EXISTENCE OF SOLUTIONS FOR η-GENERALIZED MIXED VECTOR EQUILIBRIUM PROBLEMS IN BANACH SPACES

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ABSTRACT
In this paper, some generalized mixed vector equilibrium problems in Banach spaces are introduced and the equivalence of two classes of η-generalized mixed vector equilibrium problems is proved under suitable assumptions. By using the equivalence theorem, some results on the existence of solutions for η-generalized mixed vector equilibrium problems are obtained.

Keywords: *η-Generalized mixed vector equilibrium problems, Pseudomonotone, KKM-mapping, Hausdorff metric*

1. INTRODUCTION
The theory of vector variational inequalities has become a very powerful and effective tool in many fields like mechanics and physics, economics and finance, transportation and operations research, optimization, game theory and control problem, etc. Giannessi [1] was first to introduce the concept of vector variational inequality in a finite-dimensional Euclidean space. Later G.Y. Chen and G.M. Cheng [2], Lee *et al.* [3] and Q.H.Ansari *et al.* [4] have intensively studied some kinds of vector variational inequalities, vector quasi-variational inequalities and vector complementarity problems in Banach spaces. Recently, Lee *et al*. [5]and Siddiqi *et al.* [6] established some existence theorems for solutions of some kinds of vector-valued inequality equations and vector variational inequalities in Banach spaces and in Hausdorff topological vector spaces.

On the other hand, the generalized vector equilibrium problems have been extensively studied by many authors (see,e.g., [16,20-23]) since they include as special cases generalized vector variational inequality problems, generalized vector variational-like inequality problems, generalized vector complementarity problems and generalized vector equilibrium problems.

In 2008, Ansari *et al.* [4] considered the Stampacchia-type vector variational inequality problems (in short,SVVIP); In 2005, Vasil *et al.* [7] proved the existence of the solutions of variational-like inequalities with relaxed η-α semimonotone mappings in arbitrary Banach spaces. Recently, Li *et al*. [8] introduced and considered a class of η-vector variational-like inequalities and gave some existence results.

Motivated by the above works in this field, we first obtain the equivalence of two η-generalized mixed vector equilibrium problems. Furthermore, the equivalence theorem can be applied to deduce the existence results of η-generalized mixed vector equilibrium problems in Banach spaces. In fact, we prove the equivalence of η-generalized mixed vector equilibrium problems under certain pseudomonotonity assumptions.

2. PROBLEM FORMULATION
In this paper, Let X and Y be two real Banach spaces, K be a nonempty convex subset of X. Let $\{C(u):u\in K\}$ be a family of closed convex and pointed cones of Y with $\mathrm{int}C(u)\neq\varnothing$, where $\mathrm{int}C(u)$ is the interior of $C(u)$. Let $T:K\to 2^{L(X,Y)}$ be a nonempty compact-valued multifunction, where $L(X,Y)$ denotes the space of all continuous linear operators from X into Y. Given mappings $F:L(X,Y)\times K\to Y$, $A:L(X,Y)\to L(X,Y)$, $\eta:K\times K\to X$ and $h:K\to Y$, we will study the following η-generalized mixed vector equilibrium problem η-GMVEP: Find $u_0\in K$ and $s_0\in T(u_0)$ such that

$$F(As_0,\eta(v,u_0))+h(v)-h(u_0)\notin -\mathrm{int}C(u_0),\forall v\in K. \quad (1)$$

Special cases:
(I) If we take $F(As_0,\eta(v,u_0))=\langle As_0,\eta(v,u_0)\rangle$, then the problem (1) deduces to the following generalized vector variational-like inquality problem: find $u_0\in K$ and $s_0\in T(u_0)$ such that

$$\langle As_0,\eta(v,u_0)\rangle+h(v)-h(u_0)\notin -\mathrm{int}C(u_0),\forall v\in K.$$

which was considered by M.F. Khan and Salahuddin [9].
(II) If we take A is an identity mapping, $h=0$, then the problem (I) is equivalent to the following vector

variational-like inequality problem: find $u_0 \in K$ and $s_0 \in T(u_0)$ such that

$$\langle s_0, \eta(v,u_0)\rangle \notin -\mathrm{int}C(u_0), \forall v \in K.$$

which was considered by Q.H. Ansari [10] and Lee *et al.* [11].

