SOLUTION OF TWO-DIMENSIONAL FREDHOLM INTEGRAL EQUATIONS OF THE FIRST KIND BY USING OPTIMAL HOMOTOPY ASYMPTITIC METHOD

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ABSTRACT

The purpose of this study is to present the optimal homotopy asymptotic method (OHAM) for solving two-dimensional linear and nonlinear Fredholm integral equations of the first kind. Several examples are illustrated to show that the method is effective and simple to apply for solving two-dimensional linear and nonlinear Fredholm integral equations of the first kind.

Keywords: *Optimal homotopy asymptotic method, Two-dimensional Fredholm integral equations of the first kind.*

1. INTRODUCTION

In recent years, several methods have been used to approximate the solution of one and two- dimensional linear and nonlinear integral equations of the first kind by researchers in mathematics, physics and engineering. Goswami et al. [1] presented wavelets on a bounded interval for solving integral equations of the first kind. An algorithm to handle the ill-posed linear and non-linear Fredholm integral equations of the first kind was proposed by Molabahrami [2]. Tari and Shahmorad [3] used operational Tau method to approximate the solution of two-dimensional linear Volterra integral equations of the first kind. An adaptive multiscale moment method for solving two-dimensional Fredholm integral equation of the first kind was developed by Su and Sarkar [4]. McKee et al. [5] presented Euler method for solving two-dimensional Volterra integral equations of the first kind. Su and Sarkar [6] used multiscale moment method to solve Fredholm integral equation of the first kind whilst Bazrafshan et al. [7] used homotopy analysis method to solve two-dimensional integral equations.

The optimal homotopy asymptotic method (OHAM) has been used by Almousa and Ismail [8] to solve linear Fredholm integral equations of the first kind and they reported that OHAM was effective and simple in solving these equations.

The purpose of this study is to present the OHAM, which was introduced by Marinca and Herişanu [9-12], for solving two-dimensional linear and nonlinear integral equations of the first kind. Consider the general form of two-dimensional Fredholm integral equation of the first kind is

$$f(x,y) = \int_a^b \int_c^d k(x,y,s,t)(g(s,t))^n ds dt, \quad n \in N \tag{1}$$

where a, b, c and d are constants, $k(x,y,s,t)$ is the given kernel and $f(x,y)$ is a known function. When $n = 1$ we have the linear case and when $n \geq 2$ we have the nonlinear case.

2. DESCRIPTION OF THE METHOD TO TWO-DIMENSIONAL FREDHOLM LINEAR AND NONLINEAR INTEGRAL EQUATIONS OF THE FIRST KIND

To explain the optimal homotopy asymptotic method for solving two-dimensional linear and nonlinear integral equations of the first kind, we shall follow the approach of [9-12] and other papers. Let us rewrite Eq. (1) in the following form:

$$f(x,y) - \int_a^b \int_c^d k(x,y,s,t)(g(s,t))^n ds dt = 0, \quad n \in N \tag{2}$$

To derive the solutions by using the OHAM, consider a family of equations for an embedding parameter $p \in [0,1]$ as follows:

$$L\big(g(x,y,p)\big) = g(x,y)$$

$$N\big(g(x,y,p)\big) = -\int_a^b \int_c^d k(x,y,s,t)(g(s,t))^n ds dt, \ n \in N$$

$$g(x,y,p): \Omega \times [0,1] \to R$$

which satisfies

$$(1-p)\big[L\big(g(x,y,p)\big)+f(x,y)\big] = H(p)\big[L\big(g(x,y,p)\big)+f(x,y)+N\big(g(x,y,p)\big)\big], \tag{3}$$

where the non-zero auxiliary function $H(p)=\sum_{j=1}^{m} c_j p^j$ where c_j , $j=1,2,\ldots$ are auxiliary constants. When $p=0$ and $p=1$, it holds that

$$g(x,y,0)=g_0(x,y), \qquad g(x,y,1)=g(x,y) \tag{4}$$

respectively. Expand the approximate solution in Taylor's series about p as follows:

$$g(x,y,p,c_j)=g_0(x,y)+\sum_{m=1}^{\infty} g_m(x,y,c_j)p^m, j=1,2,\ldots \tag{5}$$

