ON JENSEN'S AND HERMITE-HADAMARD'S INEQUALITY

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ABSTRACT
The article deals with the generalizations of Jensen's inequality in the discrete and integral form. One generalization of Hermite-Hadamard's inequality is also presented. The transition from discrete to integral means is realized using the integral method with convex combinations.

Keywords: *26A51, 26D15, 28A25, 52A40 convex combination, affine combination, convex function, Jensen's inequality, Hermite-Hadamard's inequality*.

1. INTRODUCTION
The integral inequalities have a special place in the branch of mathematical inequalities. With an extensive measure theory these inequalities can be widely used in mathematics and physics. Besides, using discrete measures integral inequalities can be easily presented in the discrete form. The reverse procedure that starts from the discrete case and produces the integral, can also be implemented by applying the integral method with convex combinations. The latter is more difficult.

Recall the two well-known and widely used integral inequalities for a real valued convex function defined on a bounded closed interval of real numbers.

Every convex function $f:[a,b]\to\mathbb{R}$ verifies the inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{\int_a^b f(x)dx}{b-a}\le\frac{f(a)+f(b)}{2} \tag{1}$$

known as Hermite-Hadamard's inequality, and the inequality

$$f\left(\frac{\int_a^b xdx}{b-a}\right)\le\frac{\int_a^b f(x)dx}{b-a} \tag{2}$$

known as the integral form of Jensen's inequality.

Hermite-Hadamard's inequality refers to the estimation of the integral arithmetic mean of a convex function. In 1883, Hermite has discovered the double inequality in (1) which was published in [3]. Ten years later, Hadamard has rediscovered the left-hand side of this inequality which was presented in [2]. More details on this dramatic story, see [7] or [8, pages 62-63].

In the discrete case Jensen's inequality is applied to convex combinations of vectors from a convex set. In the integral case Jensen's inequality is widely applied to barycenters and integral means. In 1906, Jensen has proved the version of the inequality in (2) using Lebesgue measure, which was published in [4].

2. INEQUALITIES WITH AFFINE COMBINATIONS
This section is prepared according to Theorem A which is one of the main results of the submitted paper [10].

Convex sets are generally observed in a real vector space $\mathcal{X}$. Affiliation to some vector set is analytically expressed by combinations of vectors (points) $x_i\in\mathcal{X}$ and scalars (coefficients) $\alpha_i\in\mathbb{R}$. The sum

$$\sum_{i=1}^{n}\alpha_i x_i \tag{3}$$

belongs to the vector subspace $\text{lin}\{x_i\}$ (the smallest vector space that contains all x_i), and it is called the linear combination. If $\sum_{i=1}^{n}\alpha_i = 1$, the sum in (3) belongs to the affine hull $\text{aff}\{x_i\}$ (the smallest translated vector space that contains all x_i), and it is called the affine combination. If $\sum_{i=1}^{n}\alpha_i = 1$ and all $\alpha_i \in [0,1]$, the sum in (3) belongs to the convex hull $\text{co}\{x_i\}$ (the smallest convex vector set that contains all x_i), and it is called the convex combination.

In what follows, we use a real interval $[a,b]$ assuming $a < b$. Every $x \in \mathbb{R}$ can be uniquely presented as the affine combination

$$x = \alpha_x a + \beta_x b \tag{4}$$

where

$$\alpha_x = \frac{b-x}{b-a}, \beta_x = \frac{x-a}{b-a}. \tag{5}$$

Let $f:\mathbb{R}\to\mathbb{R}$ be a convex function, and $f_{\{a,b\}}^{\text{cho}}$ be the chord line passing through the points $A(a, f(a))$ and $B(b, f(b))$ of the graph of f. If $x \in [a,b]$, then the above combination is convex, and we have the chord inequality

$$f(x) \le \alpha_x f(a) + \beta_x f(b) = f_{\{a,b\}}^{\text{cho}}(x). \tag{6}$$

If $x \notin [a,b]$, then the reverse inequality is valid in (6).