(III) If $T: K \to L(X,Y)$ is a single-valued mapping, then the problem (II) reduces to the problem: find $u_0 \in K$ such that

$$\langle T(u_0), \eta(v,u_0)\rangle \notin -\mathrm{int}C(u_0), \forall v \in K.$$

which was considered by Siddiqi *et al.* [6] and D.Y. Jung [12].

(IV) If $\eta(v,u_0) = v - g(u_0)$, $g: K \to K$ is a mapping, then the problem (II) is equivalent to finding $u_0 \in K$ and $s_0 \in T(u_0)$ such that

$$\langle s_0, v - g(u_0)\rangle \notin -\mathrm{int}C(u_0), \forall v \in K.$$

which was considered by A. Khaliq *et al.* [13].

(V) If $\eta(v,u_0) = v - u_0$, then the above problem (IV) is equivalent to the following generalized vector variational inequality problem: find $u_0 \in K$ and $s_0 \in T(u_0)$ such that

$$\langle s_0, v - u_0\rangle \notin -\mathrm{int}C(u_0), \forall v \in K.$$

which was considered by G.M. Lee *et al.*[5].

(VI) If $\eta(v,u_0) = v - u_0$, then the above problem (III) is equivalent to vector variational inequality problem: find $u_0 \in K$ such that

$$\langle T(u_0), v - u_0\rangle \notin -\mathrm{int}C(u_0), \forall v \in K.$$

which was considered by G.Y Chen *et al.* [14].

Now, we recall some notations, definitions and lemmas, which will be used in the next section.

Definition 2.1 ([15]) Let K be a nonempty subset of topological vector space X. A multifunction $F: K \to 2^Y$ is called KKM-mapping if, for every nonempty finite subset $\{u_1, u_2, \cdots, u_n\}$ of K, $\mathrm{co}\{u_1, u_2, \cdots, u_n\} \subseteq \bigcup_{i=1}^{n} F(u_i)$.

Definition 2.2 ([16]) Let $C: K \to 2^Y$ be a multifunction such that $C(u)$ is a proper closed and convex moving cone with $\mathrm{int}C(u) \neq \varnothing$, then a mapping $g: K \to 2^Y$ is said to be $C(u)$-convex if, for all $u, v \in K$ and $\lambda \in [0,1]$,

$$g(\lambda u + (1-\lambda)v) \subseteq \lambda g(u) + (1-\lambda)g(v) - C(u).$$

and g is said to be affine if, for each $u, v \in K$ and $\lambda \in [0,1]$,

$$g(\lambda u + (1-\lambda)v) = \lambda g(u) + (1-\lambda)g(v).$$

Remark 2.3 If $g: K \to Y$ is a $C(u)$-convex vector-valued function, then $\sum_{i=1}^{n} \lambda_i g(u_i) - g(\sum_{i=1}^{n} \lambda_i u_i) \in C(u)$, for all $u_i \in K$, $\lambda_i \in [0,1]$, $i = 1, 2, \cdots, n$ with $\sum_{i=1}^{n} \lambda_i = 1$.

Definition 2.4 ([17]) A mapping $\eta: K \times K \to X$ is said to be skew if, for each $u, v \in K$, $\eta(u,v) + \eta(v,u) = 0$.

Definition 2.5 Let $F: L(X,Y) \times X \to Y$, $\eta: K \times K \to X$, $h: K \to Y$ be functions and $T: K \to 2^{L(X,Y)}$ be a multifunction, then F is said to be $\eta - C(u)$ pseudomonotone in K if for each $u, v \in K$,

$$\exists s \in T(u), \quad F(As, \eta(v,u)) + h(v) - h(u) \notin -\mathrm{int}C(u)$$
$$\Rightarrow \forall t \in T(v), \quad F(At, \eta(u,v)) + h(u) - h(v) \notin \mathrm{int}C(u)$$

