (3) is nonlinear. Only the cases $m=0,$ $m=1$ and $m=5$ have an analytical solution [3] while for other values of m the solution must be found using numerical or approximation techniques.

The Lane-Emden equation of the second kind is similar in form to equation (3), yet depends on an exponential nonlinearity. The Lane-Emden equation of the second kind reads

$$u''(x)+\frac{2}{x}u'(x)+e^{u(x)}=0, \tag{5}$$

and this ordinary differential equation is held subject to the initial conditions

$$u(0)=0 \text{ and } u'(0)=0. \tag{6}$$

Equation (5) was derived by Bonnor (1956) to describe what are now commonly known as Bonnor-Ebert (Bonnor, 1956; Ebert, 1955) gas spheres – isothermal gas spheres embedded in a pressurized medium at the maximum possible mass allowing for hydrostatic equilibrium.

The aim of the present paper is to apply Bernstein polynomials operational matrices (BPOMs) to propose a reliable numerical technique for solving the Lane-Emden equation of the second kind. In this work we find the solution of the problem as the truncated Bernstein series defined by

$$u(x)=\sum_{n=0}^{N} u_n B_{n,N}(x), \qquad a \leq x \leq b, \tag{7}$$

where $B_{n,N}(x), n=0,\ 1,\ 2,\ ...,\ \mathrm{N}$ denotes the Bernstein polynomials; $u_n,\ \ 0 \leq \mathrm{n} \leq \mathrm{N}$ are unknown Bernstein coefficients, and N is any positive integer chosen such that $N \geq 2.$ To find a numeric solution in the form equation (7) of the problem (5), we use the collocation points defined by

$$x_i = a+\frac{b-a}{2N-1}i, \quad \text{i=0, 1, ..., N.} \tag{8}$$

The Bernstein polynomials of degree N are defined by

$$B_{n,N}(x)=\binom{N}{n}\frac{(x-a)^n(b-x)^{N-n}}{(b-a)^N}, \tag{9}$$

$$a \leq x \leq b, \quad n=0,\ 1,\ ...,\ N.$$

The outline of the paper is as follows: in the next section, we present the basic idea of the BPOMs method. Section 3, exhibits the final system. Section 4 illustrates some numerical examples to show the accuracy of this method. Finally section 5 concludes the paper.

2. BASIC IDEA OF BPOMS

2.1 Approximate of u(x)

A function $u(x)$ square integrable in $[a,\mathrm{b}],$ maybe expressed in terms of Bernstein basis [20]. In practice, only the first $(N+1)-$ terms Bernstein polynomials are considered. Hence, if we write

$$u(x) \approx \sum_{i=0}^{N} u_i B_{i,N}(x)=\Phi(x)\ U, \tag{10}$$

where

$$U=[u_0,\ \mathrm{u}_1,\ ...,\ \mathrm{u}_N]^T, \tag{11}$$

$$\Phi(x)=[B_{0,N}(x),\ B_{1,N}(x),\ ...,\ B_{N,N}(x)],$$

(12)

then

$$U = Q^{-1}\big(\Phi(x),\ u(x)\big), \tag{13}$$

where Q is said dual matrix of $\Phi(x)$ and is given in [20]. Now, we can write the unknown function $u(x)$ in the matrix form as follows

$$u(x) = \Phi(x)U, \tag{14}$$

Using the expression (9) and taking $n = 0,\ 1,\ ...,\ N,$ we find the corresponding matrix relation as

$$\Phi(x) = \Delta(x)A, \tag{15}$$

where

$$\Delta(x) = [1,\ x,\ ...,\ x^N], \tag{16}$$

and A is the $(N+1)\times(N+1)$ matrix. For example, on interval $[-1,1],$ we have

$$A = \begin{bmatrix} (-1)^0\binom{N}{0} & (-1)^0\binom{N-1}{0} & \cdots & \binom{N}{0} \\ (-1)^1\binom{N}{1} & (-1)^1\binom{N-1}{1} + (-1)^0\binom{N-1}{0} & \cdots & \binom{N}{1} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^N\binom{N}{N} & (-1)^{N-1}\binom{N-1}{N-1} & \cdots & \binom{N}{N} \end{bmatrix},$$

