LINEAR 3-ARMENDARIZ RINGS

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ABSTRACT
In the present note we study the properties of linear 3-Armendariz rings, which are generalization of 3-Armendariz rings and linear Armendariz rings and the connections among linear 3-Armendariz rings, 3-Armendariz rings and ring satisfies condition (P). We prove that a right Ore ring R is linear 3-Armendariz if and only if so is $Q,$ where Q is the classical right quotient ring of $R.$ With the help of this result we can show that a commutative ring R is linear 3-Armendariz if and only if the total quotient ring of R is linear 3-Armendariz.

Keywords: *Armendariz ring; 3-Armendariz ring; linear 3-Armendariz ring.*
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1. INTRODUCTION

Throughout this paper all rings are associative rings which maybe not have identity. Given a ring R, the polynomial ring over R is denoted by $R[x]$. The study of Armendariz ring was initiated by Armendariz [5] and Rege and Chhawchharia [8]. A ring R is called Armendariz if whenever polynomials, $f(x)=a_0+a_1x+\cdots+a_nx^n, g(x)=b_0+b_1x+\cdots+b_mx^m \in R[x]$ satisfy $f(x)g(x)=0$, then $a_ib_j=0,$ for all i, j. (The converse is always true.) Some properties of Armendariz rings have been studied in Rege and Chhawchharia [8], Anderson and Camillo [4], Kim and Lee [9], Huh et al. [2], and Lee and Wong [10]. Hong et al. [3], have studied a generalization of Armendariz rings, which they called α-skew Armendariz rings, where α is an endomorphism of $R.$ Suiyi [12] introduced the notion of 3-Armendariz ring. He defined a ring R is called 3-Armendariz ring if whenever polynomials:

$$f(x)=a_0+a_1x+\cdots+a_nx^n, g(x)=b_0+b_1x+\cdots+b_mx^m, h(x)=c_0+c_1x+\cdots+c_rx^r \in R[x]$$

satisfy $f(x)g(x)h(x)=0$, then $a_ib_jc_k=0$, for all i, j, k. Due to Lee and Wong [10], a ring R is called weak Armendariz (or linear Armendariz) if for given $f(x)=a_0+a_1x$ and $g(x)=b_0+b_1x \in R[x],$ such that $f(x)g(x)=0$ then $a_ib_j=0,$ for all $0\le i\le 1, 0\le j\le 1.$ (the converse is obviously true.) A ring R is called reduced if it has no nonzero nilpotent elements. Reduced rings are Armendariz by [5, Lemma 1] and subrings of Armendariz rings are also Armendariz ring. It is obviously that Armendariz rings are linear Armendariz and that subrings of linear Armendariz rings are still linear Armendariz. There is weak Armendariz (or linear Armendariz) ring but not Armendariz by [10, Example 3.2]. The structure of linear Armendariz rings was also observed by Anderson and Camillo [4], containing the relation closely related rings. A ring is called an abelian if every idempotent is central. Weak Armendariz (or linear Armendariz) rings are abelian by [10, Lemma 3.4(3)]. Due to Jeon et al. [6], the class of weak Armendariz (or linear Armendariz) rings is closed under direct products. Motivated by results in Suiyi [12], Jeon et al. [6], Lee and Wong [10], Kim and Lee [9], and Rege and Chhawchharia [8], we investigate a generalization of linear Armendariz ring and 3-Armemdariz rings which we called linear 3-Armendariz ring.

2. LINEAR 3-ARMENDARIZ RINGS

Definition 2.1 *A ring R is called linear 3-Armendariz if whenever polynomials* $f(x)=a_0+a_1x$, $g(x)=b_0+b_1x$ *and* $h(x)=c_0+c_1x \in R[x],$ *satisfy* $f(x)g(x)h(x)=0$ *then* $a_ib_jc_k=0,$ *for all* $0\le i\le 1, 0\le j\le 1, 0\le k\le 1.$

Condition (P): For all $a,b,c\in R,$ if $(abc)^2=0,$ then $abc=0.$ (see [12])

Lemma 2.2 *[11, Proposition 1]. If* R *is a reduced ring, then* R *satisfies the condition* (P)*, but the converse is not true.*

Lemma 2.3 *[12, Theorem 1]. If* R *satisfies the condition* (P)*, then* R *is 3-Armendariz.*

Lemma 2.4 *If* R *is 3-Armendariz, then for any* $f_1(x), f_2(x), \cdots, f_n(x) \in R[x], n \geq 3,$ *if* $f_1(x) f_2(x) f_3 \cdots f_n(x) = 0,$ *then* $C_{f_1(x)} C_{f_2(x)} \cdots C_{f_n(x)} = 0.$

