SOLITARY WAVE SOLUTIONS FOR SOME NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS USING SINE-COSINE METHOD

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ABSTRACT
This paper implemented the sine-cosine methodfor solving Nonlinear partial differential equations PDES. Method has been successfully tested on Combined Kdv-Mkdv equation, Sharma-Tasso-Olver equation, Coupled Klien-Gordon equation, Generalized-Pochhammer-Chree equation (GPC). The calculations demonstrate the effectiveness and convenience of the method for nonlinear system of PDEs.

Keywords: *sine-cosine method, Nonlinear PDEs, Combined Kdv-Mkdv equation, Sharma-Tasso-Olver equation, Coupled Klien-Gordon equation, Generalized-Pochhammer-Chree equation (GPC).*

1. Introduction

Nonlinear evolution equations (NLEEs) have been the subject of study in various branches of Mathematical-physical sciences such as physics, biology, and chemistry. The analytical solutions of such equations are of fundamental importance since a lot of mathematical physical models are described by NLEEs.

The nonlinear wave phenomena observed in the above mentioned scientific fields, are often modeled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The availability of these exact solutions, for those nonlinear equations can greatly facilitate the verification of numerical solvers on the stability analysis of the solution. The investigation of exact solutions of NLPDEs plays an important role in the study of these phenomena. In the past several decades, many effective methods for obtaining exact solutions of NLPDEs have been presented. In the literature, there is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions[1-12].

This paper outlines the implementation of efficient and reliable technique which is sine-cosine method for solving nonlinear partial differential equations which are very important in applied sciences.

2. Description of the sine-cosine method

Consider the nonlinear partial differential equation in the form

$$F\left(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{xy}, u_{yy}, \dots\dots\dots\dots\right) = 0 \quad (1)$$

where u(x, y, t) is a traveling wave solution of nonlinear partial differential equation Eq. (1). We use the transformations,

$$u(x, y, t) = f(\xi) \quad (2)$$

Where

$$\xi = x + y - \lambda t \quad (3)$$

This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = -\lambda \frac{d}{d\xi}(.),\ \frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.),\ \frac{\partial}{\partial y}(.) = \frac{d}{d\xi}(.) \quad (4)$$

Using Eq. (3) to transfer the nonlinear partial differential equation Eq. (1) to nonlinear ordinary differential equation

$$Q(f, f', f'', f''', \dots\dots\dots\dots\dots.) = 0 \quad (5)$$

The ordinary differential equation (5) is then integrated as long as all terms contain derivatives, where we neglect the integration constants. The solutions of many nonlinear equations can be expressed in the form:[10]

$f(\xi) = \alpha \sin^{\beta}(\mu\xi)$, $|\xi| \le \frac{\pi}{2\mu}$(6)

or in the form

$f(\xi) = \alpha \cos^{\beta}(\mu\xi)$, $|\xi| \le \frac{\pi}{2\mu}$(7)

Where α ,μ, and β are parameters to be determined, μ and λ are the wave number and the wave speed, respectively. Anwar[10] use

$$\left.\begin{aligned} f(\xi) &= \alpha \sin^{\beta}(\mu\xi) \\ f'(\xi) &= \alpha\beta\mu \sin^{\beta-1}(\mu\xi)\cos(\mu\xi) \quad (8) \\ f''(\xi) &= \alpha\beta(\beta-1)\mu^2 \sin^{\beta-2}(\mu\xi) - \alpha\beta^2\mu^2 \sin^{\beta}(\mu\xi) \end{aligned}\right\}$$

and their derivative. Or use

$$\left.\begin{aligned} f(\xi) &= \alpha \cos^{\beta}(\mu\xi) \\ f'(\xi) &= -\alpha\beta\mu \cos^{\beta-1}(\mu\xi)\sin(\mu\xi) (9) \\ f''(\xi) &= \alpha\beta(\beta-1)\mu^2 \cos^{\beta-2}(\mu\xi) - \alpha\beta^2\mu^2 \cos^{\beta}(\mu\xi) \\ f'''(\xi) &= -\alpha\beta(\beta-1)(\beta-2)\mu^3 \cos^{\beta-3}(\mu\xi)\sin(\mu\xi) \\ &+\alpha\beta^3\mu^3 \cos^{\beta-1}(\mu\xi)\sin(\mu\xi) \end{aligned}\right\}$$

and so on. Substitute (8) or (9) into the reduced equation (5), balance the terms of the sine functions when (8) are used, or balance the terms of the cosine functions when (9) are used, and solve the resulting system of algebraic equations by using computerized symbolic packages. We next collect all terms with the same power in $\sin^{k}(\mu\xi)$ or $\cos^{k}(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknown's α , μ and β, and solve the subsequent system.

Applications: The sine-cosine method is generalized on four PDEs.