From relations (12), (14) and (15), we obtain the following matrix forms

$$u(x) = \Phi(x)U = \Delta(x)AU, \tag{17}$$

2.2 Approximate of f(x,u(x))

We can also approximate the function $f(x,\ u(x))$ by the Bernstein polynomials as

$$f(x,u) \approx f(x, \Phi(x)U) = \Phi(x)H, \tag{18}$$

where the unknown is

$$H = [h_0,\ h_1,\ ...,\ h_N]^T, \tag{19}$$

similarly (13), we have

$$H = Q^{-1}(\Phi(x),\ f(x,\ \Phi(x)U)), \tag{20}$$

3. MATRIX RELATIONS FOR u"(x) AND u'(x)

The differentiation of vector $\Phi(x)$ in equation (12) can be expressed as

$$\Phi'(x) = \Phi(x)D, \tag{21}$$

where D is the $(N+1)\times(N+1)$ operational matrix of derivatives for Bernstein polynomials [21]. From (15), we have $\Phi(x) = \Delta(x)A$ and then

$$\Phi'(x) = [0, 1,\ 2x,\ ...,\ Nx^{N-1}]\,A, \tag{22}$$

Defining $(N)\times(N+1)$ matrix V and vector $\Delta^*(x)$ as

$$V=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 2 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & N\end{bmatrix},\quad \Delta^*(x)=[1,\ \mathrm{x},\ \mathrm{x}^2,\ ...,\ \mathrm{x}^{N-1}], \tag{23}$$

equation (22) may then be restated as

$$\Phi'(x)=\Delta^*(x)\ VA, \tag{24}$$

We now expand vector $\Delta^*(x)$ in terms of $\Phi(x)$. We get $\Delta^*(x)=\Phi(x)\ \mathrm{B}^*$ where

$$B^*=[A_{[1]}^{-1},\ A_{[2]}^{-1},\ A_{[3]}^{-1},\ ...,\ A_{[N]}^{-1}], \tag{25}$$

where $A_{[k]}^{-1}$ is the $(k+1)$ the row of A^{-1} for $k=1,\ 2,\ ...,\ \mathrm{N}$. So

$$\Phi'(x)=\Phi(x)\ \mathrm{B}^*\ \mathrm{V}\ \mathrm{A}, \tag{26}$$

Therefore, we have the operational matrix of derivative as

$$\mathrm{D}=B^*\ \mathrm{V}\ \mathrm{A}. \tag{27}$$

If we approximate $u(x)\approx\Phi(x)\ U$, then for $k\geq 2$ (k is the order of derivatives), we get

$$u^{(k)}(x)\approx\Phi^{(k)}(x)\ \mathrm{U}=\Phi(\mathrm{x})\ \mathrm{D}^k\ \mathrm{U}. \tag{28}$$

3.1. Matrix Relation For Initial Conditions

We obtain the corresponding matrix forms for initial conditions (6) by means of the relations (15) and (28) as

$$\Delta(0)\ \mathrm{A}\ \mathrm{U}=0,\quad \Delta(0)\ \mathrm{A}\ \mathrm{D}\ \mathrm{U}=0, \tag{29}$$

Briefly, the matrix form for conditions (6) is

$$I_i\ \mathrm{U}=0,\ \ \mathrm{i}=0,1, \tag{30}$$

where

$$\begin{aligned} I_0&=\Delta(0)\ \mathrm{A}=[\mathrm{z}_{00},\mathrm{z}_{01},\ ...,\ \mathrm{z}_{0N}\ ],\\ I_1&=\Delta(0)\ \mathrm{A}\ \mathrm{D}=[\mathrm{z}_{10},\mathrm{z}_{11},\ ...,\ \mathrm{z}_{1N}\ ].\end{aligned} \tag{31}$$

3.2. Final System

We substitute and simplify the matrix relations (15), (18) and (29) into equation (5), and obtain the fundamental matrix equation as

$$\Delta(x)\ \mathrm{A}\ \mathrm{D}^2\ \mathrm{U}+\frac{2}{x}\Delta(x)\ \mathrm{A}\ \mathrm{D}\ \mathrm{U}+\Delta(x)\ \mathrm{A}\ \mathrm{H}=0, \tag{32}$$

