EXP-FUNCTION METHOD FOR SOLUTIONS OF (3+1) DIMENSIONAL BREAKING SOLITON AND GARDNER-KP EQUATIONS

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ABSTRACT
In this paper,Exp-function method is used for constructing analytical solutions of nonlinear partial differential equations(NPDE$_s$). For illustrative examples, we choose the (3+1) dimensional breaking soliton equation and Gardner-KP equation.

Kewords: *(3+1) dimensional breaking soliton equation, Gardner-KP equation, complex solutions.*

1. INTRODUCTION

The nonlinear partial differential equations (NPDE$_s$) play an important role in applied mathematics, physics and engineering. These equations are mathematical models of some complex physical occurrences that arise in mechanics, chemistry, biology, etc. Analytical solutions of nonlinear differential equations give some useful information about character of the equations. Therefore, it has become more interesting that searching for analytical solutions of nonlinear partial differential equations (NPDE$_s$) by using Maple, Matlab, Mathematica.

Nonlinear partial differential equations (NPDEs) are widely used to describe complex phenomena in several aspects of physics as well as other natural and applied sciences. One of the most important tasks in the study of NPDEs is to construct exact solutions.Many effective and reliable methods are used in the literature to investigate solitons, and in particular multiple soliton solutions of completely integrable equations such asthe invers scattering method [1], the Backlund transformation method [2], the truncated Painleve expansion [3], the sine-cosine function method [4], the Jacobi elliptic function expansion method [5].Recently, many methods are developed for travelling wave solutions of NPDE by scientist. In 1992tanh method was presented by Malfiet [6], extended tanh function method by Fan in 2000 [7], modified extended tanh function method by Elwakil in 2002 [8], generalized extended tanh function method by Zheng in 2003 [9], generalized tanh function method by Chen and Zhang in 2004 [10], generalized Jacobi elliptic function method by Chen and Zhang in 2004 [11],Jacobi elliptic rational expansion methodin 2004 [12], WeierstrassJacobi elliptic function expansion method in 2006 [13], Exp-function method by He [14] in 2006, $\frac{G'}{G}$ -expansion method by Wang [15] in 2008, extended $\frac{G'}{G}$ -expansion method by Guo and Zhou in 2010 [16] and generalized $\frac{G'}{G}$ -expansion method by Lü in 2010 [17]. Most of these methods are based on finding balance term with balancing of the highest order linear and nonlinear derivative terms. So, these methods can be only applied to nonlinear partial differential equations.

In this study, we obtain the analytical solutions of the (3+1) dimensional breaking soliton equation [18] and Gardner-KP equation [19]by usingthe Exp-function method [14].

This paper is organized as follows:

In section 2, we present the Exp-function method. In section 3, we take two illustrative examples and bring out some new solutions.

2. ANALYSIS OF THE METHOD

In this section, we simply describe the Exp-function method. Consider a given nonlinear partial differential equation (NPDE) for $u(x,t)$ in the form

$$H(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0 \qquad (2.1)$$

where,H is a polynomial of its arguments. The main steps of the method are outlined in the following:

STEP 1

Using the formal travelling wave transformation

$$u(x,t) = U(\xi),\ \xi = kx \pm wt \quad (2.2)$$

where k and w are the wave number and the wave speed, respectively, then Eq. (2.1) is reduced to the following nonlinear ordinary differential equation (NODE) with transformEq.(2.2)

$$G(U', U'', U''', \ldots) = 0 \quad (2.3)$$

where the prime denotes the derivative with respect to ξ.

STEP 2

Integrating Eq. (2.3), term by term one or more times yield constant(s) of integration and these integration constant(s) can be set to zero for simplicity.

STEP 3

The solution of the Eq. (2.3) which we are looking for is expressed as

$$u(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)}, \quad (2.4)$$

where c, d, p and q are unknown positive integers, which to be further determined, and a_n and b_m are unknown constants. We suppose that the solution of Eq. (2.3) can be expressed as

$$u(\xi) = \frac{a_c \exp(c\xi) + \cdots + a_{-d} \exp(-d\xi)}{a_p \exp(p\xi) + \cdots + a_{-q} \exp(-q\xi)} \quad (2.5)$$

where c, d, p and q are positive integers that can be determined by balancing the highest order derivative and with the highest nonlinear terms in Eq. (2.3). Substituting solution (2.5) into Eq. (2.3) yields a set of algebraic equations for $\exp(\xi)$. After this separated algebraic equation, we can find a_n and b_m constants.