Proof. Assume that $f_1(x) f_2(x) f_3(x) \cdots f_n(x) = 0,$ then $f_1(x) f_2(x)(f_3(x) \cdots f_n(x)) = 0.$ Thus $C_{f_1(x)} C_{f_2(x)} \cdots C_{f_n(x)} = 0.$ Since R is 3-Armendariz. So

$$C_{f_1(x)} f_2(x) f_3(x) \cdots f_n(x) = 0.$$

Thus $(C_{f_1(x)} f_2(x)) f_3(x)(f_4(x) \cdots f_n(x)) = 0.$ By the hypothesis, we have

$$C_{f_1(x)} C_{f_2(x)} C_{f_3(x)} C_{f_4(x) \cdots f_n(x)} = 0.$$

Hence $C_{f_1(x)} C_{f_2(x)} f_3(x) f_4(x) f_5(x) \cdots f_n(x) = 0,$ so

$$(C_{f_1(x)} C_{f_2(x)} f_3(x)) f_4(x)(f_5(x) \cdots f_n(x)) = 0.$$

Repeating this process, we can show that $C_{f_1(x)} C_{f_2(x)} \cdots C_{f_n(x)} = 0.$

Theorem 2.5 *Let* R *be a ring. Then* R *satisfies the condition* (P) *if and only if* $R[x]$ *satisfies the condition* (P).

Proof. $\Rightarrow$). Assume that $f(x), g(x), h(x) \in R[x]$ be such that $(f(x)g(x)h(x))^2 = 0$ where $f(x) = \Sigma_{i=0}^{n} a_i x^i, g(x) = \Sigma_{j=0}^{m} b_j x^j, h(x) = \Sigma_{k=0}^{r} c_k x^k.$ Then

$$f(x)g(x)h(x) f(x)g(x)h(x) = 0.$$

Since R satisfies the condition $(P), a_i b_j c_k a_s b_t c_q = 0$ for any $i, j, k, s, t, q,$ by Lemma 2.3 and Lemma 2.4. In particular $(a_i b_j c_k)^2 = 0$ for any $i, j, k.$ Thus $a_i b_j c_k = 0$ for any $i, j, k.$ Since R satisfies the condition $(P).$ There for $f(x)g(x)h(x) = 0,$ which implies that $R[x]$ satisfies the condition (P).

$\Leftarrow$). It is clear that R is a subring of $R[x]$ and the subrings of a ring satisfies the condition (P) is also satisfies the condition $(P).$

Remark 2.6 *Let* R *be a ring with identity.* R *is Armendariz if and only if* R *is 3-Armendariz. (see [12])*

Clearly, any 3-Armendariz ring is linear 3-Armendariz, but the converse is not true by the following example.

Example 2.7 *[6, Example 1.2(1)]. Let* $R = Z_3[x, y]/(x^3, x^2 y^2, y^3)$ *where* Z_3 *is a Galois field of order 3, with identity 1,* $Z_3[x, y]$ *is the polynomial ring with two indeterminates* x, y *over* $Z_3,$ *and* $(x^3, x^2 y^2, y^3)$ *is the ideal of* $Z_3[x, y]$ *generated by* $x^3, x^2 y^2, y^3$*. Let* $R[t]$ *be the polynomial ring with an indeterminate* t *over* $R.$ *Since* $(\bar{x} + \bar{y}t)^3 = (\bar{x} + \bar{y}t)(\bar{x}^2 + 2\bar{x}\bar{y}t + \bar{y}^2 t^2) = 0$ *with* $\bar{x}\bar{y}^2 \neq 0,$ R *is not Armendariz, by Remark 2.6,* R *is not 3-Armendariz. But* R *is linear 3-Armendariz.*

Lemma 2.8 *Let S be a subring of R. If R is linear 3-Armendariz. Then so is S.*

Proof. Let $f(x)=\Sigma_{i=0}^{1}a_i x^i, g(x)=\Sigma_{j=0}^{1}b_j x^j, h(x)=\Sigma_{k=0}^{1}c_k x^k \in S[x],$ be such that $f(x)g(x)h(x)=0.$ Then $f(x),g(x),h(x)\in R[x].$ Since R is linear 3-Armendariz, then $a_i b_j c_k=0.$ This means that S is linear 3-Armendariz.

Proposition 2.9 *The class of linear 3-Armendariz rings is closed under finite direct products.*