By plugging the collocation points x_i defind by (8), we get the system of matrix equations

$$\Delta(x_i)\ \mathrm{A}\ \mathrm{D}^2\ \mathrm{U}+\frac{2}{x_i}\Delta(x_i)\ \mathrm{A}\ \mathrm{D}\ \mathrm{U}+\Delta(x_i)\ \mathrm{A}\ \mathrm{H}=0,\quad i=0,1,...,N, \tag{33}$$

or briefly the fundamental matrix equation

$$\left\{\Delta\ \mathrm{A}\ \mathrm{D}^2+\Gamma\ \Delta\ \mathrm{A}\ \mathrm{D}\right\}\ \mathrm{U}=-\Delta\ \mathrm{A}\ \mathrm{H}, \tag{34}$$

where

$$\Delta=\begin{bmatrix}\Delta(x_0)\\ \Delta(x_1)\\ \vdots\\ \Delta(x_N)\end{bmatrix}=\begin{bmatrix}1 & x_0 & x_0^2 & \cdots & x_0^N\\ 1 & x_1 & x_1^2 & \cdots & x_1^N\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & x_N & x_N^2 & \cdots & x_N^N\end{bmatrix}, \tag{35}$$

and

$$\Gamma=diag\left(\frac{2}{x_0},\ \frac{2}{x_1},\ ...,\ \frac{2}{x_N}\right). \tag{36}$$

Hence, equation (34) can be written in the form

$$\mathrm{W}\ \mathrm{U}=\mathrm{F},\quad \mathrm{W}=[\mathrm{w}_{pq}],\quad \mathrm{p},\mathrm{q}=0,1,...,\mathrm{N}, \tag{37}$$

where

$$\mathrm{W}=\Delta\ \mathrm{A}\ \mathrm{D}^2+\Gamma\ \Delta\ \mathrm{A}\ \mathrm{D},\quad \mathrm{F}=-\Delta\ \mathrm{A}\ \mathrm{H}, \tag{38}$$

Finally, to obtain the solution of equation (5) under the initial condition (6), we replace the row matrices (31) by the last two rows of the matrix (37), and get the new augmented matrix

$$[\,\overline{\mathrm{W}};\overline{\mathrm{F}}\,]=\begin{bmatrix}w_{00} & w_{01} & \cdots & w_{0N} & ; & -\Delta(x_0)\,\mathrm{A}\,\mathrm{H}\\ w_{10} & w_{11} & \cdots & w_{1N} & ; & -\Delta(x_1)\,\mathrm{A}\,\mathrm{H}\\ \cdots & \cdots & \cdots & \cdots & ; & \cdots\\ w_{N,0} & w_{N,1} & \cdots & w_{N,N} & ; & -\Delta(x_N)\,\mathrm{A}\,\mathrm{H}\\ & & & & & \\ z_{00} & z_{01} & \cdots & z_{0N} & ; & 0\\ z_{10} & z_{11} & \cdots & z_{1N} & ; & 0\end{bmatrix}. \tag{39}$$

If rank $\overline{\overline{\mathrm{W}}}=rank\ [\ \overline{\overline{\mathrm{W}}}\ ;\ \overline{\mathrm{F}}\ ]=\mathrm{N}+1,$ then we can write

$$U=(\overline{\overline{\mathrm{W}}})^{-1}\ \overline{\mathrm{F}}, \tag{40}$$

thus, the matrix U is uniquely determined. Also the equation (5) under the initial conditions (6) has a unique solution. This solution is given by truncated Bernstein series (7).

4. NUMERICAL EXAMPLES

To illustrate the effectiveness of the proposed method in the present paper, three examples are presented in this section.

Example 1. Consider the following isothermal gas spheres equation [10, 18, 22- 24],

$$\begin{cases} u''(x)+\dfrac{2}{x}u'(x)+8\,e^{u(x)}+4\,e^{\frac{u(x)}{2}}=0, \\ u(0)=0, \quad u'(0)=0, \end{cases} \tag{41}$$

The exact solution is $u(x)=-2\ln(1+x^2)$. The numerical results of proposed method for this example are exhibited in Table 1.

Table 1. Approximate and exact solutions for example 1.

x	Present method with N=2	Present method with N=4	Present method with N=6	Exact solution
0	0	0	0	0
0.1	-0.132363	-0.015819	-0.019921	-0.019901
0.2	-0.271471	-0.073367	-0.078466	-0.078441
0.3	-0.417325	-0.166968	-0.172380	-0.172355
0.4	-0.569925	-0.291037	-0.296859	-0.296840
0.5	-0.729272	-0.440082	-0.446311	-0.446287
0.6	-0.895363	-0.608701	-0.614998	-0.614969
0.7	-1.068201	-0.791586	-0.797578	-0.797552
0.8	-1.247785	-0.983519	-0.989539	-0.989392
0.9	-1.434114	-1.179377	-1.187516	-1.186654
1	-1.627189	-1.374124	-1.389509	-1.386294