Simplicity of the chord line $f_{\{a,b\}}^{\text{cho}}(x)$ as the affine function $h(x) = kx + l$ will be very useful to our work. For this reason, here is the following lemma:

Lemma 2.1 *Let* $x_i \in \mathbb{R}$ *be points. Let* $\alpha_i \in \mathbb{R}$ *be coefficients of the sum* $\sum_{i=1}^{n}\alpha_i = 1$*. Then every affine function* $h:\mathrm{R}\to\mathrm{R}$ *verifies the equality*

$$h\left(\sum_{i=1}^{n}\alpha_i x_i\right) = \sum_{i=1}^{n}\alpha_i h(x_i). \tag{7}$$

We will examine the behavior of convex functions on some special types of affine combinations.

Lemma 2.2 *Let* $c \in [a,b]$ *be a point. Let* $\alpha, \beta \in [0,1]$, $\gamma \in [-1,1]$ *be coefficients of the sum* $\alpha + \beta + \gamma = 1$*. Then every convex function* $f:[a,b]\to\mathbb{R}$ *verifies the inequality*

$$f(\alpha a + \beta b + \gamma c) \le \alpha f(a) + \beta f(b) + \gamma f(c). \tag{8}$$

Proof. First of all let us show that the affine combination $\alpha a + \beta b + \gamma c$ belongs to $[a,b]$. Since $c \in [a,b]$, it has to be $c = \alpha_c a + \beta_c b$ for coefficients α_c and β_c taken from the formulas in (5). Then we have

$$\begin{aligned} \alpha a + \beta b + \gamma c &= \alpha a + \beta b + (1-\alpha-\beta)(\alpha_c a + \beta_c b) \\ &= [\alpha(1-\alpha_c) + (1-\beta)\alpha_c]a + [\beta(1-\beta_c) + (1-\alpha)\beta_c]b. \end{aligned} \tag{9}$$

The coefficients in square brackets are non-negative with the sum equals 1, so the observed expression $\alpha a + \beta b + \gamma c$ belongs to $[a,b]$.

If $\gamma \ge 0$, the inequality in (8) is the Jensen inequality for the three-membered convex combination

$\alpha a+\beta b+\gamma c$.

If $\gamma \le 0$, we use the inequality in (6) and the affinity of the chord line $f^{\text{cho}}_{\{a,b\}}$, in this way:

$$\begin{aligned} f(\alpha a+\beta b+\gamma c) &\le f^{cho}_{\{a,b\}}(\alpha a+\beta b+\gamma c) \\ &= \alpha f^{cho}_{\{a,b\}}(a)+\beta f^{cho}_{\{a,b\}}(b)+\gamma f^{cho}_{\{a,b\}}(c) \\ &\le \alpha f(a)+\beta f(b)+\gamma f(c) \end{aligned}$$

respecting that $f^{\text{cho}}_{[a,b]}(a)= f(a)$, $f^{\text{cho}}_{\{a,b\}}(b)= f(b)$ and $\gamma f^{cho}_{\{a,b\}}(c)\le \gamma f(c)$.

Sufficient conditions on the coefficients in Lemma 2.2 are: $\alpha,\beta\in[0,1]$ and $\alpha+\beta+\gamma=1$. From these conditions follows $\gamma\in[-1,1]$.

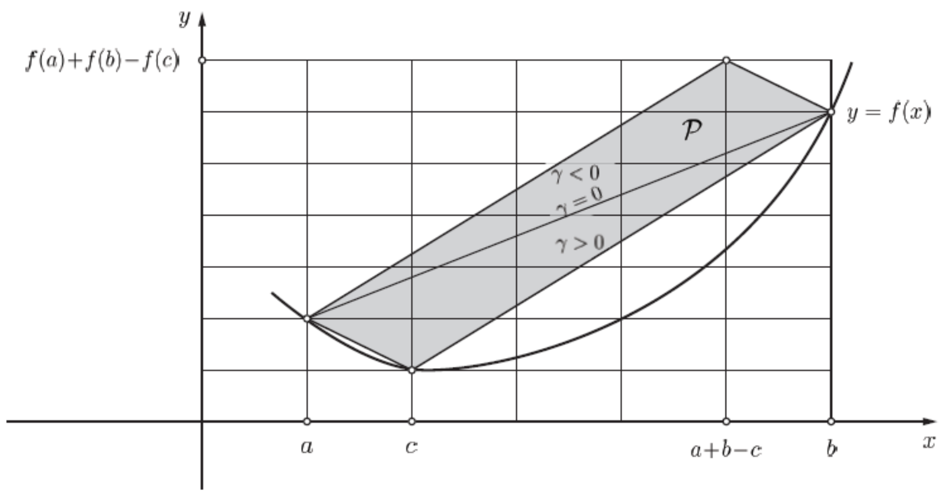

Figure 1: Geometric interpretation of the inequality in (8)