Proof. Let $R=\prod_{s\in\beta}R_s$ be the finite direct product of R_s where $\beta=\{1,2,\cdots,n\},$ R_s is linear 3-Armendariz ring. Suppose $f(x)g(x)h(x)=0$ for some polynomials $f(x)=a_0+a_1x, g(x)=b_0+b_1x$ and $h(x)=c_0+c_1x\in R[x]$, where $a_i=(a_{i1},a_{i2},\cdots,a_{in}),b_j=(b_{j1},b_{j2},\cdots,b_{jn}),c_k=(c_{k1},c_{k2},\cdots,c_{kn}),$ are elements of the product ring R. Set $f_s(x)=\Sigma_{i=0}^{1}a_{is}x^i$, $g_s(x)=\Sigma_{j=0}^{1}b_{js}x^j$ and $h_s(x)=\Sigma_{k=0}^{1}c_{ks}x^k\in R[x].$ Since $f(x)g(x)h(x)=0$ then $\Sigma_{i+j+k=l}a_i b_j c_k=0, 0\le l\le i+j+k,$ so $\Sigma_{i+j+k=l}(a_{i1}b_{j1}c_{k1},\cdots,a_{in}b_{jn}c_{kn})=0,$ and so $\Sigma_{i+j+k=l}(a_{is}b_{js}c_{ks})=0, 1\le s\le n.$ Thus $f_s(x)g_s(x)h_s(x)=0$ in $R_s[x], 1\le s\le n.$ Since R_s is linear 3-Armendariz rings, then we have $a_{is}b_{js}c_{ks}=0$ in $R_s, 1\le s\le n.$ Then it is clear that $a_i b_j c_k=a_{i1}b_{j1}c_{k1},\cdots,a_{in}b_{jn}c_{kn}=0.$ Therefore $a_i b_j c_k=0.$ Thus R is linear 3-Armendariz.

Proposition 2.10 *If R satisfies condition (P), then R is linear 3-Armendariz.*

Proof. If R satisfies condition (P) it easy proof,

$$abc=0\Leftrightarrow bca\Leftrightarrow cab=0.$$

Let $f(x)=a_0+a_1x, g(x)=b_0+b_1x, h(x)=c_0+c_1x\in R[x],$ be satisfy $f(x)g(x)h(x)=0.$ Next we only need to Proof $a_i b_j c_k=0,$ for all $0\le i,j,k\le 1$

$$f(x)g(x)h(x)=(a_0+a_1x)(b_0+b_1x)(c_0+c_1x)$$
$$=a_0b_0c_0+(a_0b_0c_1+a_0b_1c_0+a_1b_0c_0)x+(a_0b_1c_1+a_1b_0c_1+a_1b_1c_0)x^2+(a_1b_1c_1)x^3.$$

Then we have the following equation

$$a_0b_0c_0=0,(1)$$

$$a_0b_0c_1+a_0b_1c_0+a_1b_0c_0=0,(2)$$

$$a_0b_1c_1+a_1b_1c_0+a_1b_0c_1=0,(3)$$

$$a_1b_1c_1=0.(4)$$

Multiplying (2) on left side by c_0 then we have

$$c_0a_0b_1c_0+c_0a_1b_0c_0=0(5)$$

Multiplying (5) on left side by b_0 yields

$$b_0c_0a_1b_0c_0=0.(6)$$

Then $(a_1b_0c_0)^2=0.$ Thus, $a_1b_0c_0=0.$ Since $R,$ satisfies condition (P) for all $i=0,1.$ Now (5) be comes $c_0a_0b_1c_0=0.$ Thus, $c_0a_0b_1=0$, for all $j=0,1.$ Now from (2) it follows $a_0b_0c_1=0$ for all $k=0,1.$

Multiplying (3) on left side by b_0c_0 then we have

$$b_0c_0a_1b_1c_0 + b_0c_0a_1b_0c_1 = 0.(7)$$

Multiplying (7) on right side by b_0 yields

$$b_0c_0a_1b_1c_0b_0 = 0.$$

Thus, $a_1b_1c_0 = 0$ for all $i, j = 0,1.$ Since R satisfies condition $(P).$ Thus, (7) becomes $b_0c_0a_1b_0c_1 = 0.$ Thus, $a_1b_0c_1 = 0$ for all $i,k = 0,1.$ Now (3) it follows $a_0b_1c_1 = 0$ for all $j,k = 0,1.$ Therefore $a_ib_jc_k = 0$, for all $i, j, k.$ This means that R is linear 3-Armendariz ring.

The following example show that the converse of Proposition 2.10, is not true.

Example 2.11 *Let*

$$R = \left\{ \begin{pmatrix} 0 & a & b & c & d \\ 0 & 0 & e & 0 & f \\ 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mid a,b,c,d,e,f,g,h \in \mathbb{Z} \right\}.$$

Next, we proof R is linear 3-Armendariz ring, but it doesn't satisfies condition $(P).$

Proof. For any $\alpha = \alpha_0 + \alpha_1 x, \beta = \beta_0 + \beta_1 x, \gamma = \gamma_0 + \gamma_1 x \in R[x],$ and $\alpha\beta\gamma = 0.$ We have the following ring isomorphism

$$\phi : R[x] \to \left\{ \begin{pmatrix} 0 & a & b & c & d \\ 0 & 0 & e & 0 & f \\ 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mid a,b,c,d,e,f,g,h \in \mathbb{Z}[x] \right\}.$$