3. APPLICATIONS

In this section, we study two important nonlinear evolution equations using the Exp- function method to illustrate its success.

3.1. Solutions of the (3+1) Dimensional Breaking Soliton Equation

Firstly, let's consider (2+1) dimensional breaking soliton equation [18] as following

$$u_{xt} - 4u_{xy}u_x - 2u_{xx}u_y - u_{xxxy} = 0. \quad (3.1.1)$$

This equation was used to describe the (2 + 1)-dimensional interaction of the Riemann wave propagated along the y-axis with a long wave propagated along the x-axis [20, 21]. A class of overturning soliton solutions has been introduced in [20]. For $y = x$, and by integrating the resulting equation in (3.1.1), the equation is reduced to the KdV equation. However, we consider an extension to equation (3.1.1) by adding the last three terms with y replaced by z. This enables us to establish the (3 + 1)-dimensional breaking soliton equation

$$-u_{xxxy} - u_{xxxz} + u_{xt} - 2u_{xx}u_y - 2u_{xx}u_z - 4u_x u_{xy} - 4u_x u_{xz} = 0 \quad (3.1.2)$$

where u_{kx} denotes the partial derivative $\frac{\partial^k u}{\partial x^k}$.

Now, we seek for the travelling wave solutions of Eq. (3.1.2) by the Exp-function method. Using the transformation $u(x,t) = U(\xi),\ \xi = k(x + ly + sz - \lambda t)$ where u is the wave speed, then we get

$$-k^2(l+s)U''' - \lambda U' - 3k(l+s)(U')^2 \quad (3.1.3)$$

Here primes and $U^{(k)}$ denote the derivatives with respect to ξ, $U' = \frac{\partial U}{\partial \xi}$. This reduced NODE (3.1.3) is of the (3+1) dimensional breaking solitonequation. After this step, we make the ansatz (2.5) for the solution of Eq. (3.1.3).Balancingthe term u''' and $(u')^2$ and obtain

$$\frac{c_1 \exp[(7p + c)\xi] + \cdots}{c_2 \exp[8p\xi] + \cdots} = \frac{c_3 \exp[2p + 2c\xi] + \cdots}{c_4 \exp[4p\xi] + \cdots}.$$

Thisgives $p = c$. Similarly to determine values of d and q we balance the terms u''' and $(u')^2$

$$\frac{\cdots d_1 \exp[-(7q+d)\xi]}{\cdots d_2 \exp[-8q\xi]} = \frac{\cdots d_3 \exp[-(2q+2d)\xi]}{\cdots d_4 \exp[-4q\xi]}$$

and we find $q = d$. For simplicity, setting $p = c = 1$ and $q = d = 1$, equation (2.5) reduces to

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{\exp(\xi) + b_0 + b_{-1}\exp(-\xi)} \quad (3.1.4)$$

Substituting equation (3.1.4) into (3.1.3) yields the following set of algebraic equations for $a_0, a_1, a_{-1}, b_0, b_{-1}$