As it can be observed, we have almost obtained the exact solution for N = 6 through BPOMs. The approximated solution for N = 2, N = 4 and N = 6 are as follows:

$$u_2(x)=-2.1079\times10^{-14}-1.2899x-0.3373x^2,$$
$$u_4(x)=8.8406\times10^{-14}+0.6961x-2.3745x^2+0.9691x^3-0.3829x^4,$$
$$u_6(x)=5.0657\times10^{-14}-0.0008x-1.9875x^2-0.0981x^3+1.4019x^4-0.8552x^5+0.1503x^6,$$

Since the exact solution is $u(x)=-2\ln(1+x^2)$, if we write Taylor expansion, we will have:

$$u(x)\approx\ -2x^2+x^4-0.6667x^6+O(x^8),$$

As it can be observed, as N increases, the approximate solution gets closer to the exact solution. By applying the method in section 2 and 3, for N = 2, we have

$$A=\begin{bmatrix}1&-2&1\\0&2&-2\\0&0&1\end{bmatrix},\quad Q^{-1}=\begin{bmatrix}9&-9&3\\-9&21&-9\\3&-9&9\end{bmatrix},\quad D=\begin{bmatrix}-2&-1&0\\2&0&-2\\0&1&2\end{bmatrix},\quad \Delta=\begin{bmatrix}1&\frac{1}{3}&\frac{1}{9}\\1&\frac{2}{3}&\frac{4}{9}\\1&1&1\end{bmatrix},\quad \Gamma=\begin{bmatrix}6&0&0\\0&3&0\\0&0&2\end{bmatrix},\quad W=\begin{bmatrix}-6&0&6\\0&-6&6\\2&-8&6\end{bmatrix},$$

$$\Phi(x)=[1-2x+x^2\quad 2x-2x^2\quad x^2],\quad [\ \overline{\overline{W}}\ ;\ \overline{\overline{F}}\ ]=\begin{bmatrix}-6&0&6&\frac{10}{3}-\frac{2}{3}\pi-16\ln(2)\\0&-6&6&-\frac{2}{3}+\frac{10}{3}\pi-24\ln(2)\\2&-8&6&-18-6\pi+48\ln(2)\\1&0&0&0\\-2&2&0&0\end{bmatrix},\quad U=\begin{bmatrix}0\\\frac{172}{87}+\frac{28}{87}\pi-\frac{152}{29}\ln(2)\\\frac{179}{261}+\frac{17}{261}\pi-\frac{316}{87}\ln(2)\end{bmatrix}.$$

Example 2. Consider the following isothermal gas spheres equation

$$\begin{cases} u''(x)+\dfrac{2}{x}u'(x)+e^{u(x)}=\dfrac{e^{\sin x}\,x-x\sin x+2\cos x}{x}, \\ u(-1)=\sin(-1), \quad \mathrm{u}'(1)=\cos(1), \end{cases} \tag{42}$$

where the exact solution is $u(x)=\sin(x)$. The approximated solution for N = 3, N = 5 and N = 7 are as follows:

$$u_3(x)=-\,0.0019+0.9978x+0.0006x^2-0.1577x^3,$$

$$u_5(x)=0.00001+0.9999x-6.885\times10^{-7}x^2-0.1666x^3-0.00003x^4+0.0080x^5,$$

$$u_7(x)=7.436\times10^{-7}+0.9999x+0.000003x^2-0.1666x^3-0.00003x^4+0.0083x^5+0.00002x^6$$
$$-0.0002x^7,$$

Since the exact solution is $u(x)=\sin(x)$, if we write Taylor expansion, we will have:

$$u(x)\approx x-0.1667x^3+0.0083x^5-0.0002x^7+O(x^9),$$

As it can be observed, as N increases, the approximate solution gets closer to the exact solution. By applying the method in section 2 and 3, for N = 3, we have