$$\begin{pmatrix} 0 & a_0 & b_0 & c_0 & d_0 \\ 0 & 0 & e_0 & 0 & f_0 \\ 0 & 0 & 0 & 0 & g_0 \\ 0 & 0 & 0 & 0 & h_0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_1 & b_1 & c_1 & d_1 \\ 0 & 0 & e_1 & 0 & f_1 \\ 0 & 0 & 0 & 0 & g_1 \\ 0 & 0 & 0 & 0 & h_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} x$$

$$\to \begin{pmatrix} 0 & a_0 + a_1x & b_0 + b_1x & c_0 + c_1x & d_0 + d_1x \\ 0 & 0 & e_0 + e_1x & 0 & f_0 + f_1x \\ 0 & 0 & 0 & 0 & g_0 + g_1x \\ 0 & 0 & 0 & 0 & h_0 + h_1x \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By ring isomorphism we have

$$0=\phi(\alpha)\phi(\beta)\phi(\gamma)=\begin{pmatrix}0&0&0&0&A_1B_2G_3\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$

with $A_1=a_{10}+a_{11}x, B_2=b_{20}+b_{21}x, G_3=g_{30}+g_{31}x, a_{1i}$ is (1,2) entry of α_i, b_{2j} is (2,3) entry of β_j, g_{3k} is (3,5) entry of γ_k. By [12, Example 3], Z is 3-Armendariz. Therefore Z is linear 3-Armendariz. so $a_{1i}b_{2j}g_{3k}=0,$ then $\alpha_i\beta_j\gamma_k=0$. Thus, R is linear 3-Armendariz ring. But it is clear that R don't satisfies condition (P).

Proposition 2.12 *Let* R *be a ring satisfies condition* (P)*, then*

$$S=\left\{\begin{pmatrix}a&b&c\\0&a&d\\0&0&a\end{pmatrix} \mid a,b,c,d\in R\right\}$$

is linear 3-Armendariz ring.

Proof. We notice that

$$\begin{pmatrix}a_1&b_1&c_1\\0&a_1&d_1\\0&0&a_1\end{pmatrix},\begin{pmatrix}a_2&b_2&c_2\\0&a_2&d_2\\0&0&a_2\end{pmatrix}\in S.$$

The operations of additive and multiplication are denoted by

$$(a_1,b_1,c_1,d_1)+(a_2,b_2,c_2,d_2)=(a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2)$$

and

$$(a_1,b_1,c_1,d_1)(a_2,b_2,c_2,d_2)=(a_1a_2,a_1b_2+b_1a_2,a_1c_2+b_1d_2+c_1a_2,a_1d_2+d_1a_2)$$

respectively. So every polynomial in $S[x]$ can be expressed in the form

$$(p_0(x),p_1(x),p_2(x),p_3(x))$$

for some $p_i(x)$' s in $R[x]$. For any three polynomials of $S[x]$,

$$f(x)=(f_0(x),f_1(x),f_2(x),f_3(x);$$

$$g(x)=(g_0(x),g_1(x),g_2(x),g_3(x);$$

$$h(x)=(h_0(x),h_1(x),h_2(x),h_3(x).$$

Suppose that $f(x)g(x)h(x)=0$ then we have the following Equations

$$f_0(x)g_0(x)h_0(x)=0,(1)$$

$$f_0(x)g_1(x)h_0(x)+f_0(x)g_0(x)h_1(x)+f_1(x)g_0(x)h_0(x)=0, \quad (2)$$

$$\begin{aligned} & f_0(x)g_0(x)h_2(x)+f_0(x)g_1(x)h_3(x)+f_1(x)g_0(x)h_3(x) \\ & + f_0(x)g_2(x)h_0(x)+f_1(x)g_3(x)h_0(x)+f_2(x)g_0(x)h_0(x)=0, \end{aligned} \quad (3)$$

$$f_0(x)g_0(x)h_3(x)+f_0(x)g_3(x)h_0(x)+f_3(x)g_0(x)h_0(x)=0. \quad (4)$$

Since $R[x]$ satisfies condition (P), by Theorem 2.5, then from (1), we have $h_0(x)f_0(x)g_0(x)=g_0(x)h_0(x)f_0(x)=0.$ Multiplying (2) on left side by $h_0(x)$ then we have

$$h_0(x)f_0(x)g_1(x)h_0(x)+h_0(x)f_1(x)g_0(x)h_0(x)=0. \quad (5)$$

Multiplying (5) on right side by $f_0(x)$ then we have

$$h_0(x)f_0(x)g_1(x)h_0(x)f_0(x)=0. \quad (6)$$

Thus, $f_0(x)g_1(x)h_0(x)=0.$ From (5), we have $h_0(x)f_1(x)g_0(x)h_0(x)f_0(x)=0.$ Thus $f_1(x)g_0(x)h_0(x)=0,$ since $R[x]$ satisfies the condition (P), again by Theorem 2.5. Now, from (2) it follows $f_0(x)g_0(x)h_1(x)=0.$