$$-12kla_{-1}^2 - 12ksa_{-1}^2 - 32k^2la_{-1}b_{-1} - 32k^2sa_{-1}b_{-1} + 4\lambda a_{-1}b_{-1}$$
$$+6kla_0^2b_{-1} + 6ksa_0^2b_{-1} + 24kla_{-1}a_1b_{-1} + 24ksa_{-1}a_1b_{-1}$$
$$+32k^2la_1b_{-1}^2 + 32k^2sa_1b_{-1}^2 - 4\lambda a_1b_{-1}^2 - 12kla_1^2b_{-1}^2$$
$$-12ksa_1^2b_{-1}^2 - 6kla_{-1}a_0b_0 - 6ksa_{-1}a_0b_0 - 6kla_0a_1b_{-1}b_0$$
$$-6ksa_0a_1b_{-1}b_0 + 4k^2la_{-1}b_0^2 + 4k^2sa_{-1}b_0^2 + 4\lambda a_{-1}b_0^2 + 6kla_{-1}a_1b_0^2$$
$$+6ksa_{-1}a_1b_0^2 - 4k^2la_1b_{-1}b_0^2 - 4k^2sa_1b_{-1}b_0^2 - 4\lambda a_1b_{-1}b_0^2 = 0$$
$$k^2la_0 + k^2sa_0 + \lambda a_0 - k^2la_1b_0 - k^2sa_1b_0 - \lambda a_1b_0 = 0$$
$$-k^2la_0b_{-1}^3 - k^2sa_0b_{-1}^3 - \lambda a_0b_{-1}^3 + k^2la_{-1}b_{-1}^2b_0 + k^2sa_{-1}b_{-1}^2b_0 + \lambda a_{-1}b_{-1}^2b_0 = 0$$
$$8k^2la_{-1} + 8k^2sa_{-1} + 2\lambda a_{-1} - 3kla_0^2 - 3ksa_0^2 - 8k^2la_1b_{-1} - 8k^2sa_1b_{-1}$$
$$-2\lambda a_1b_{-1} - 4k^2la_0b_0 - 4k^2sa_0b_0 + 2\lambda a_0b_0 + 6kla_0a_1b_0$$
$$+6ksa_0a_1b_0 + 4k^2la_1b_0^2 + 4k^2sa_1b_0^2 - 2\lambda a_1b_0^2 - 3kla_1^2b_0^2 - 3ksa_1^2b_0^2 = 0$$
$$8k^2la_{-1}b_{-1}^2 + 8k^2sa_{-1}b_{-1}^2 + 2\lambda a_{-1}b_{-1}^2 - 3kla_0^2b_{-1}^2 - 3ksa_0^2b_{-1}^2 - 8k^2la_1b_{-1}^3$$
$$-8k^2sa_1b_{-1}^3 - 2\lambda a_1b_{-1}^3 + 6kla_{-1}a_0b_{-1}b_0 + 6ksa_{-1}a_0b_{-1}b_0$$
$$+4k^2la_0b_{-1}^2b_0 + 4k^2sa_0b_{-1}^2b_0 - 2\lambda a_0b_{-1}^2b_0 - 3kla_{-1}^2b_0^2$$
$$-3ksa_{-1}^2b_0^2 - 4k^2la_{-1}b_{-1}b_0^2 - 4k^2sa_{-1}b_{-1}b_0^2 + 2\lambda a_{-1}b_{-1}b_0^2 = 0$$
$$12kla_{-1}a_0b_{-1} + 12ksa_{-1}a_0b_{-1} + 23k^2la_0b_{-1}^2 + 23k^2sa_0b_{-1}^2 - \lambda a_0b_{-1}^2 - 12kla_0a_1b_{-1}^2$$
$$-12ksa_0a_1b_{-1}^2 - 12kla_{-1}^2b_0 - 12ksa_{-1}^2b_0 - 18k^2la_{-1}b_{-1}b_0 - 18k^2sa_{-1}b_{-1}b_0$$
$$+6\lambda a_{-1}b_{-1}b_0 + 12kla_{-1}a_1b_{-1}b_0 + 12ksa_{-1}a_1b_{-1}b_0 - 5k^2la_1b_{-1}^2b_0 - 5k^2sa_1b_{-1}^2b_0$$
$$-5\lambda a_1b_{-1}^2b_0 - k^2la_0b_{-1}b_0^2 - k^2sa_0b_{-1}b_0^2 - \lambda a_0b_{-1}b_0^2 + k^2la_{-1}b_0^3$$
$$+k^2sa_{-1}b_0^3 + \lambda a_{-1}b_0^3 = 0 \quad (3.1.5)$$
$$-12kla_{-1}a_0 - 12ksa_{-1}a_0 - 23k^2la_0b_{-1} - 23k^2sa_0b_{-1} + \lambda a_0b_{-1} + 12kla_0a_1b_{-1}$$
$$+12ksa_0a_1b_{-1} + 5k^2la_{-1}b_0 + 5k^2sa_{-1}b_0 + 5\lambda a_{-1}b_0 + 12kla_{-1}a_1b_0$$
$$+12ksa_{-1}a_1b_0 + 18k^2la_1b_{-1}b_0 + 18k^2sa_1b_{-1}b_0 - 6\lambda a_1b_{-1}b_0$$
$$-12kla_1^2b_{-1}b_0 - 12ksa_1^2b_{-1}b_0 + k^2la_0b_0^2 + k^2sa_0b_0^2 + \lambda a_0b_0^2 - k^2la_1b_0^3$$
$$-k^2sa_1b_0^3 - \lambda a_1b_0^3 = 0$$

We can solve the system of nonlinear algebraic equations (3.1.5), by Mathematica and we have the following set of solutions:

Case 1:

$$a_1 = 0, \quad a_{-1} = 0, \quad k \neq 0, \quad b_0 = -\frac{a_0}{2k}, \quad b_{-1} = 0, \quad \lambda = -k^2l - k^2s, \quad la_0 + sa_0 \neq 0$$

$$u(x,y,z,t) = \frac{a_0}{\text{Cosh}[k((k^2l + k^2s)t + x + ly + sz)] + \text{Sinh}[k((k^2l + k^2s)t + x + ly + sz)] - \frac{a_0}{2k}}$$