$$A=\begin{bmatrix} \frac{1}{8} & \frac{-3}{8} & \frac{3}{8} & \frac{-1}{8} \\ \frac{3}{8} & \frac{-3}{8} & \frac{-3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{-3}{8} & \frac{-3}{8} \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}, \qquad \mathrm{D}=\begin{bmatrix} \frac{-3}{2} & \frac{-1}{2} & 0 & 0 \\ \frac{3}{2} & \frac{-1}{2} & -1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{-3}{2} \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}, \qquad \Delta=\begin{bmatrix} 1 & \frac{-3}{5} & \frac{9}{25} & \frac{-27}{125} \\ 1 & \frac{-1}{5} & \frac{1}{25} & \frac{-1}{125} \\ 1 & \frac{1}{5} & \frac{1}{25} & \frac{1}{125} \\ 1 & \frac{3}{5} & \frac{9}{25} & \frac{27}{125} \end{bmatrix},$$

$$\Gamma=\begin{bmatrix} \frac{-10}{3} & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & \frac{10}{3} \end{bmatrix}, \quad \mathrm{Q}^{-1}=\begin{bmatrix} 8 & -12 & 8 & -2 \\ -12 & \frac{104}{3} & \frac{-86}{3} & 8 \\ 8 & \frac{-86}{3} & \frac{104}{3} & -12 \\ -2 & 8 & -12 & 8 \end{bmatrix}, \quad U=\begin{bmatrix} -0.841471 \\ -0.492474 \\ 0.488121 \\ 0.838765 \end{bmatrix},$$

$$[\;\overline{\mathrm{W}}\;;\;\overline{\mathrm{F}}\;]=\begin{bmatrix} \frac{22}{5} & \frac{-37}{10} & \frac{-4}{5} & \frac{1}{10} & -2.1759 \\ \frac{63}{10} & \frac{3}{5} & \frac{-51}{10} & \frac{-9}{5} & -9.6066 \\ \frac{-9}{5} & \frac{-51}{10} & \frac{3}{5} & \frac{63}{10} & 9.5934 \\ \frac{1}{10} & \frac{-4}{5} & \frac{-37}{10} & \frac{22}{5} & 2.1992 \\ 1 & 0 & 0 & 0 & -0.8415 \\ 0 & 0 & \frac{-3}{2} & \frac{3}{2} & 0.5403 \end{bmatrix}.$$

The numerical results of proposed method for this example are exhibited in Table 2.

Table 2. Approximate and exact solutions for example 2.

x	Present method with N=3	Present method with N=5	Present method with N=7	Exact solution
-1	-0.841471	-0.841471	-0.841471	-0.841471
-0.8	-0.719085	-0.717342	-0.717356	-0.717356
-0.6	-0.566373	-0.564632	-0.564642	-0.564642
-0.4	-0.390904	-0.389407	-0.389418	-0.389418
-0.2	-0.200247	-0.198657	-0.198668	-0.198669
0	-0.001970	0.000011	7.45×10^{-7}	0
0.2	0.196355	0.198678	0.198670	0.198669
0.4	0.387161	0.389427	0.389419	0.389418
0.6	0.562877	0.564645	0.564639	0.564642
0.8	0.715935	0.717336	0.717346	0.717356
1	0.838765	0.841427	0.841454	0.841471

Example 3. Consider the isothermal gas spheres equation [23]

$$\begin{cases} u''(x)+\frac{1}{x}u'(x)+e^{u(x)}=0, \\ u(1)=0, \quad u'(0)=0, \end{cases}$$

This problem has the exact solution $u(x)=2\ln\left(\frac{4-2\sqrt{2}}{(3-2\sqrt{2})x^2+1}\right)$.

The numerical results of proposed method for this example are exhibited in Table 3.

Table 3. Approximate and exact solutions for example 3.

x	Present method with N=2	Present method with N=4	Present method with N=6	Exact solution
0	0.345499	0.316399	0.316722	0.316694
0.1	0.333069	0.313019	0.313293	0.313266
0.2	0.315724	0.302787	0.303043	0.303015
0.3	0.293463	0.285830	0.286075	0.286047
0.4	0.266287	0.262328	0.262559	0.262531
0.5	0.234195	0.232507	0.232724	0.232697
0.6	0.197187	0.196645	0.196854	0.196827
0.7	0.155263	0.155065	0.155276	0.155248
0.8	0.108425	0.108141	0.108349	0.108323
0.9	0.056670	0.056295	0.056459	0.056439
1	3.5×10^{-15}	-1.6×10^{-14}	-4.2×10^{-14}	-2×10^{-10}