Definition 2.6 $T: X \to 2^Y$. The graph of T, denoted by $\mathrm{Gr}(T)$, is the following set:

$$\mathrm{Gr}(T) = \{(u,v): v \in T(u)\}.$$

Lemma 2.7 (Fan's lemma[18]) Let K be a nonempty subset of Hausdorff topological vector space X. Let $G: K \to 2^X$ be a KKM-mapping, such that for any $y \in K$, $G(y)$ is closed and $G(y^*)$ is compact for some $y^* \in K$. Then

$$\bigcap_{y \in K} G(y) \neq \varnothing.$$

Lemma 2.8 ([18]) Let Y be a topological vector space with a closed convex and pointed cone C such that

$\mathrm{int}C \neq \varnothing$.

Then for all $x, y, z \in Y$, we have

(i) $x - y \in -\mathrm{int}C$ and $x \notin -\mathrm{int}C \Rightarrow y \notin -\mathrm{int}C$;

(ii) $x + y \in -C$ and $x + z \notin \mathrm{int}C \Rightarrow z - y \notin -\mathrm{int}C$;

(iii) $x + z - y \notin -\mathrm{int}C$ and $-y \in -C \Rightarrow x + z \notin -\mathrm{int}C$;

(iv) $x + y \notin -\mathrm{int}C$ and $y - z \in -C \Rightarrow x + z \notin -\mathrm{int}C$.

Lemma 2.9 (Nadler's theorem [19]) Let (X , Y) be a normed vector space and H be a Hausdorff metric which is defined by

$$H(A,B) = \max\left(\sup_{u\in A}\inf_{v\in B}\|u-v\|, \sup_{v\in B}\inf_{u\in A}\|u-v\|\right), \forall A, B \in CB(X),$$

where $CB(X)$ is the collection of all closed and bounded subsets of X. If A and B are any two compact sets in X, then for each $\epsilon > 0$ and each $u \in A$, there exists $v \in B$ such that

$$\|u - v\| \leq (1+\epsilon)H(A,B).$$

In particular, if A and B are compact sets in X, then for each $u \in A$, there exists $v \in B$ such that

$$\|u - v\| \leq H(A,B).$$

3. MAIN RESULT

In this section, we first state and prove the equivalent of two η-generalized mixed vector equilibrium problems.

Theorem 3.1 Let X and Y be two real Banach spaces, K be a nonempty convex subset of X and $\{C(u) : u \in K\}$ be a family of closed convex and pointed cones of Y. Let $T : K \to 2^{L(X,Y)}$ be a nonempty compact-valued multifunction such that for each $u, v \in K$,

$$H(T(u + \lambda(v-u)), T(u)) \to 0 \qquad \lambda \to 0^+$$

where H is the Hausdorff metric defined on $CB(L(X,Y))$, $\eta : K \times K \to X$ be an operator. Suppose that the following conditions hold:

(a) $A : L(X,Y) \to L(X,Y)$ is a continuous mapping;

(b) for each $u, v \in K$, $F(At_\lambda, \eta(v_\lambda, v_\lambda)) \in C(u)$, $\forall t_\lambda \in T(v_\lambda)$, where $v_\lambda := u + \lambda(v-u)$, $\lambda \in (0,1)$;

(c) for each $u, v \in K$, $F(w_\lambda, \eta(v, v_\lambda)) + h(v) - h(v_\lambda) \to F(w_0, \eta(v,u)) + h(v) - h(u)$ as $\lambda \to 0^+$ for each $\{w_\lambda\} \subseteq L(X,Y)$ with $w_\lambda \to w_0$ where $v_\lambda := u + \lambda(v-u)$ and $\lambda \in (0,1)$;

(d) for each $v \in K$ and $w \in L(X,Y)$, $F(w,\cdot)$, $\eta(\cdot, v)$, $h(\cdot)$ are affine on K;

(e) for each $u, v \in K$, F is $\eta - C(u)$ pseudomonotone on K if

$$\exists s \in T(u), \quad F(As, \eta(v,u)) + h(v) - h(u) \notin -\mathrm{int}C(u)$$

implies

$$\forall t \in T(v), \quad F(At, \eta(u,v)) + h(u) - h(v) \notin \mathrm{int}C(u).$$