The analytic inequality written in the formula in (8) can be described by a geometric figure. Given $c\in[a,b]$, take the graph points $A(a, f(a))$, $B(b, f(b))$ and $C(c, f(c))$, and determine the position of the points

$$P(\alpha a+\beta b+\gamma c,\alpha f(a)+\beta f(b)+\gamma f(c)) \tag{10}$$

for $\alpha,\beta\in[0,1]$ and $\alpha+\beta+\gamma=1$. We have the radius-vectors equality

$$\vec r_P=\alpha\vec r_A+\beta\vec r_B+\gamma\vec r_C=(\alpha+\gamma)\vec r_A+(\beta+\gamma)\vec r_B-\gamma(\vec r_A+\vec r_B-\vec r_C). \tag{11}$$

If $\gamma\ge 0$, the left-hand side of the equality in (11) represents the convex combinations of the vectors $\vec r_A$, $\vec r_B$ and $\vec r_C$. Interpreted geometrically, the points P belong to the triangle $\text{co}\{A,B,C\}$.

If $\gamma\le 0$, the coefficients $\alpha+\gamma,\beta+\gamma,-\gamma\in[0,1]$ with the sum equal to 1. Take the point $D(a+b-c, f(a)+f(b)-f(c))$. The right-hand side of the equality in (11) represents the convex combinations of the vectors $\vec r_A$, $\vec r_B$ and $\vec r_D$. This means that the geometric location of the points P is just the triangle $\text{co}\{A,B,D\}$.

So, given a function f, point $c\in[a,b]$ and coefficients α,β,γ of the sum $\alpha+\beta+\gamma=1$, the points of the formula in (10) belong to the convex quadrangle

$$\mathcal{P}=co\{A(a, f(a)),C(c, f(c)),D(a+b-c, f(a)+f(b)-f(c)),B(b, f(b))\},$$

illustrated in Figure 1.

Theorem A *Let* $x_i \in [a,b]$ *be points. Let* $\alpha, \beta, p_i \in [0,1]$, $\gamma \in [-1,1]$ *be coefficients of the sums* $\alpha + \beta + \gamma = \sum_{i=1}^{n} p_i = 1$. *Then every convex function* $f : [a,b] \to \mathbb{R}$ *verifies the inequality*

$$f\left(\alpha a + \beta b + \gamma \sum_{i=1}^{n} p_i x_i\right) \le \alpha f(a) + \beta f(b) + \gamma \sum_{i=1}^{n} p_i f(x_i). \tag{12}$$

For $\alpha = \beta = 0$ and $\gamma = 1$, the inequality in (12) is reduced to the Jensen inequality. For $\alpha = \beta = 1$ and $\gamma = -1$, the inequality in (12) is reduced to the Jensen-Mercer inequality, see [5].

Corollary 2.3 *Let* $x_i \in [a,b]$ *be points. Let* $\alpha, \beta \in [0,1]$, $\gamma_i \le 0$ *be coefficients of the sum* $\alpha + \beta + \sum_{i=1}^{n} \gamma_i = 1$. *Then every convex function* $f : [a,b] \to \mathrm{R}$ *verifies the inequality*

$$f\left(\alpha a + \beta b + \sum_{i=1}^{n} \gamma_i x_i\right) \le \alpha f(a) + \beta f(b) + \sum_{i=1}^{n} \gamma_i f(x_i). \tag{13}$$

Proof. Put $\gamma = \sum_{i=1}^{n} \gamma_i$, and suppose $\gamma < 0$. Then the affine combination,

$$\alpha a + \beta b + \sum_{i=1}^{n} \gamma_i x_i = \alpha a + \beta b + \gamma \sum_{i=1}^{n} \frac{\gamma_i}{\gamma} x_i ,$$

coincides with that of Theorem A because $p_i = \gamma_i / \gamma \ge 0$. The inequality in (13) now follows after applying Theorem A.