If Eq. (5) is convergent at $p=1$, then we obtain:

$$g(x,y,1,c_j)=g_0(x,y)+\sum_{m=1}^{\infty} g_m(x,y,c_j), j=1,2,\ldots. \tag{6}$$

By replacing Eq. (6) into Eq. (3), we obtain the zeroth order problem, the first order problem and the *m*th -order problem as follows:

$$O(p^0): g_0(x,y)=-f(x,y). \tag{7}$$

$$O(p^1): g_1(x,y)=-c_1\int_a^b\int_c^d k(x,y,s,t)(g_0(s,t))^n dsdt, \quad n\in N. \tag{8}$$

$$O(p^m): g_m(x,y)=g_{m-1}(x,y)+\sum_{i=1}^{m-1} c_i\, g_{m-i}(x,y)+\sum_{i=1}^{m-1} c_i\, N_{m-i}\big(g_0(x,y),g_1(x,y),\ldots,g_{m-1}(x,y)\big)$$
$$-c_m\int_a^b\int_c^d k(x,y,s,t)\,\big(g_0(s,t)\big)^n, m=2,3,\ldots \tag{9}$$

where $N_{m-i}\,(g_0(x,y),g_1(x,y),\ldots,g_{m-1}(x,y))$ are the coefficient of p^{m-i} in the expansion of $N(g(x,p))$ about p:

$$N(g(x,y,p,c_j))=N_0\big(g_0(x,y)\big)+\sum_{m=1}^{\infty} N_m\big(g_0(x,y),g_1(x,y),\ldots,g_m(x,y)\big)p^m, j=1,2,\ldots. \tag{10}$$

For calculating the constants $c_1,c_2,c_3,\ldots$, consider the result of the *m*th-order approximations as follows:

$$g^m(x,y,c_j)=f(x,y)+\sum_{k=1}^{m} g_k(x,y,c_j), j=1,2,\ldots,m. \tag{11}$$

Using Eq. (11) into Eq. (1), we can obtain the residual for $j=1,2,\ldots$

$$\mathcal{R}(x,y,c_j)=g_0(x,y)-\int_a^b\int_c^d k(x,y,s,t)(g^m(s,t))^n dsdt. \tag{12}$$

If $\mathcal{R}(x,y,c_j)=0$, then $g^m(x,y,c_j)$ represent the exact solution. The least squares can be used to determine $c_1,c_2,c_3,\ldots$. At first, we consider the functional

$$J(c_j)=\int_a^b\int_c^d \mathcal{R}^2(x,y,c_j)dxdy, \tag{13}$$

and by using Galerkin's method, we have

$$\frac{dJ}{dc_m}=\int_a^b\int_c^d \mathcal{R}(x,y,c_j)\frac{\partial\mathcal{R}}{\partial c_m}dxdy, \tag{14}$$

and then minimizing it to obtain the values of $c_1,c_2,c_3,\ldots,c_m$, we have

$$\frac{dJ}{dc_1} = \frac{dJ}{dc_2} = \cdots = \frac{dJ}{dc_m} = 0. \quad (15)$$

3. NUMERICAL EXAMPLE AND DISCUSSION

In this section, two examples of two-dimensional Fredholm integral equations of the first kind were solved to show effectiveness of the OHAM as an approximate analytical solution method.

Example 1. Consider the two-dimensional linear Fredholm integral equation of the first kind with the exact solution $g(x,y) = \frac{2e^{x+y}}{e^2-1}$. [2]

$$\frac{1}{2}(e^2-1)e^{x+y} = \int_0^1\int_0^1 e^{x+y+s+t} g(s,t)\,ds\,dt, \quad (16)$$

To derive the solutions by using the OHAM, let

$$L\big(g(x,y,p)\big) = g(x,y) \quad (17)$$

$$N\big(g(x,y,p)\big) = -\int_0^1\int_0^1 e^{x+y+s+t} g(s,t)\,ds\,dt \quad (18)$$

$$f(x,y) = \left(\frac{1}{2}(e^2-1)e^{x+y}\right) \quad (19)$$

which satisfies

$$(1-p)[(g_0(x,y) + pg_1(x,y) + p^2g_2(x,y) + \cdots) + (\tfrac{1}{2}(e^2-1)e^{x+y})] = (pc_1 + p^2c_2 + p^3c_3 + \cdots)[(g_0(x,y) + pg_1(x,y) + p^2g_2(x,y) + \cdots) + \left(\tfrac{1}{2}(e^2-1)e^{x+y}\right) - \int_0^1\int_0^1 e^{x+y+s+t}(g_0(s,t) + pg_1(s,t) + p^2g_2(s,t) + \cdots)\,ds\,dt]. \quad (20)$$