Multiplying (4) on left side by $h_0(x)$ yields

$$h_0(x)f_0(x)g_3(x)h_0(x)+h_0(x)f_3(x)g_0(x)h_0(x)=0. \quad (7)$$

Multiplying (7) on left side by $f_0(x)$ then we have

$$f_0(x)h_0(x)f_0(x)g_3(x)h_0(x)=0.$$

Thus, $f_0(x)g_3(x)h_0(x)=0.$ Since $R[x]$ satisfies the condition (P), also Theorem 2.5, from (7) we have

$$h_0(x)g_0(x)f_3(x)h_0(x)=0.$$

Thus, $g_0(x)f_3(x)h_0(x)=0.$ Now from (4) it follows $f_0(x)g_0(x)h_3(x)=0.$

Now, multiplying (3) on left side by $h_0(x)$ then we have

$$h_0(x)f_0(x)g_2(x)h_0(x)+h_0(x)f_1(x)g_3(x)h_0(x)+h_0(x)f_2(x)g_0(x)h_0(x)=0 \quad (8)$$

Multiplying (8) on left side by $g_0(x)$ yields

$$g_0(x)h_0(x)f_2(x)g_0(x)h_0(x)=0.$$

Thus, $g_0(x)f_2(x)h_0(x)=0.$ Thus (8) becomes

$$h_0(x)f_0(x)g_2(x)h_0(x)+h_0(x)f_1(x)g_3(x)h_0(x)=0. \quad (9)$$

Multiplying (9) on left side by $g_3(x)$ yields

$$g_3(x)h_0(x)f_1(x)g_3(x)h_0(x)=0.$$

Thus, $f_1(x)g_3(x)h_0(x)=0.$ Thus from (9) we have

$$h_0(x)f_0(x)g_2(x)h_0(x)=0.$$

Thus, $f_0(x)g_2(x)h_0(x)=0.$ Now from (8) it follows $f_0(x)g_2(x)h_0(x)=0.$ Now (3) becomes

$$f_0(x)g_0(x)h_2(x)+f_0(x)g_1(x)h_3(x)+f_1(x)g_0(x)h_3(x)=0. \quad (10)$$

Multiplying (10) on left side by $h_0(x)$ yields

$$h_0(x)f_0(x)g_1(x)h_3(x)+h_0(x)f_1(x)g_0(x)h_3(x)=0. \quad (11)$$

Multiplying (11) on left side by $g_0(x)h_3(x)$ yields

$$g_0(x)h_3(x)h_0 f_1(x)g_0(x)h_3(x) = 0.$$

Thus, $f_1(x)g_0(x)h_3(x) = 0.$ Then from (11) we have $h_0(x) f_0(x)g_1(x)h_3(x) = 0.$ Thus, $f_0(x)g_1(x)h_3(x) = 0.$ Now from (10) it follows $f_0(x)g_0(x)h_2(x) = 0.$ Now let

$$f(x) = \Sigma_{i=0}^{1}\begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix}x^i, g(x) = \Sigma_{j=0}^{1}\begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a'_j & d'_j \\ 0 & 0 & a'_j \end{pmatrix}x^j$$

and

$$h(x) = \Sigma_{k=0}^{1}\begin{pmatrix} a''_k & b''_k & c''_k \\ 0 & a''_k & d''_k \\ 0 & 0 & a''_k \end{pmatrix}x^k$$

where

$$f_0(x) = \Sigma_{i=0}^{1} a_i x^i,\ f_1(x) = \Sigma_{i=0}^{1} b_i x^i,\ f_2(x) = \Sigma_{i=0}^{1} c_i x^i,\ f_3(x) = \Sigma_{i=0}^{1} d_i x^i,$$

$$g_0(x) = \Sigma_{j=0}^{1} a'_j x^j, g_1(x) = \Sigma_{j=0}^{1} b'_j x^j, g_2(x) = \Sigma_{j=0}^{1} c'_j x^j, g_3(x) = \Sigma_{j=0}^{1} d'_j x^j$$

$$h_0(x) = \Sigma_{k=0}^{1} a''_k x^k, h_1(x) = \Sigma_{k=0}^{1} b''_k x^k, h_2(x) = \Sigma_{k=0}^{1} c''_k x^k, h_3(x) = \Sigma_{k=0}^{1} d''_k x^k.$$

Then we obtain that,

$a_i a'_j a''_k = 0, a''_k a_i a'_j = 0, a'_j b_i a''_k = 0, b_i a'_j a''_k = 0, a_i a_i b''_k = 0, d_i a'_j a''_k = 0, a_i d'_j a''_k = 0, a_i a'_j d''_k = 0, a_i c'_j a''_k = 0, b_i d'_j a''_k = 0, c_i a'_j a''_k = 0, a_i a'_j c''_k = 0,$ and $b_i a'_j d''_k = 0.$ for all $i, j, k = 0,1,$ by the results of Proposition 2.10, the condition that R satisfies condition (P), we have

$$\begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix}\begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a'_j & d'_j \\ 0 & 0 & a'_j \end{pmatrix}\begin{pmatrix} a''_k & b''_k & c''_k \\ 0 & a''_k & d''_k \\ 0 & 0 & a''_k \end{pmatrix} = 0.$$

For all i, j, k and therefore S is linear 3-Armendariz ring.