By applying the method in section 2 and 3, for N = 4, we have

$$A=\begin{bmatrix}1&-4&6&-4&1\\0&4&-12&12&-4\\0&0&6&-12&6\\0&0&0&4&-4\\0&0&0&0&1\end{bmatrix},\Phi^T(x)=\begin{bmatrix}1-4x+6x^2-4x^3+x^4\\4x-12x^2+12x^3-4x^4\\6x^2-12x^3+6x^4\\4x^3-4x^4\\x^4\end{bmatrix},\mathrm{D}=\begin{bmatrix}-4&-1&0&0&0\\4&-2&-2&0&0\\0&3&0&-3&0\\0&0&2&2&-4\\0&0&0&1&4\end{bmatrix},$$

$$\mathrm{Q}^{-1}=\begin{bmatrix}25&-50&50&-25&5\\-50&175&\frac{-425}{2}&\frac{235}{2}&-25\\50&\frac{-425}{2}&330&\frac{-425}{2}&50\\-25&\frac{235}{2}&\frac{-425}{2}&175&-50\\5&-25&50&-50&25\end{bmatrix},\Delta=\begin{bmatrix}1&\frac{1}{7}&\frac{1}{49}&\frac{1}{343}&\frac{1}{2401}\\1&\frac{2}{7}&\frac{4}{49}&\frac{8}{343}&\frac{16}{2401}\\1&\frac{3}{7}&\frac{9}{49}&\frac{27}{343}&\frac{81}{2401}\\1&\frac{4}{7}&\frac{16}{49}&\frac{64}{343}&\frac{256}{2401}\\1&\frac{5}{7}&\frac{25}{49}&\frac{125}{343}&\frac{625}{2401}\end{bmatrix},\Gamma=\begin{bmatrix}7&0&0&0&0\\0&\frac{7}{2}&0&0&0\\0&0&\frac{7}{3}&0&0\\0&0&0&\frac{7}{4}&0\\0&0&0&0&\frac{7}{5}\end{bmatrix},$$

$$\left[\overline{\mathrm{W}};\overline{\mathrm{F}}\right]=\begin{bmatrix}\frac{-432}{49}&\frac{-288}{49}&\frac{516}{49}&\frac{188}{49}&\frac{16}{49}&-1.363091\\\frac{50}{49}&\frac{-410}{49}&\frac{48}{49}&\frac{248}{49}&\frac{64}{49}&-1.334886\\\frac{320}{147}&\frac{-608}{147}&\frac{-228}{49}&\frac{180}{49}&\frac{144}{49}&-1.289953\\\frac{81}{49}&\frac{-9}{49}&\frac{-312}{49}&\frac{-16}{49}&\frac{256}{49}&-1.230855\\\frac{208}{245}&\frac{512}{245}&\frac{-204}{49}&\frac{-340}{49}&\frac{400}{49}&-1.160555\\0&0&0&0&1&0\\-4&4&0&0&0&0\end{bmatrix},\qquad U=\begin{bmatrix}0.316399\\0.316595\\0.258971\\0.145878\\-1.598721\end{bmatrix}.$$

We obtain the approximate solution of the problem for N= 2, 4 and N = 6 respectively,

$$u_2(x)=0.3455-0.0997x-0.2458x^2,$$
$$u_4(x)=0.3164+0.0008x-0.3469x^2+0.0094x^3+0.0203x^4,$$
$$u_6(x)=0.3167-0.000002x-0.3431x^2-0.0003x^3+0.0305x^4-0.0019x^5-0.0019x^6,$$

Since the exact solution is

$$u(x)=2\ln\left(\frac{4-2\sqrt{2}}{(3-2\sqrt{2})x^2+1}\right),$$ if we write Taylor expansion, we will have:

$$u(x)\approx 0.3167-0.3431x^2+0.0294x^4-0.0034x^6+O(x^8),$$

As it can be observed, as N increases, the approximate solution gets closer to the exact solution.

5. CONCLUSIONS

In this paper, we have proposed a numerical solution to solve isothermal gas spheres equations with initial or boundary conditions by Bernstein polynomials operational matrices and derived operational matrix. We use formula for numerical examples and it is obvious that the numerical solution coincides with the exact solution even with a few Bernstein polynomials used in the approximation. Finally, we show that the approximation becomes more accurate when N is increased. Therefore, for better results, it is recommended to use a larger N.

6. ACKNOWLEDGEMENTS

The authors would like to thank Islamic Azad University Birjand Branch for partially financially supporting this research and providing facilities and encouraging this work.