Case 2:

$$a_1 = 2k, \quad k \neq 0, \quad a_{-1} = \frac{a_0^2}{k}, \quad b_0 = \frac{a_0}{2k}, \quad b_{-1} = 2b_0^2, \quad \lambda = -k^2 l - k^2 s, \quad la_0 + sa_0 \neq 0$$

$$u(\xi) = -\frac{2ka_0((\mathrm{Cosh}[\xi] - \mathrm{Sinh}[\xi])a_{-1} + a_0)}{a_0(-2k(\mathrm{Cosh}[\xi] + \mathrm{Sinh}[\xi]) + a_0) + a_{-1}(-2k + (\mathrm{Cosh}[\xi] - \mathrm{Sinh}[\xi])a_0)}$$

where $\xi = k((k^2 l + k^2 s)t + x + ly + sz)$.

Case 3:

$$a_1 = 4k, \quad a_0 = 0, \quad a_{-1} = 0, \quad b_0 = 0, \quad b_{-1} \neq 0, \quad \lambda = -4(k^2 l + k^2 s), \quad kl + ks \neq 0$$

$$u(\xi) = \frac{4k(\mathrm{Cosh}[\xi] + \mathrm{Sinh}[\xi])}{\mathrm{Cosh}[\xi] + \mathrm{Sinh}[\xi] + (\mathrm{Cosh}[\xi] - \mathrm{Sinh}[\xi])b_{-1}}$$

where $\xi = k(4(k^2 l + k^2 s)t + x + ly + sz)$.

3.2. Solutions of the Gardner-Kp Equation

We consider the following Gardner-KP equationin the form

$$(u_t + 6uu_x + 6u^2 u_x + u_{xxx})_x + u_{yy} = 0 \qquad (3.2.1)$$

that describe internal solitary waves in shallow seas. The two models will be classified as positive Gardner equation and negative Gardner equation depending on the sign of the cubic nonlinear term [22]. Gardner equation is widely used in various branches of physics, such as plasma physics, fluid physics, quantum field theory [23–25]. It also describes a variety of wave phenomena in plasma and solid state [26, 27]. Studies of various physical structures of nonlinear equations had attracted much attention in connection with the important problems that arise in scientific applications.Using the transformation $u(x,t) = U(\xi),\ \xi = k(x + y - \lambda t)$ and integrating we find that

$$(1-\lambda)U + 3U^2 + 2U^3 + k^2 U'' = 0 \qquad (3.2.2)$$

wheretheintegration constant is taken as zero. Using gives the ansatz (2.5) for the solution of Eq. (3.2.2)and balancingthe termsu'' andu^3

$$\frac{c_1 \exp[(3p+c)\xi] + \cdots}{c_2 \exp[4p\xi] + \cdots} = \frac{c_3 \exp[3c\xi] + \cdots}{c_4 \exp[3p\xi] + \cdots}.$$

andthen gives $p = c$. Similarly to determine values of d and q use balance the terms u'' andu^3. This gives

$$\frac{\cdots d_1 \exp[-(3q+d)\xi]}{\cdots d_2 \exp[-4q\xi]} = \frac{\cdots d_3 \exp[-(3d)\xi]}{\cdots d_4 \exp[-3q\xi]}$$

and then we find $q = d$. For simplicity, setting $p = c = 1$ and $q = d = 1$, equation (2.5) reduces to

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{\exp(\xi) + b_0 + b_{-1}\exp(-\xi)} \qquad (3.2.3)$$

Substituting equation (3.2.3) into (3.2.2) yields the following set of algebraic equations for $a_0, a_1, a_{-1}, b_0, b_{-1}$

$$a_{-1} + 4k^2 a_{-1} - \lambda a_{-1} + 3a_0^2 + 6a_{-1}a_1 + 6a_0^2 a_1 + 6a_{-1}a_1^2 + 2a_1 b_{-1} - 4k^2 a_1 b_{-1}$$
$$-2\lambda a_1 b_{-1} + 3a_1^2 b_{-1} + 2a_0 b_0 - k^2 a_0 b_0 - 2\lambda a_0 b_0 + 6a_0 a_1 b_0 + a_1 b_0^2 +$$
$$k^2 a_1 b_0^2 - \lambda a_1 b_0^2 = 0$$