Given a ring R and a bimodule ${}_R M_R$, the trivial extension of R by M is the ring $T(R, M) = R \bigoplus M$ with the usual addition and the following multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$ This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R, m \in M$ and the usual matrix operations are used.

Corollary 2.13 *If a ring* R *satisfies condition* (P), *then the trivial extension* $T(R, R)$ *is linear 3-Armendariz ring.*

Proof. Notice that $T(R, R)$ is isomorphic to

$$U = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b \in R \right\}$$

and that each subring of linear 3-Armendariz ring is also linear 3-Armendariz. Thus, $T(R,R)$ is linear 3-Armendariz ring by Proposition 2.12.

Remark 2.14
Let R be a ring satisties condition (P) and let

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}.$$

Based on Proposition 2.12, one may suspect that R_n may be also linear 3-Armendariz ring for $n \geq 4$. But the following example erases the possibility.

Example 2.15
Let a ring R satisfy $R^3 = 0.$ For any three elements $a,b,c \in R$ and $abc \neq 0$, then in R_4

$$aI_4(bE_{12} + (bE_{12} - bE_{13})x)(cE_{34} + (cE_{24} + cE_{34})x) = 0.$$

But

$$aI_4(bE_{12})(cE_{24} + cE_{34}) \neq 0.$$

So R_4 is not linear 3-Armendariz ring. Similarly, for the case of $n \geq 5,$ we have the same result.

From Proposition 2.12, one may suspect that if R is linear 3-Armendariz then every n-by-n full matrix ring $M_n(R)$ over R is linear 3-Armendariz, where $n \geq 2.$ But the following example erases the possibility.

Example 2.16 *Let* R *be a ring and let* $S = M_2(F).$ *Let*

$$f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x, \; g(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x, \qquad h(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x$$

be polynomials in $S[x].$ Then $f(x)g(x)h(x) = 0.$ But

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0.$$

Thus S is not linear 3-Armendariz.
From Corollary 2.13, one may suspect that if R is linear 3-Armendariz, then the trivial extension $T(R,R)$ is linear 3-Armendariz, but the following example eliminates the possibility.

Example 2.17 *Let* $T(R,R)$ *be a ring satisfies condition* (P). *Then* $R=\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a,b\in T(R,R)\right\}$ is linear 3-Armendariz ring by Corollary 2.13. Let

$S=\left\{\begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \mid A,B\in T(R,R)\right\}$ and let $f(x)=\begin{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \end{pmatrix}$,

$$g(x)=\begin{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \end{pmatrix}+\begin{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -b & 0 \\ 0 & -b \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \end{pmatrix}x,$$

$$h(x)=\begin{pmatrix} \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \end{pmatrix}+\begin{pmatrix} \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \end{pmatrix}x.$$

Then $f(x)g(x)h(x)=0.$ But

$$\begin{pmatrix} \begin{pmatrix} a & 0\\ 0 & a\end{pmatrix} & 0 \\ 0 & \begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}\end{pmatrix} \times \begin{pmatrix} \begin{pmatrix} 0 & b\\ 0 & 0\end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & b\\ 0 & 0\end{pmatrix}\end{pmatrix}$$

$$\times \begin{pmatrix} \begin{pmatrix}0&c\\0&0\end{pmatrix} & \begin{pmatrix}c&0\\0&c\end{pmatrix}\\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&c\\0&0\end{pmatrix}\end{pmatrix} = \begin{pmatrix} \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&abc\\0&0\end{pmatrix}\\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix}\end{pmatrix} \neq 0.$$

Thus, S is not linear 3-Armendariz ring.

Lemma 2.18 *If R is linear 3-Armendariz ring, then R is abelian.*

Proof. For any element $e, r \in R,$ with $e = e^2.$ Since we have equality of $R[x]$ $e[e - er(1-e)x][1-e+er(1-e)x] = 0,$ thus, $e.e.er(1-e) = er - ere = 0,$ that is $er = ere.$ And because $(1-e)[(1-e)-(1-e)rex][e+(1-e)rex] = 0,$ thus, $(1-e).(1-e).(1-e)re = (1-e)re = re - ere = 0,$ that is $re = ere.$ It follows that e is central element of $R,$ that is R is abelian.

The converse of Lemma 2.18, is not hold by the following example.

Example 2.19 *Let S be an abelian ring, and* $R = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in S \right\}.$ *Then R is abelian by [1, Lemma 2]. Consider*

$$f(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x,$$

$$g(x)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}+\begin{pmatrix}0&1&0&0\\0&0&0&1\\0&0&0&1\\0&0&0&0\end{pmatrix}x,$$

$$h(x)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}+\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}x$$

in $R[x]$. Then $f(x)g(x)h(x)=0,$ but

$$\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&1\\0&0&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\neq 0.$$

So R is not linear 3-Armendariz.