6. REFERENCES

[1] J. H. Lane, "On the theoretical temperature of the sun under the hypothesis of a gaseous mass maintaining its volume by its internal heat and depending on the laws of gases known to terrestrial experiment," *The American Journal of Science and Arts*, 2nd series, **50**, 57-74 (1870).

[2] R. Emden, Gaskugeln, Teubner, Leipzig and Berlin (1907).

[3] S. Chandrasekhar, "Introduction to the study of stellar structure," New York, Dover (1967).

[4] HT. Davis, "Introduction to nonlinear differential and integral equations," New York, Dover (1962).

[5] OW. Richardson, "The emission of electricity from hot bodies," London (1921).

[6] A. Yildirim, T. Öziş, "Solutions of singular IVPs of Lane-Emden type by homotopy perturbation method," *Phys. Lett. A*, **369**, 70-76 (2007).

[7] J. I. Ramos, "Series approach to the Lane-Emden equation and comparison with the homotopy perturbation method," *Chaos Soliton Fractals*, **38**, 400-408 (2008).

[8] M. S. H. Chowdhury, I. Hashim, "Solutions of Emden-Fowler equations by homotopy perturbation method," *Nonlinear Anal. RWA*, **10**, 104-115 (2009).

[9] S. A. Yousefi, "Legendre wavelets method for solving differential equations of Lane- Emden type," *Appl. Math. Comput*, **181**, 1417-1422 (2006).

[10] A. Yildirim, T. Öziş, "Solutions of singular IVPs of Lane-Emden type by the variational iteration method," *Nonlinear Anal*, **70**, 2480-2484 (2009).

[11] M. Dehghan, F. Shakeri, "Approximate solution of a differential equation arising in astrophysics using the variational iteration method," *New Astron*, **13**, 53-59 (2008).

[12] N. Caglar, H. Caglar, "B-spline solution of singular boundary value problems," *Appl. Math. Comput*, **182**, 1509-1513 (2006).

[13] A. M. Wazwaz, "A new algorithm for solving differential equations of Lane-Emden type," *Appl. Math. Comput*, **118**, 287-310 (2001).

[14] Ş. Yüzbaşı, M. Sezer, "A collocation approach to solve a class of Lane-Emden type equations," *J. Adv. Res. Appl. Math*, Vol. 3, **2**, 58-73 (2011).

[15] S. K. Vanani, A. Aminataei, "On the numerical solution of differential equations of Lane-Emden type, " *Comput. Math. Appl*, **59**, 2815-2820 (2010).

[16] K. Parand, A. Shahini, M. Dehghan, "Rational Legendre pseudospectral approach for solving nonlinear differential equations of Lane-Emden type," *J. Comput. Phys*, **228**, 8830-8840 (2009).

[17] N. T. Shawagfeh, "Nonperturbative approximate solution for Lane-Emden equation," *J. Math. Phys*, Vol. 34, **9**, 43-64 (1993).

[18] K. Parand, M. Dehghan, A. R. Rezaei, S. M. Ghaderi, "An approximation algorithm for the solution of the nonlinear Lane-Emden type equations arising in astrophysics using Hermite functions collocation method," *Comput. Phys. Comm*, **181**, 1096-1108 (2010).

[19] Ji-Huan He, "Variational approach to the Lane-Emden equation," *Appl. Math. Comput*, **143**, 539-541 (2003).

[20] K. Maleknejad, E. Hashemizadeh, B. Basirat, "Computational method based on Bernestein operational matrices for nonlinear Volterra-Fredholm- Hammerstein integral equations," *Commun. Nonlinear Sci. Numer. Simul*, Vol. 17, **1**, 52-61 (2012).

[21] B. Basirat, M. A. Shahdadi, "Numerical solution of nonlinear integro-differential equations with initial conditions by Bernstein operational matrix of derivative," *International Journal of Modern Nonlinear Theory and Application*, **2**, 141-149 (2013).

[22] A. M. Wazwaz, "A new method for solving singular initial value problems in the second-order ordinary differential equations," *Appl. Math. and Comput*, **128**, 45-57 (2002).

[23] A. H. Bhrawy, A. S. Alofi, "A Jacobi-Gauss collocation method for solving nonlinear Lane-Emden type equations," *Commun Nonlinear Sci Numer Simulat*, **17**, 62-70 (2012).

[24] J. I. Ramos, "Linearization techniques for singular initial-value problems of ordinary differential equations," *Appl. Math. and Comput*, **161**, 525-542 (2005).