Then the following are equivalent:

① there exists $u_0 \in K$ and $s_0 \in T(u_0)$ such that

$$F(As_0, \eta(v,u_0)) + h(v) - h(u_0) \notin -\mathrm{int}C(u_0), \forall v \in K.$$

② there exists $u_0 \in K$ such that

$$F(At, \eta(u_0,v)) + h(u_0) - h(v) \notin \mathrm{int}C(u_0), \forall v \in K, \forall t \in T(v).$$

Proof. Suppose that there exist $u_0 \in K$ such that, for $v \in K$, there exists $s_0 \in T(u_0)$ satisfying $F(As_0, \eta(v,u_0)) + h(v) - h(u_0) \notin -\mathrm{int}C(u_0)$. Then it follows from condition (e) that ② holds.

Conversely, suppose that there exists $u_0 \in K$ such that

$$F(At, \eta(u_0,v)) + h(u_0) - h(v) \notin \mathrm{int}C(u_0)$$

for all $v \in K$ and $t \in T(v)$. For an arbitrary $v \in K$, letting $v_\lambda := u + \lambda(v-u)$, $\lambda \in (0,1)$, we have $v_\lambda \in K$ by the convexity of K. Hence for all $t_\lambda \in T(v_\lambda)$

$$F(At_\lambda, \eta(u_0, v_\lambda)) + h(u_0) - h(v_\lambda) \notin \mathrm{int}C(u_0). \tag{2}$$

Since from condition (d), we have for each $v \in K$ and $w \in L(X,Y)$, $F(w,\cdot)$, $\eta(\cdot, v)$, $h(\cdot)$ are affine on K. Then, we derive

$$\begin{aligned}
&F(At_\lambda,\eta(v_\lambda,v_\lambda))+h(v_\lambda)-h(v_\lambda)\\
&= F(At_\lambda,\eta(\lambda v+(1-\lambda)u_0,v_\lambda))+h(\lambda v+(1-\lambda)u_0)-h(v_\lambda)\\
&=\lambda\left[F(At_\lambda,\eta(v,v_\lambda))+h(v)-h(v_\lambda)\right]+(1-\lambda)\left[F(At_\lambda,\eta(u_0,v_\lambda))+h(u_0)-h(v_\lambda)\right]
\end{aligned}$$

Hence, we deduce that

$$F(At_\lambda,\eta(v,v_\lambda))+h(v)-h(v_\lambda)\notin -\mathrm{int}C(u_0)\,. \tag{3}$$

In fact, suppose to the contrary that

$$F(At_\lambda,\eta(v,v_\lambda))+h(v)-h(v_\lambda)\in -\mathrm{int}C(u_0)\,.$$

Since $-\mathrm{int}C(u_0)$ is a convex cone, we know that

$$\lambda[F(At_\lambda,\eta(v,v_\lambda))+h(v)-h(v_\lambda)]\in -\mathrm{int}C(u_0)\,.$$

Note that condition (b) implies that

$$F(At_\lambda,\eta(v_\lambda,v_\lambda))+h(v_\lambda)-h(v_\lambda)\in C(u_0)\,,$$

thus, we derive that

$$\begin{aligned}
&(1-\lambda)\left[F(At_\lambda,\eta(u_0,v_\lambda))+h(u_0)-h(v_\lambda)\right]\\
&=\left[F(At_\lambda,\eta(v_\lambda,v_\lambda))+h(v_\lambda)-h(v_\lambda)\right]-\lambda\left[F(At_\lambda,\eta(v,v_\lambda))+h(v)-h(v_\lambda)\right]\\
&\in C(u_0)-(-\mathrm{int}C(u_0))\\
&=\mathrm{int}C(u_0).
\end{aligned}$$

Therefore,

$$F(At_\lambda,\eta(u_0,v_\lambda))+h(u_0)-h(v_\lambda)\in \mathrm{int}C(u_0)\,,$$

which contradicts (2). Hence,

$$F(At_\lambda,\eta(v,v_\lambda))+h(v)-h(v_\lambda)\notin -\mathrm{int}C(u_0)\,.$$