Proposition 2.20 *For an abelian ring R, the following conditions are equivalent:*

1. R is linear 3-Armendariz ring;
2. eR and $(1-e)R$ are linear 3-Armendariz rings for every $e=e^2\in R.$

Proof. We only need to prove (2)$\Rightarrow$(1). For any $f(x)=\Sigma_{i=0}^{1}a_ix^i, g(x)=\Sigma_{j=0}^{1}b_jx^j, h(x)=\Sigma_{k=0}^{1}c_kx^k$ nonzero polynomials in $R[x]$. Assume that $f(x)g(x)h(x)=0.$ Let

$f_1(x)=ef(x), g_1(x)=eg(x), h_1(x)=eh(x)\in(eR)[x],\ f_2(x)=(1-e)f(x), g_2(x)=(1-e)g(x), h_2(x)=(1-e)h(x)\in((1-e)R)[x].$

Then $f_1(x)g_1(x)h_1(x)=ef(x)eg(x)eh(x)=ef(x)g(x)h(x)=0$ in (eR)[x]. Since eR is linear 3-Armendariz ring, so $ea_ieb_jec_k=ea_ib_jc_k=0$ and

$f_2(x)g_2(x)h_2(x)=((1-e)f(x))((1-e)g(x))((1-e)h(x))=(f(x)-ef(x))(g(x)-eg(x))(h(x)-eh(x))=f(x)g(x)h(x)-ef(x)g(x)h(x)=0$

in $((1-e)R)[x].$ Since $(1-e)R$ is linear 3-Armendariz ring, so $(1-e)a_i(1-e)b_j(1-e)c_k=a_ib_jc_k-ea_ib_jc_k=0,$ and so $a_ib_jc_k=0,$ for all $0\leq i,j,k\leq 1.$ Therefore R is linear 3-Armendariz ring, and the result follows.

Recall that an element u of a ring R is right regular if $ur=0$ implies $r=0$ for $r\in R.$ Similarly, left regular elements can be defined. An element is regular if it is both left and right regular (and hence not a zero divisor).
A ring R is called right (resp., left) Ore if given $a,b\in R$ with b regular, there exist $a_1,b_1\in R$ with b_1 regular such that $ab_1=ba_1$ (resp., $b_1a=a_1b$). It is a well-known fact that R is a right (resp., left) Ore ring if and only

if the classical right (resp., left) quotient ring of R exists.

Theorem 2.21 *Let R be a right Ore ring with the classical right quotient ring Q. Then R is linear 3-Armendariz if and only if so is Q.*

Proof. It suffices to show by Lemma 2.8 that if R is linear 3-Armendariz rings so is Q. We apply the proof of [2, Theorem 12]. Consider $f(x)=\Sigma_{i=0}^{1}\alpha_i x^i$, $g(x)=\Sigma_{j=0}^{1}\beta_j x^j$ and $h(x)=\Sigma_{k=0}^{1}\gamma_k x^k \in Q[x]$, such that $f(x)g(x)h(x)=0$. By [7, Proposition 2.1.16], we can assume that $\alpha_i = a_i u^{-1}, \beta_j = b_j v^{-1}, \gamma_k = c_k w^{-1}$ with $a_i, b_j, c_k \in R$ for all i, j, k and a right regular elements $u, v, w \in R$. Also by [7, Proposition 2.1.16] for each j there exist $d_j \in R$ and a right regular element $s \in R$ such that $u^{-1}b_j = d_j s^{-1}$. For each k, there exist $e_k \in R$ and a right regular element $t \in R$ such that $(vs)^{-1}c_k = e_k t^{-1}$. Put $\phi(x)=\Sigma_{i=0}^{1}a_i x^i, \varphi(x)=\Sigma_{j=0}^{1}d_j x^j$ and $\psi(x)=\Sigma_{k=0}^{1}e_k x^k \in R[x]$. Then we have

$$\begin{aligned} 0 = & f(x)g(x)h(x) \\ = & \Sigma_{i=0}^{1}\Sigma_{j=0}^{1}\Sigma_{k=0}^{1}\alpha_i\beta_j\gamma_k x^{i+j+k} \\ = & \Sigma_{i=0}^{1}\Sigma_{j=0}^{1}\Sigma_{k=0}^{1}a_i(u^{-1}b_j)v^{-1}c_k w^{-1}x^{i+j+k} \\ = & \Sigma_{i=0}^{1}\Sigma_{j=0}^{1}\Sigma_{k=0}^{1}a_i d_j s^{-1}v^{-1}c_k w^{-1}x^{i+j+k} \\ = & \Sigma_{i=0}^{1}\Sigma_{j=0}^{1}\Sigma_{k=0}^{1}a_i d_j (vs)^{-1}c_k w^{-1}x^{i+j+k} \\ = & \Sigma_{i=0}^{1}\Sigma_{j=0}^{1}\Sigma_{k=0}^{1}a_i d_j e_k t^{-1}w^{-1}x^{i+j+k} \\ = & \Sigma_{i=0}^{1}\Sigma_{j=0}^{1}\Sigma_{k=0}^{1}a_i d_j e_k (wt)^{-1}x^{i+j+k} \\ = & \phi(x)\varphi(x)\psi(x)(wt)^{-1}. \end{aligned}$$