By using Eqs. (7- 9), we obtain a series of problems

$$O(P^0): g_0(x,y) = -\left(\tfrac{1}{2}(e^2-1)e^{x+y}\right). \quad (21)$$

$$O(p^1): g_1(x,y) = -c_1\int_0^1\int_0^1 e^{x+y+s+t} g_0(s,t)\,ds\,dt. \quad (22)$$

$$O(p^2): g_2(x,y) = (1+c_1)g_1(x,y) - c_1\int_0^1\int_0^1 e^{x+y+s+t} g_1(s,t)\,ds\,dt - c_2\int_0^1\int_0^1 e^{x+y+s+t} g_0(s,t)\,ds\,dt. \quad (23)$$

Hence, the solutions are

$$O(P^0): g_0(x,y) = -\left(\tfrac{1}{2}(e^2-1)e^{x+y}\right).$$

$$O(p^1): g_1(x,y) = -c_1(\tfrac{-3}{8}e^{x+y+2} + \tfrac{1}{8}e^{x+y} + \tfrac{3}{8}e^{x+y+4} - \tfrac{1}{8}e^{x+y+6}).$$

$$O(p^2): g_2(x,y) = -(1+c_1)c_1\left(\tfrac{-3}{8}e^{x+y+2} + \tfrac{1}{8}e^{x+y} + \tfrac{3}{8}e^{x+y+4} - \tfrac{1}{8}e^{x+y+6}\right) - c_1(\tfrac{5}{32}c_1e^{x+y+2} - \tfrac{1}{32}c_1e^{x+y} - \tfrac{5}{16}c_1e^{x+y+4} + \tfrac{5}{16}c_1e^{x+y+6} - \tfrac{5}{32}c_1e^{x+y+8} + \tfrac{1}{32}c_1e^{x+y+10}) - c_2\left(\tfrac{-3}{8}e^{x+y+2} + \tfrac{1}{8}e^{x+y} + \tfrac{3}{8}e^{x+y+4} - \tfrac{1}{8}e^{x+y+6}\right).$$

Adding the equations $g_0(x,y)$, $g_1(x,y)$ and $g_2(x,y)$, we obtain

$$g(x,y) = -\left(\tfrac{1}{2}(e^2-1)e^{x+y}\right) - c_1\left(\tfrac{-3}{8}e^{x+y+2} + \tfrac{1}{8}e^{x+y} + \tfrac{3}{8}e^{x+y+4} - \tfrac{1}{8}e^{x+y+6}\right) - (1+c_1)c_1\left(\tfrac{-3}{8}e^{x+y+2} + \tfrac{1}{8}e^{x+y} + \tfrac{3}{8}e^{x+y+4} - \tfrac{1}{8}e^{x+y+6}\right) - c_1(\tfrac{5}{32}c_1e^{x+y+2} - \tfrac{1}{32}c_1e^{x+y} - \tfrac{5}{16}c_1e^{x+y+4} + \tfrac{5}{16}c_1e^{x+y+6} - \tfrac{5}{32}c_1e^{x+y+8} + \tfrac{1}{32}c_1e^{x+y+10}) - c_2\left(\tfrac{-3}{8}e^{x+y+2} + \tfrac{1}{8}e^{x+y} + \tfrac{3}{8}e^{x+y+4} - \tfrac{1}{8}e^{x+y+6}\right). \quad (24)$$

By using Eqs. (11– 15), we can calculate the constants c_1 and c_2, i.e.

$$c_1 = 0.1075933438, \quad c_2 = -0.001033138488.$$

Now, we substitute the constants c_1 and c_2 in Eq. (24), so the solution is given by:

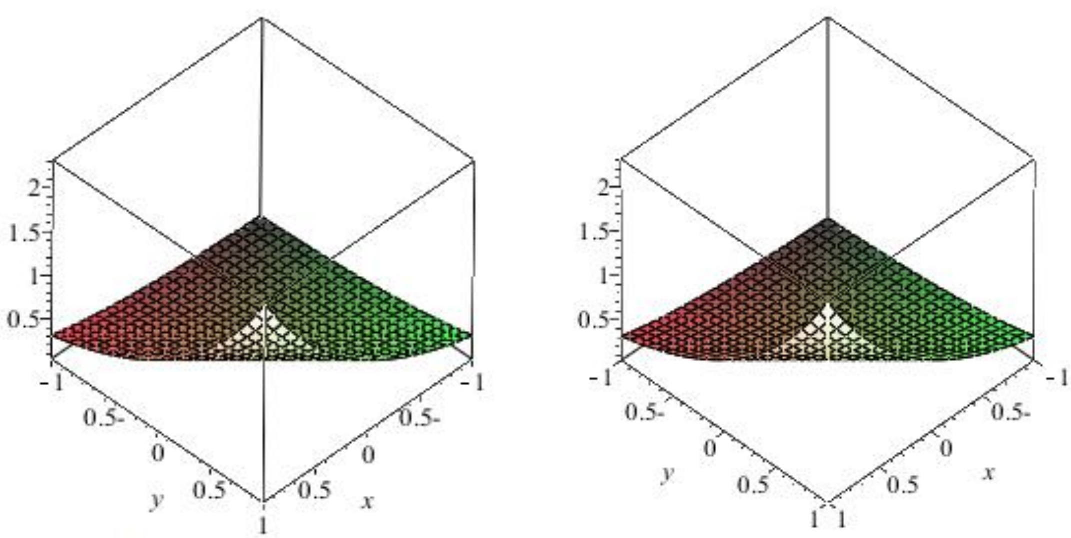
$$g(x, y) = 0.313035296\, e^{x+y}. \tag{25}$$

In the table 1, we show some numerical results of these solutions calculated according to the present method.

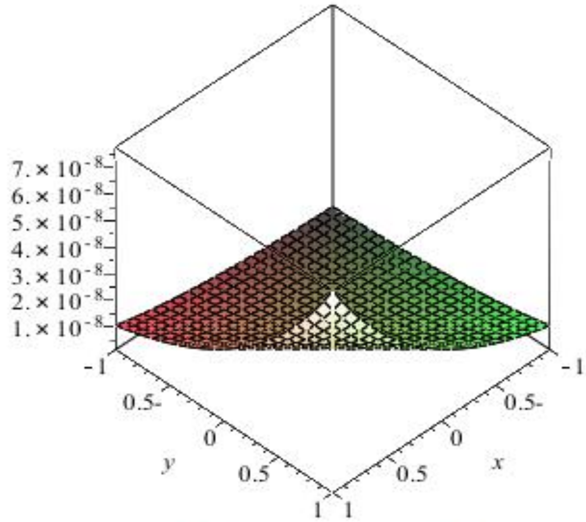
Table 1: Numerical results of Example 1.

(x,y)	g_{exact}	g_{OHAM}	$\lvert g_{exact} - g_{OHAM} \rvert$
(0,0)	0.3130352854	0.313035296	1.06×10^{-8}
(0.1,0.1)	03823426109	0.3823421739	1.30×10^{-8}
(0.2,0.2)	0.4669937701	0.4669937859	1.58×10^{-8}
(0.3,0.3)	0.5703874786	0.5703874979	1.93×10^{-8}
(0.4,0.4)	0.6966728396	0.6966728632	2.36×10^{-8}
(0.5,0.5)	0.8509181278	0.8509181566	2.88×10^{-8}
(0.6,0.6)	1.039313749	1.039313784	3.5×10^{-8}
(0.7,0.7)	1.269420679	1.269420722	4.3×10^{-8}
(0.8,0.8)	1.550473918	1.550473971	5.3×10^{-8}
(0.9,0.9)	1.893753121	1.893753185	6.4×10^{-8}
(1.0,1.0)	2.313035285	2.313035363	7.7×10^{-8}

The exact solution, OHAM solution and absolute error of the Example 1 are shown in Figure 1. This example shows the efficiency of the method for two-dimensional linear Fredholm integral equation of the first kind.



(a) Exact solution for example 1 (b) OHAM solution for example 1



(c) Absolute error for example 1

Figure 1: (a) Exact solution, (b) OHAM solution and (c) Absolute error for Example 1.