Since $T(v_\lambda),T(u_0)$ are compact subsets of $L(X,Y)$, by Lemma 2.9 for each $t_\lambda\in T(v_\lambda)$ we can find $s_\lambda\in T(u_0)$ such that

$$\|t_\lambda-s_\lambda\|\le H(T(v_\lambda),T(u_0))\,.$$

Since $T(u_0)$ is compact subset of $L(X,Y)$, without loss of generality we may assume that $s_\lambda\to s_0\in T(u_0)$ as $\lambda\to 0^+$. Moreover, we have

$$\begin{aligned}
\|t_\lambda-s_0\|&\le\|t_\lambda-s_\lambda\|+\|s_\lambda-s_0\|\\
&\le H(T(v_\lambda),T(u_0))+\|s_\lambda-s_0\|.
\end{aligned}$$

Since $H(T(v_\lambda),T(u_0))\to 0$ as $\lambda\to 0^+$, we have $t_\lambda\to s_0$. Thus from condition (a) we get $At_\lambda\to As_0$ as $\lambda\to 0^+$.
Furthermore, according to condition (c) we have

$$F(At_\lambda,\eta(v,v_\lambda))+h(v)-h(v_\lambda)\to F(As_0,\eta(v,u_0))+h(v)-h(u_0)\quad as\quad \lambda\to 0^+\,.$$

Consequently, it follows from (3) and the closedness of $Y\setminus(-\mathrm{int}C(u_0))$ that

$$F(As_0,\eta(v,u_0))+h(v)-h(u_0)\notin -\mathrm{int}C(u_0)$$

for all $v\in K$.
This completes the proof.

Next we derive an existence result of η-generalized mixed vector equilibrium problems by employing Theorem 3.1.

Theorem 3.2 Let all suppositions of Theorem 3.1 remain valid. Suppose additionally that:

(a) for each $u,v\in K$, $F(At,\eta(u,u))=0$;

(b) for each $v\in K$ and $w\in L(X,Y)$, $F(w,\cdot)$, $\eta(\cdot,v)$, $h(\cdot)$ are continuous;

(c) the mapping $W:K\to 2^Y$, defined by $W(u)=Y\setminus(\mathrm{int}C(u))$, $\forall u\in K$, has closed graph on K;

(d) for each $u\in K$ and $w\in L(X,Y)$, $F(w,\cdot)$, $\eta(u,\cdot)$, $h(\cdot)$ are affine;

(e) suppose additionally that there exists a nonempty, compact and convex subset D of K such that for each $u\in K\setminus D$ there exists $v\in D$ satisfying

$$F(At,\eta(u,v))+h(u)-h(v)\in \mathrm{int}C(u),\forall t\in T(v).$$

Then there exists $u_0 \in K$ and $s_0 \in T(u_0)$ such that

$$F(As_0, \eta(v,u_0)) + h(v) - h(u_0) \notin -\text{int}C(u_0), \forall v \in K.$$

Proof. We define $G: K \to 2^D$

$$G(v) = \{u \in D : F(At, \eta(u,v)) + h(u) - h(v) \notin \text{int}C(u), \forall t \in T(v)\}, \forall v \in K.$$

Firstly, we prove that $G(v)$ is closed for each $v \in K$. Indeed, let $\{u_n\}$ be a sequence in $G(v)$ such that $u_n \to \hat{u}$. Then $\hat{u} \in D$ since D is compact. It follows from $u_n \in G(v)$ that for each $t \in T(v)$,

$$F(At, \eta(u_n, v)) + h(u_n) - h(v) \notin \text{int}C(u_n).$$

Hence

$$F(At, \eta(u_n, v)) + h(u_n) - h(v) \in W(u_n) = Y \setminus \text{int}C(u_n).$$

Since from condition (b) it follows that for each $v \in K$ and $w \in L(X,Y)$, $F(w,\cdot)$, $\eta(\cdot,v)$, $h(\cdot)$ are continuous, we have

$$F(At, \eta(u_n, v)) + h(u_n) - h(v) \to F(At, \eta(\hat{u}, v)) + h(\hat{u}) - h(v).$$

Note that the graph $Gr(W)$ is closed in $X \times Y$ by condition (c), we obtain that

$$F(At, \eta(\hat{u}, v)) + h(\hat{u}) - h(v) \in W(\hat{u}).$$

Hence, we have

$$F(At, \eta(\hat{u}, v)) + h(\hat{u}) - h(v) \notin \text{int}C(\hat{u}).$$

This shows that $\hat{u} \in G(v)$ and $G(v)$ is closed.