Hence $\phi(x)\varphi(x)\psi(x)=\Sigma_{i=0}^{1}\Sigma_{j=0}^{1}\Sigma_{k=0}^{1}a_i d_j e_k x^{i+j+k}=0$, in $R[x]$. Since R is linear 3-Armendariz rings, $a_i d_j e_k = 0$ for all i, j, k and so
$\alpha_i\beta_j\gamma_k = a_i u^{-1}b_j v^{-1}c_k w^{-1} = a_i(u^{-1}b_j)v^{-1}c_k w^{-1} = a_i d_j s^{-1}v^{-1}c_k w^{-1} = a_i d_j (vs)^{-1}c_k w^{-1} = a_i d_j e_k t^{-1}w^{-1} = a_i d_j e_k (wt)^{-1} = 0,$
for all i, j, k. Therefore Q, is linear 3-Armendariz ring.

Proposition 2.22 *Let R be a ring and Δ be a multiplicative closed subset in R consisting of central regular elements. Then R is linear 3-Armendariz rings if and only if $\Delta^{-1}R$ is linear 3-Armendariz rings.*

Proof. Let R be linear 3-Armendariz ring and $S=\Delta^{-1}R$. Put $f(x)g(x)h(x)=0$, where $f(x)=\Sigma_{i=0}^{1}\alpha_i x^i$, $g(x)=\Sigma_{j=0}^{1}\beta_j x^j$ and $h(x)=\Sigma_{k=0}^{1}\gamma_k x^k \in S[x]$, We may assume that $\alpha_i = a_i u^{-1}, \beta_j = b_j v^{-1}, \gamma_k = c_k w^{-1}$ with $a_i, b_j, c_k \in R$ for all i, j, k and $u, v, w \in \Delta$. Then we have

$$\begin{aligned} 0 = & f(x)g(x)h(x) \\ = & \Sigma_{i=0}^{1}\Sigma_{j=0}^{1}\Sigma_{k=0}^{1}\alpha_i\beta_j\gamma_k x^{i+j+k} \\ = & \Sigma_{i=0}^{1}\Sigma_{j=0}^{1}\Sigma_{k=0}^{1}a_i u^{-1}b_j v^{-1}c_k w^{-1}x^{i+j+k} \\ = & \Sigma_{i=0}^{1}\Sigma_{j=0}^{1}\Sigma_{k=0}^{1}a_i b_j c_k u^{-1}v^{-1}w^{-1}x^{i+j+k} \\ = & \Sigma_{i=0}^{1}\Sigma_{j=0}^{1}\Sigma_{k=0}^{1}a_i b_j c_k x^{i+j+k}(uvw)^{-1}. \end{aligned}$$

Hence $\Sigma_{i=0}^{1}\Sigma_{j=0}^{1}\Sigma_{k=0}^{1}a_i b_j c_k x^{i+j+k}=0$, in $R[x]$. Since R is linear 3-Armendariz ring, $a_i b_j c_k = 0$ for all i, j, k and so $\alpha_i\beta_j\gamma_k = a_i b_j c_k (uvw)^{-1} = 0$, for all i, j, k. Thus, S is linear 3-Armendariz ring. The converse is obtained by Lemma 2.8 we can complete The proof.

The ring of Laurent polynomials in $x,$ with coefficients in a ring $R,$ consists of all formal sum $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k,n are (possibly negative) integers; denote it by $R[x;x^{-1}].$

Corollary 2.23 *A commutative ring R is linear 3-Armendariz if and only if so is the total quotient ring of $R.$*

Proof. It suffices to show the necessity by Lemma 2.8. Let Δ be the multiplicative closed subset of all regular elements in R. Then $\Delta^{-1}R$ is the total quotient ring of R and hence the result holds by Proposition 2.22.

Corollary 2.24 *Let R be a ring. $R[x]$ is linear 3-Armendariz if and only if $R[x;x^{-1}]$ is linear 3-Armendariz.*

Proof. It suffices to establish necessity since $R[x]$ is a subring of $R[x;x^{-1}].$ Let $\Delta = \{1, x, x^2, \cdots\},$ then clearly Δ is a multiplicatively closed subset of $R[x].$ Since $R[x;x^{-1}] = \Delta^{-1}R[x],$ it follows that $R[x;x^{-1}]$ is linear 3-Armendariz by Proposition 2.22.

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