Remark 2.4 *Let* $\sum_{i=1}^{n} \alpha_i x_i$ *be an affine combination with the smallest point* x_1 *and the largest point* x_n *, and noting that* $\sum_{i=1}^{n} \alpha_i = 1$. *Suppose that*

$$\sum_{i=1}^{n} \alpha_i x_i = \alpha_1 x_1 + \alpha_n x_n + \beta \sum_{i=2}^{k} \frac{\alpha_i}{\beta} x_i + \gamma \sum_{i=k+1}^{n-1} \frac{\alpha_i}{\gamma} x_i \tag{14}$$

where $\alpha_1, \alpha_n, \alpha_2, \ldots, \alpha_k \in [0,1]$ and $\alpha_{k+1}, \ldots, \alpha_{n-1} \in [-1,0]$ with $\beta = \sum_{i=2}^{k} \alpha_i > 0$ and $\gamma = \sum_{i=k+1}^{n-1} \alpha_i < 0$

If $\alpha_1 + \beta \le 1$ and $\alpha_n + \beta \le 1$, then the observed affine combination belongs to the interval $\mathrm{co}\{x_1, x_n\}$.

It is true because the convex combinations $\sum_{i=2}^{k} (\alpha_i / \beta) x_i$ and $\sum_{i=k+1}^{n-1} (\alpha_i / \gamma) x_i$ ($\alpha_i / \beta \ge 0$ and $\alpha_i / \gamma \ge 0$) belong to $\mathrm{co}\{x_1, x_n\}$. Putting

$$\sum_{i=2}^{k} (\alpha_i / \beta) x_i = \beta_1 x_1 + \beta_n x_n \, , \, \sum_{i=k+1}^{n-1} (\alpha_i / \gamma) x_i = c$$

we have

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i x_i &= \alpha_1 x_1 + \alpha_n x_n + \beta(\beta_1 x_1 + \beta_n x_n) + \gamma c \\ &= (\alpha_1 + \beta\beta_1) x_1 + (\alpha_n + \beta\beta_n) x_n + \gamma c \\ &= \bar{\alpha} x_1 + \bar{\beta} x_n + \gamma c, \end{aligned}$$

that is, the affine combination with the coefficients $\bar{\alpha} = \alpha_1 + \beta\beta_1 \le \alpha_1 + \beta \le 1$, $\bar{\beta} = \alpha_n + \beta\beta_n \le \alpha_n + \beta \le 1$ and γ . The combination $\bar{\alpha} x_1 + \bar{\beta} x_n + \gamma c \in \mathrm{co}\{x_1, x_n\}$ by Lemma 2.2, and the same is true for the combination in

(14).

3. TRANSITION FROM CONVEX COMBINATIONS TO INTEGRALS

The natural way of transition from discrete to integral means is one that involves convex combinations in the integral method. Integral analogy of the concept of convex combinations is the concept of barycenter.

Let us show how the convex combinations can be implemented into the integral method. Let μ be a measure (positive measure according to [11]) on a set $\mathcal{S} \subseteq \mathbb{R}$ with $\mu(\mathcal{S}) > 0$. Given a positive integer n, let $\mathcal{S} = \cup_{i=1}^{n} \mathcal{S}_{ni}$ be a partition of pairwise disjoint μ-measurable sets $\mathcal{S}_{ni}$, and $x_{ni} \in \mathcal{S}_{ni}$ be points. If the sequence $(s_n)_n$ of the convex combination centers

$$s_n = \sum_{i=1}^{n} \frac{\mu(\mathcal{S}_{ni})}{\mu(\mathcal{S})} x_{ni}$$

converges, we get the set barycenter as the limit $\lim_{n\to\infty} s_n$. So, the μ-barycenter of $\mathcal{S}$ can be defined with

$$B(\mathcal{S},\mu) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\mu(\mathcal{S}_{ni})}{\mu(\mathcal{S})} x_{ni} = \frac{\int_{\mathcal{S}} x\,d\mu(x)}{\mu(\mathcal{S})}. \tag{15}$$