Now we have to prove that $\bigcap_{v \in K} G(v) \neq \varnothing$. Indeed, since D is compact, it is sufficient to prove that the family $\{G(v)\}_{v \in K}$ has the finite intersection property. Let $\{v_1, v_2, \cdots, v_n\}$ be a finite subset of K, we claim that $\bigcap_{j=1}^{n} G(v) \neq \varnothing$. Indeed, note that $V := co(D \cup \{v_1, v_2, \cdots, v_n\})$ is a compact convex subset of K.

We define $I: V \to 2^V$

$$I(v) = \{u \in V : F(At, \eta(u,v)) + h(u) - h(v) \notin \text{int}C(u), \forall t \in T(v)\}, \forall v \in V.$$

From condition (a), we have $0 = F(At, \eta(u,u)) \notin \text{int}C(u)$, this shows $v \in I(v)$ and $I(v)$ is nonempty for each $v \in V$. Since $I(v)$ is a closed subset of a compact set V, we know that $I(v)$ is a compact.

Next, we assert that $I(v)$ is a KKM mapping. If not, there exists a finite subset $\{y_1, y_2, \cdots, y_n\}$ of V and $\lambda_i \geq 0$, $i = 1, 2, \cdots, n$, with $\sum_{i=1}^{n} \lambda_i = 1$ such that

$$\sum_{i=1}^{n} \lambda_i y_i \notin \bigcup_{j=1}^{n} G(y_j).$$

Then

$$F(At, \eta(\sum_{i=1}^{n} \lambda_i y_i, y_j)) + h(\sum_{i=1}^{n} \lambda_i y_i) - h(y_j) \in \text{int}C(\sum_{i=1}^{n} \lambda_i y_i).$$

From the cindition (d) we have or each $u \in K$ and $w \in L(X,Y)$, $F(w,\cdot)$, $\eta(u,\cdot)$, $h(\cdot)$ are affine, then

$$F(At, \eta(\sum_{i=1}^{n} \lambda_i y_i, \sum_{i=1}^{n} \lambda_i y_i)) + h(\sum_{i=1}^{n} \lambda_i y_i) - h(\sum_{i=1}^{n} \lambda_i y_i) \in \text{int}C(\sum_{i=1}^{n} \lambda_i y_i).$$

From condition (a), we have

$$0 = F(At, \eta(\sum_{i=1}^{n} \lambda_i y_i, \sum_{i=1}^{n} \lambda_i y_i)) \in \text{int}C(\sum_{i=1}^{n} \lambda_i y_i).$$

which contradicts $C(u) \neq Y$. Therefore $I(v)$ is a KKM mapping. According to lemma 2.7, there exists $u^* \in V$ such that $u^* \in I(v)$ for all $v \in V$; that is, there exists there exists $u^* \in V$,

$$F(At, \eta(u^*, v)) + h(u^*) - h(v) \notin \text{int}C(u^*), \forall v \in K, t \in T(v).$$

By condition (e), we get $u^* \in D$ and $u^* \in G(v_j)$, $j = 1, 2, \cdots, n$. Hence, $\{G(v)\}_{v \in K}$ has the finite intersection property and so $\bigcap_{v \in K} G(v) \neq \varnothing$, that is, there exists $u_0 \in D \subseteq K$, for all $v \in K$ and $t \in T(v)$ such that

$$F(At, \eta(u_0, v)) + h(u_0) - h(v) \notin \mathrm{int}C(u_0) .$$

For the remainder of the proof, we can derive the conclusion of Theorem 3.2 by the same proof as in Theorem 3.1.