3.1. Combined Kdv-Mkdv equation:

Consider the following combined Kdv-Mkdvequation[13],

$u_t - 2a_1 u u_x - 3a_2 u^2 u_x + u_{xxx} = 0$(10)

Where a_1 and a_2 are non-zero constant, and by letting $u(x,t) = u(\xi)$where,

$u(\xi) = \alpha \sin^{\beta}(\mu\xi)$(11)

Where α, μ, and β are parameters that must be determined and by applying the wave variable$\xi = kx + \lambda t$, equation (10) turns to be the following ordinary differential equation:

$\lambda u' - a_1 k(u^2)' - a_2 k(u^3)' + k^3 u''' = 0$ (12)

By integrating equation (12) once with zero constant, we have

$\lambda u - a_1 k u^2 - a_2 k u^3 + k^3 u'' = 0$(13)

Applying sine-cosine method, equation (13) becomes,

$$\lambda\alpha \sin^{\beta}(\mu\xi) - a_1 k\alpha^2 \sin^{2\beta}(\mu\xi) - a_2 k\alpha^3 \sin^{3\beta}(\mu\xi) + k^3\alpha\beta\mu^2(\beta-1)\sin^{\beta-2}(\mu\xi) - k^3\alpha\beta^2\mu^2\sin^{\beta}(\mu\xi) = 0$$

(14)

Equating the identical powers and the coefficients of each pair of the sine function, leads us to the following algebraic system:

$$3\beta = \beta - 2$$

$$\lambda\alpha - k^3\alpha\beta^2\mu^2 = 0$$

$$-a_1 k\alpha^2 = 0$$

$-a_2 k\alpha^3 + k^3\alpha\beta\mu^2(\beta - 1) = 0$(15)

Solving the algebraic system of (15), we obtain

$\beta = -1$, $\alpha = \pm\left(\frac{\sqrt{2}k\mu}{\sqrt{c}}\right)$ and $\mu = \pm\frac{1}{k}\sqrt{\frac{\lambda}{k}}$

Then by substituting the values ofβ, α and μin equation (14), the solution of equation (10) will beobtained and written in the form:

$u(x,t) = \pm\left(\sqrt{\frac{2\lambda}{ck}}\right)\csc\left(\frac{1}{k}\sqrt{\frac{\lambda}{k}}\,(kx + \lambda t)\right)$(16)

3.2. Sharma-Tasso-Olver equation:

Consider the following Sharma-Tasso-Olverequation[14],

$u_t + 3au_x^2 + 3au^2u_x + 3auu_{xx} + au_{xxx} = 0$ (17)

Where a is a non-zero constant, and by letting $u(x,t) = u(\xi)$ where,

$u(\xi) = \alpha\sin^{\beta}(\mu\xi)$ (18)

Where α, μ, and β are parameters that must be determined and by applying the wave variable $\xi = kx + \lambda t$, equation (17) turns to be the following ordinary differential equation:

$\lambda u' + 3ak^2(uu')' + ak(u^3)' + ak^3u''' = 0$ (19)

By integrating equation (19) once with zero constant, we have

$\lambda u + 3ak^2uu' + aku^3 + ak^3u'' = 0$(20)

Applying sine-cosine method, equation (20) becomes,

$\lambda\alpha\sin^{\beta}(\mu\xi) + 3ak^2\alpha^2\mu\beta\sin^{\beta-1}(\mu\xi)\cos(\mu\xi) + ak\alpha^3\sin^{3\beta}(\mu\xi) + ak^3\alpha\beta\mu^2(\beta - 1)\sin^{\beta-2}(\mu\xi) - ak^3\alpha\beta^2\mu^2\sin^{\beta}(\mu\xi) = 0$(21)

Equating the identical powers and the coefficients of each pair of the sine function, leads us to the following algebraic system:

$$3\beta = \beta - 2$$

$$\lambda\alpha - ak^3\alpha\beta^2\mu^2 = 0$$

$$3ak^2\alpha^2\mu\beta = 0$$

$ak\alpha^3 + ak^3\alpha\beta\mu^2(\beta - 1) = 0$(22)

Solving the algebraic system of (22), we obtain

$\beta = -1,\ \ \alpha = \pm\sqrt{2}\mu k\, i$ and $\mu = \pm\frac{1}{k}\sqrt{\frac{\lambda}{ak}}$

Then by substituting the values of β, α and μ in equation (22), the solution of equation (17) will be obtained and written in the form:

$u(x,t) = \ \pm i\sqrt{\frac{2\lambda}{ak}}\ \csc\left(\frac{1}{k}\sqrt{\frac{\lambda}{ak}}(kx+\lambda t)\right)$ (23)

3.3. Coupled Klien-Gordon equation:
Consider the following Coupled Klien-Gordon equation[15],

$u_{tt} - u_x + a_1 u + a_2 u^3 + a_3 vu = 0$ (24)

$v_t - v_x + 4u_t = 0$ (25)

Where a_1,a_2 and a_3 are non-zero constant, and by letting $u(x,t) = u(\xi)$ where,

$u(\xi) = \alpha\sin^{\beta}(\mu\xi)$(26)

Where α, μ, and β are parameters that must be determined and by applying the wave variable $\xi = kx + \lambda t$, equations (24) and (25) turn to be the following ordinary differential equations:

$\lambda^2 u'' - ku' + a_1 u + a_2 u^3 + a_3 vu = 0$ (27)