If $p : \mathcal{S} \to \mathbb{R}$ is either non-negative or non-positive function with $\int_{\mathcal{S}} p(x)\,d\mu(x) \neq 0$, the μ-barycenter of p on $\mathcal{S}$ can be defined with

$$\begin{aligned} B(p,\mathcal{S},\mu) &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\mu(\mathcal{S}_{ni})\,p(x_{ni})}{\sum_{i=1}^{n}\mu(\mathcal{S}_{ni})\,p(x_{ni})} x_{ni} \\ &= \lim_{n\to\infty} \frac{\sum_{i=1}^{n}\mu(\mathcal{S}_{ni})\,p(x_{ni})x_{ni}}{\sum_{i=1}^{n}\mu(\mathcal{S}_{ni})\,p(x_{ni})} \\ &= \frac{\int_{\mathcal{S}} p(x)x\,d\mu(x)}{\int_{\mathcal{S}} p(x)\,d\mu(x)} \end{aligned} \tag{16}$$

provided that the limit exists. Note that the above integral sum is the n-membered convex combination of the points $x_{ni} \in \mathcal{S}$ with non-negative coefficients

$$p_{ni} = \frac{\mu(\mathcal{S}_{ni})\,p(x_{ni})}{\sum_{i=1}^{n}\mu(\mathcal{S}_{ni})\,p(x_{ni})}.$$

If $\mathcal{S}$ is convex, in our case the interval, then $B(\mathcal{S},\mu)$ and $B(p,\mathcal{S},\mu)$ belong to $\mathcal{S}$. In this way it is possible to make the transition from discrete form of Jensen's inequality to the integral form

$$f\left(\frac{\int_{\mathcal{S}} p(x)x\,d\mu(x)}{\int_{\mathcal{S}} p(x)\,d\mu(x)}\right) \leq \frac{\int_{\mathcal{S}} p(x)f(x)\,d\mu(x)}{\int_{\mathcal{S}} p(x)\,d\mu(x)}. \tag{17}$$

If $g : \mathcal{S} \to \mathbb{R}$ is a μ-integrable function, the μ-arithmetic mean of g on $\mathcal{S}$ is defined with

$$M(g,\mathcal{S},\mu) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\mu(\mathcal{S}_{ni})}{\mu(\mathcal{S})} g(x_{ni}) = \frac{\int_{\mathcal{S}} g(x)\,d\mu(x)}{\mu(\mathcal{S})} \tag{18}$$

provided that the limit exists. If g is the continuous function on the interval $\mathcal{S}$, its μ-arithmetic mean belongs to

$g(\mathcal{S})$.

4. APPLICATIONS

The main result in this section is Theorem 4.5 which extends the Hermite-Hadamard inequality to affine combinations from the interval.

The discrete inequality in (12) can be applied to known integral inequalities mentioned in Section 1, and to integral quasi-arithmetic means.

4.1 Application to Jensen's Inequality

Applying the inequality in (12) with the convex combinations of the definition in (16), we have the following result:

Corollary 4.1 *Let μ be a measure on the interval $[a,b]$. Let $p:[a,b]\to\mathbb{R}$ be either non-negative or non-positive μ-integrable function with $\int_a^b p(x)d\mu(x)\neq 0$. Let $\alpha,\beta\in[0,1]$, $\gamma\in[-1,1]$ be coefficients of the sum $\alpha+\beta+\gamma=1$. Then the inequality*

$$f\left(\alpha a+\beta b+\gamma\frac{\int_a^b p(x)x\,d\mu(x)}{\int_a^b p(x)d\mu(x)}\right)\leq \alpha f(a)+\beta f(b)+\gamma\frac{\int_a^b p(x)f(x)d\mu(x)}{\int_a^b p(x)d\mu(x)} \tag{19}$$

holds for every convex function $f:[a,b]\to\mathbb{R}$ provided that the functions $p(x)x$ and $p(x)f(x)$ are μ-integrable.