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5. REFERENCES

[1] F. Giannessi, Theorems of the alternative, quadratic programs, and complementarity problems, *In Variational Inequalities and Complementarity Problems*, 151-186(1980).

[2] G.Y. Chen and G.M. Cheng, Vector variational inequalities and vector optimization, *In Lecture Notes Economics and Mathematical Systems*, 285, 408-416(1987).

[3] G.M. Lee, B.S. Lee and S.S. Chang, On vector quasi-variational inequalities, *J. Math. Anal. Appl.* **203**, 626-639(1996).

[4] Q.H. Ansari, A.P.Farajzadeh and S.Schaible, Existence of Solutions of a Vector Variational Inequalities and Vector Complementary Problem, *J. Optim. Theory Appl.* **74**, 445-456(2008).

[5] G.M. Lee, D.S. Kim, B.S. Lee and S.J. Cho, Generalized vector variational inequality and fuzzy extension, *Appl. Math. Left*. **6**, Issue 6, 45-51(1993).

[6] A.H. Siddiqi, Q.H. Ansari and A. Khaliq, On vector variational inequalities, *J. Optim. Theory Appl*. **84,** 171-180(1995).

[7] V. Preda, M. Beldiman and A. Batatorescu, On Variational-like Inequalities with Generalized Monotone Mappings, *Grant CNCSIS*, **27694**, 415-431(2005).

[8] X. Li, J.K. Kim, N.J. Huang, Existence of Solutions for η-Generalized Vector Variational-Like Inequalities, *Journal of Inequalities and Applications*, **2010** (2010), 1-13.

[9] M.F. Khan , Salahuddin, On generalized vector variational like inequalities, *Nonlinear Anal: TMA*. **59**, 879-889(2004).

[10] Q.H. Ansari, Extended generalized vector variational-like inequalities for nonmonotone multivalued maps, Ann. Sci. Math. 21, 1-11(1997) .

[11] B.S Lee and G.M Lee, A Vector Version of Minty's Lemma and Application, *Applied Mathematics Letters*. **12**, 43-50(1999).

[12] D.Y. Jung, Some generalization of minty's lemma, *Pure Appl. Math.* **1**, 33-37(1999).

[13] A.Khaliq and M.Rashid, On generalized vector guasi-variational-like inequality problems, *Fixed Point Theory and Applications*. **3**, 243-255(2005).

[14] G.Y. Chen and S.J. Li, Existence of solutions for a generalized vector variational inequality, J. *Optim.Theory Appl*. **90**, 321-334(1996).

[15] B.Knaster, C.Kuratowski, S.Mazurkiewicz, Ein beweis des lxpunktsatzes fur n-dimensionale simplexe, *Fund Math*. **11** (1935), 767-776.

[16] L.C. Zeng and J.C. Yao, An existence result for generalized vector equilibrium problems without pseudomonotonicity, *J. Optim. Theory Appl*. **74**, 445-456(1992).

[17] B.Chen and N.J. Huang, Vector variational-like and vector optimization problems in Asplund spaces, *Optimization Letters*. **74**, 1513-1525(2012).

[18] K. Fan, A generalization of Tychono_'s _xed point theorem, *J. Math. Anal. Appl.* **142**, 305-310(1961).

[19] S.B. Nadler Jr., Multi-valued contraction mappings, *Pacific J. Math*. **30**, 475-488(1969).

[20] N.J. Huang , J.Li, H.B.Thompson, Implicit vector equilibrium problems with applications, *Math Comput Modelling.* **20**,463-473(2008).

[21] I.V. Konnov, J.C. Yao, Existence of solutions for generalized vector equilibrium problems, *J. Math. Anal. Appl.* **223**, 328-335(1999).

[22] Q.H. Ansari, S. Schaible, J.C. Yao, The system of vector equilibrium problems and its applications, *J. Optim. Theory Appl*. **107,** 547C557(2000).

[23] L.C. Zeng and J.C. Yao, Generalized Minty's lemma for generalized vector equilibrium, *ScienceDirect. Applied Mathematics Letters*, **19**, 1320-1326(2006).

[24] P. T. Harker and J.S. Pang, Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications. *Math. Programming, Ser. B*. **48**, 161-220(1990).