$\lambda v' - kv' + 4\lambda u' = 0$ (28)

By integrating equation (28) once with zero constant, we have

$(\lambda - k)v + 4\lambda u = 0$(29)

Where,

$v = \frac{-4\lambda u}{\lambda - k}$(30)

Therefore, equation (27) will become:

$\lambda^2 u'' - ku' + a_1 u + a_2 u^3 - a_3\frac{4\lambda u^2}{\lambda - k} = 0$ (31)

Applying sine-cosine method, equation (31) becomes,

$\lambda^2\alpha\beta\mu^2(\beta-1)\sin^{\beta-2}(\mu\xi) - \lambda^2\alpha\beta^2\mu^2\sin^{\beta}(\mu\xi) - k\alpha\mu\beta\sin^{\beta-1}(\mu\xi)\cos(\mu\xi) + a_1\alpha\sin^{\beta}(\mu\xi) + a_2\alpha^3\sin^{3\beta}(\mu\xi) - a_3\frac{4\lambda}{\lambda-k}\alpha^2\sin^{2\beta}(\mu\xi) = 0$(32)

Equating the identical powers and the coefficients of each pair of the sine function, leads us to the following algebraic system:

$$3\beta = \beta - 2$$

$$-\lambda^2\mu^2\alpha\beta^2 + a_1\alpha = 0$$

$$-k\alpha\mu\beta = 0$$

$$-a_3\frac{4\lambda}{\lambda - k}\alpha^2 = 0$$

$\lambda^2\alpha\beta\mu^2(\beta-1)+a_2\alpha^3=0$ (33)

Solving the algebraic system of (33), we obtain

$\beta=-1,\ \alpha=\pm\sqrt{\frac{2}{a_2}}\lambda\mu$ i and $\mu=\pm\sqrt{\frac{a_1}{\lambda}}$

Then by substituting the values of β,α and μ in equation (32), the solution of equation (31) will be obtained and written in the form:

$u(x,t)=\pm i\sqrt{\frac{2a_1}{a_2}}\csc\left(\sqrt{\frac{a_1}{\lambda}}(kx+\lambda t)\right)$ (34)

3.4 Generalized-Pochhammer-Chree equation (GPC):
Consider the following Generalized-Pochhammer-Chreeequation[16],

$u_{tt}-u_{ttxx}-(a_1u+a_2u^{n+1}+a_3u^{2n+1})_{xx}=0$ (35)

Where a_1, a_2 and a_3 are non-zero constant, and by letting $u(x,t)=u(\xi)$ where,

$u(\xi)=\alpha\cos^{\beta}(\mu\xi)$(36)

Where α, μ, and β are parameters that must be determined and by applying the wave variable $\xi=kx+\lambda t$, equation (35) turns to be the following ordinary differential equation:

$\lambda^2u''-\lambda^2k^2u''''-k^2(a_1u+a_2u^{n+1}+a_3u^{2n+1})''=0$ (37)

By integrating equation (37) twice with zero constant, we have

$\lambda^2u-\lambda^2k^2u''-k^2(a_1u+a_2u^{n+1}+a_3u^{2n+1})=0$ (38)

Applying sine-cosine method, equation (38) becomes,

$\lambda^2\alpha\cos^{\beta}(\mu\xi)-\lambda^2k^2\left(\alpha\beta\mu^2(\beta-1)\cos^{\beta-2}(\mu\xi)-\alpha\beta^2\mu^2\cos^{\beta}(\mu\xi)\right)-k^2\left(a_1\alpha\cos^{\beta}(\mu\xi)+a_2\alpha^{n+1}\cos^{\beta(n+1)}(\mu\xi)+a_3\alpha^{2n+1}\cos^{\beta(2n+1)}(\mu\xi)\right)=0$ (39)

Equating the identical powers and the coefficients of each pair of the cosine function, leads us to the following algebraic system:

$$\beta(2n+1)=\beta-2$$

$$\lambda^2\alpha+\lambda^2k^2\alpha\beta^2\mu^2-a_1k^2\alpha=0$$

$$-a_2k^2\alpha^{n+1}=0$$

$-\lambda^2k^2\alpha\beta\mu^2(\beta-1)-a_3k^2\alpha^{2n+1}=0$ (40)

Solving the algebraic system of (40), we obtain

$\beta=\frac{-1}{n},\ \alpha=\mp i\left(\frac{\lambda\mu}{n}\right)^{\frac{1}{n}}\left(\frac{1+n}{a_3}\right)^{\frac{1}{2n}}$ and $\mu=\pm n\frac{\sqrt{a_1k^2-\lambda^2}}{\lambda k}$

Then by substituting the values of β,α and μ in equation (39), the solution of equation (37) will be obtained and written in the form:

$u(x,t)=\pm i\left(\frac{\sqrt{a_1k^2-\lambda^2}}{k}\right)^{\frac{1}{n}}\left(\frac{1+n}{a_3}\right)^{\frac{1}{2n}}\sec^{\frac{1}{n}}\left(n\frac{\sqrt{a_1k^2-\lambda^2}}{\lambda k}(kx+\lambda t)\right)$ (41)

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