Example 2. Consider the two-dimensional nonlinear Fredholm integral equation of the first kind with the exact solution $g(x,y) = \frac{0.684178548x}{1+y}$. [2]

$$\frac{x}{6(1+y)} = \int_0^1\int_0^1 \frac{x(1+s+t)}{1+y}(g(s,t))^2 ds dt, \quad (26)$$

To derive the solutions by using the OHAM, let

$$L\big(g(x,y,p)\big) = g(x,y) \quad (27)$$

$$N\big(g(x,y,p)\big) = -\int_0^1\int_0^1 \frac{x(1+s+t)}{1+y}(g(s,t))^2 ds dt \quad (28)$$

$$f(x,y) = \left(\frac{x}{6(1+y)}\right) \quad (29)$$

which satisfies

$$(1-p)[(g_0(x,y) + pg_1(x,y) + p^2 g_2(x,y) + \cdots) + (\frac{x}{6(1+y)})] = (pc_1 + p^2 c_2\, p^3 c_3 + \cdots)[(g_0(x,y) + pg_1(x,y)$$
$$+p^2 g_2(x,y) + \cdots) + \left(\frac{x}{6(1+y)}\right) - \int_0^1\int_0^1 \frac{x(1+s+t)}{1+y}(g_0(s,t) + pg_1(s,t) + p^2 g_2(s,t) + \cdots)^2 ds dt]. \quad (30)$$

By using Eqs. (7-9), we obtain a series of problems

$$O(P^0): g_0(x,y) = -\left(\frac{x}{6(1+y)}\right). \quad (31)$$

$$O(p^1): g_1(x,y) = -c_1\int_0^1\int_0^1 \frac{x(1+s+t)}{1+y}(g_0(s,t))^2 ds dt. \quad (32)$$

$$O(p^2): g_2(x,y) = (1+c_1)g_1(x,y) - 2c_1\int_0^1\int_0^1 \frac{x(1+s+t)}{1+y} g_0(s,t) g_1(s,t) ds dt - c_2\int_0^1\int_0^1 \frac{x(1+s+t)}{1+y}(g_0(s,t))^2 ds dt. \quad (33)$$

Hence, the solutions are

$$O(P^0): g_0(x,y) = -\left(\frac{x}{6(1+y)}\right).$$

$$O(p^1): g_1(x,y) = -\frac{1}{864} c_1 \frac{x(3+8\ln(2))}{1+y}.$$

$$O(p^2): g_2(x,y) = -\frac{1}{864}(1+c_1)c_1\left(\frac{x(3+8\ln(2))}{1+y}\right) - \frac{1}{62208} {c_1}^2 \left(\frac{x(9+48\ln(2)+64(\ln(2))^2)}{1+y}\right) - \frac{1}{864} c_2 x \left(\frac{x(3+8\ln(2))}{1+y}\right).$$

Adding the equations $g_0(x,y)$, $g_1(x,y)$ and $g_2(x,y)$, we obtain

$$g(x,y) = -\left(\frac{x}{6(1+y)}\right) - \frac{1}{864}c_1\frac{x(3+8\ln(2))}{1+y} - \frac{1}{864}(1+c_1)c_1\left(\frac{x(3+8\ln(2))}{1+y}\right) - \frac{1}{62208}{c_1}^2\left(\frac{x(9+48\ln(2)+64(\ln(2))^2)}{1+y}\right) - \frac{1}{864}c_2 x \left(\frac{x(3+8\ln(2))}{1+y}\right). \quad (34)$$

By using Eqs. (11– 15), we can calculate the constants c_1 and c_2, i.e.

$$c_1 = -16.85161027, \quad c_2 = -370.0054405.$$

Now, we substitute the constants c_1 and c_2 in Eq. (34), so the solution is given by:

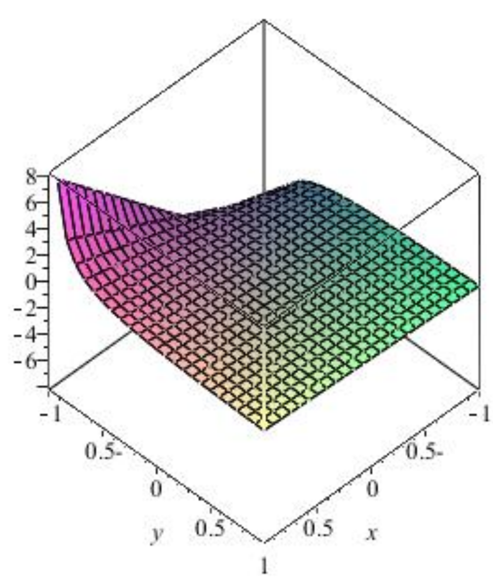
$$g(x,y) = \frac{0.6841785490x}{1+y}. \quad (35)$$

In the table 2, we show some numerical results of these solutions calculated according to the present method.