$$3a_{-1}^2 + 6a_{-1}a_0^2 + 6a_{-1}^2 a_1 + 2a_{-1}b_{-1} - 4k^2 a_{-1}b_{-1} - 2\lambda a_{-1}b_{-1} + 3a_0^2 b_{-1} +$$
$$6a_{-1}a_1 b_{-1} + a_1 b_{-1}^2 + 4k^2 a_1 b_{-1}^2 - \lambda a_1 b_{-1}^2 + 6a_{-1}a_0 b_0 + 2a_0 b_{-1} b_0 -$$
$$k^2 a_0 b_{-1} b_0 - 2\lambda a_0 b_{-1} b_0 + a_{-1} b_0^2 + k^2 a_{-1} b_0^2 - \lambda a_{-1} b_0^2 = 0$$

$$6a_{-1}^2 a_0 + 6a_{-1}a_0 b_{-1} + a_0 b_{-1}^2 + k^2 a_0 b_{-1}^2 - \lambda a_0 b_{-1}^2 + 3a_{-1}^2 b_0 +$$
$$2a_{-1}b_{-1}b_0 - k^2 a_{-1}b_{-1}b_0 - 2\lambda a_{-1}b_{-1}b_0 = 0 \qquad (3.2.4)$$

$$a_0 + k^2 a_0 - \lambda a_0 + 6a_0 a_1 + 6a_0 a_1^2 + 2a_1 b_0 - k^2 a_1 b_0 - 2\lambda a_1 b_0 + 3a_1^2 b_0 = 0$$

$$2a_{-1}^3+3a_{-1}^2b_{-1}+a_{-1}b_{-1}^2-\lambda a_{-1}b_{-1}^2=0$$
$$a_1-\lambda a_1+3a_1^2+2a_1^3=0$$
$$6a_{-1}a_0+2a_0^3+12a_{-1}a_0a_1+2a_0b_{-1}-6k^2a_0b_{-1}-2\lambda a_0b_{-1}+6a_0a_1b_{-1}+$$
$$2a_{-1}b_0+3k^2a_{-1}b_0-2\lambda a_{-1}b_0+3a_0^2b_0+6a_{-1}a_1b_0+2a_1b_{-1}b_0+$$
$$3k^2a_1b_{-1}b_0-2\lambda a_1b_{-1}b_0+a_0b_0^2-\lambda a_0b_0^2=0$$

We can solve the system of nonlinear algebraic equations (3.2.4), by Mathematicaandwe have the following set of solutions:

Case 1:

$$k=i,\qquad a_1=\frac{1}{4},\qquad a_{-1}=\frac{a_0^2}{5},\qquad a_{-1}\neq 0,\qquad b_0=-\frac{12a_0}{5},$$
$$b_{-1}=\frac{1}{8}(32a_{-1}-48a_0^2-8a_0b_0+5b_0^2),\qquad \lambda=\frac{15}{8}$$
$$u(x,y,t)=\frac{5(\mathrm{Cos}[2\mu]+i\mathrm{Sin}[2\mu])+20(\mathrm{Cos}[\mu]-i\mathrm{Sin}[\mu])a_0+4a_0^2}{4(5(\mathrm{Cos}[2\mu]+i\mathrm{Sin}[2\mu])-12(\mathrm{Cos}[\mu]-i\mathrm{Sin}[\mu])a_0+4a_0^2)}$$

where $\mu=-\frac{15t}{8}+x+y$.

Case 2:

$$k=-i,\qquad a_1=\frac{1}{4},\qquad a_{-1}=\frac{a_0^2}{5},\qquad a_{-1}\neq 0,\qquad b_0=-\frac{12a_0}{5},$$
$$b_{-1}=\frac{1}{8}(32a_{-1}-48a_0^2-8a_0b_0+5b_0^2),\qquad \lambda=\frac{15}{8}$$
$$u(x,y,t)=\frac{5\mathrm{Cos}[2\mu]-5i\mathrm{Sin}[2\mu]+20\mathrm{Cos}[\mu]a_0+20i\mathrm{Sin}[\mu]a_0+4a_0^2}{20\mathrm{Cos}[2\mu]-20i\mathrm{Sin}[2\mu]-48\mathrm{Cos}[\mu]a_0-48i\mathrm{Sin}[\mu]a_0+16a_0^2}$$

where $\mu=-\frac{15t}{8}+x+y$.

3. CONCLUSIONS

In this study, we obtain analytical solutions of the (3+1) dimensional breaking soliton equation and Gardner-KP equation.The method used in this work can be used to search for the solutions of other $\mathrm{NPDE_s}$, but also it is a computerizable method which allows us to perform complicated and tedious algebraic calculation on a computer.

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