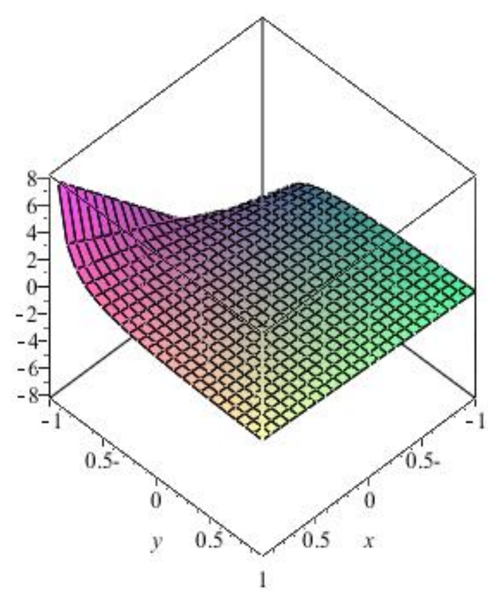
Table 2: Numerical results of Example 2.

(x,y)	g_{exact}	g_{OHAM}	$\lvert g_{exact} - g_{OHAM}\rvert$
(0,0)	0	0	0
(0.1,0.1)	0.06219804982	0.06219804991	9×10^{-11}
(0.2,0.2)	0.1140297580	0.1140297582	2×10^{-10}
(0.3,0.3)	0.1578873572	0.1578873575	3×10^{-10}
(0.4,0.4)	0.1954795851	0.1954795854	3×10^{-10}
(0.5,0.5)	0.2280595160	0.2280595163	3×10^{-10}
(0.6,0.6)	0.2565669555	0.2565669559	4×10^{-10}
(0.7,0.7)	0.2817205786	0.2817205790	4×10^{-10}
(0.8,0.8)	0.3040793547	0.3040793551	4×10^{-10}
(0.9,0.9)	0.3240845754	0.3240845758	4×10^{-10}
(1,1)	0.3420892740	0.3420892745	5×10^{-10}

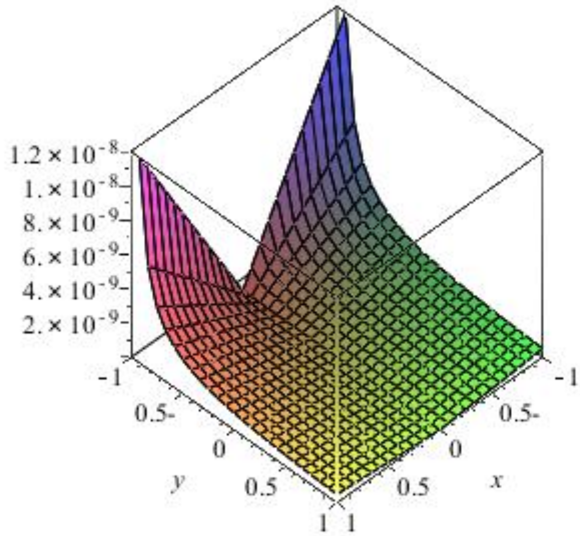
The exact solution, OHAM solution and absolute error of the Example 2 are shown in Figure 2. This example shows the efficiency of the method for two-dimensional nonlinear Fredholm integral equation of the first kind.



(a) Exact solution for example 2



(b) OHAM solution for example 2



(c) Absolute error for Example 2

Figure 2: (a) Exact solution, (b) OHAM solution and (c) Absolute error for Example 2.

4. CONCLUSIONS

In this paper, we have described the application of the OHAM for solving two-dimensional linear and nonlinear Fredholm integral equations of the first kind. The results indicated that the method is feasible, effective and simple for solving two-dimensional linear and nonlinear Fredholm integral equations of the first kind. Maple software with long format and double accuracy was used to carry out the computations.

5. ACKNOWLEDGEMENTS

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