The inequality in (17) is still valid if we substitute the identity function with a function $g:[a,b]\to\mathbb{R}$. Then the inequality in (19) can be extended using the function g that has extreme values:

Corollary 4.2 *Let μ be a measure on the interval $[a,b]$. Let $p:[a,b]\to\mathbb{R}$ be either non-negative or non-positive μ-integrable function with $\int_a^b p(x)d\mu(x)\neq 0$. Let $g:[a,b]\to\mathbb{R}$ be a μ-measurable function with extreme values $g(a)$ and $g(b)$, so $g(x)\in\mathrm{co}\{g(a),g(b)\}=\mathcal{I}$ for every $x\in[a,b]$. Then the inequality*

$$\begin{aligned}&f\left(\alpha g(a)+\beta g(b)+\gamma\frac{\int_a^b p(x)g(x)d\mu(x)}{\int_a^b p(x)d\mu(x)}\right)\\&\leq \alpha f(g(a))+\beta f(g(b))+\gamma\frac{\int_a^b p(x)f(g(x))d\mu(x)}{\int_a^b p(x)d\mu(x)}\end{aligned} \tag{20}$$

holds for every convex function $f:\mathcal{I}\to\mathbb{R}$ provided that the functions $p(x)g(x)$ and $p(x)f(g(x))$ are μ-integrable.

4.2 Application to Hermite-Hadamard's Inequality

Lemma 4.3 *Let μ be a measure on the interval $[a,b]$ with $\mu([a,b])>0$. If coefficients α_1 and β_1 of the sum $\alpha_1+\beta_1=1$ satisfy*

$$\alpha_1 a+\beta_1 b=\frac{\int_a^b x\,d\mu(x)}{\mu([a,b])}, \tag{21}$$

then the double inequality

$$f(\alpha_1 a+\beta_1 b)\le \frac{\int_a^b f(x)d\mu(x)}{\mu([a,b])}\le \alpha_1 f(a)+\beta_1 f(b) \tag{22}$$

holds for every convex function $f:[a,b]\to\mathbb{R}$.

Proof. The left-hand side of the inequality in (22) follows from Jensen's inequality:

$$f(\alpha_1 a+\beta_1 b) = f\left(\frac{\int_a^b x d\mu(x)}{\mu([a,b])}\right)\le \frac{\int_a^b f(x)d\mu(x)}{\mu([a,b])}.$$

The right-hand side of the inequality in (22) follows from the chord inequality in (6): if $f^{\text{cho}}_{\{a,b\}}(x)=kx+l$, then

$$\frac{\int_a^b f(x)d\mu(x)}{\mu([a,b])}\le k\frac{\int_a^b x d\mu(x)}{\mu([a,b])}+l = f^{cho}_{\{a,b\}}(\alpha_1 a+\beta_1 b)=\alpha_1 f(a)+\beta_1 f(b)$$

because $f^{\text{cho}}_{\{a,b\}}(a)=f(a)$ and $f^{\text{cho}}_{\{a,b\}}(b)=f(b)$.

The inequality in (22) was stated for convex continuous functions and real Borel measures in [1], and also for continuous measures in [9, Corollary 3.8].

Remark 4.4 *The coefficients* α_1 *and* β_1 *of the inequality in (22) are unique and can be calculated by the formulas in (5). If* $c=\int_a^b x d\mu(x)/\mu([a,b])$*, the alternative presentation of the inequality in (22) reads as follows:*

$$f\left(\frac{b-c}{b-a}a+\frac{c-a}{b-a}b\right)\le \frac{\int_a^b f(x)d\mu(x)}{\mu([a,b])}\le \frac{b-c}{b-a}f(a)+\frac{c-a}{b-a}f(b). \tag{23}$$

The inequality in (22) can be generalized by application of the formula in (8):

Theorem 4.5 *Let* μ *be a measure on the interval* $[a,b]$ *with* $\mu([a,b])>0$. *Let* $\alpha,\beta\in[0,1]$, $\gamma\in[-1,1]$ *be coefficients of the sum* $\alpha+\beta+\gamma=1$. *Let* $f:[a,b]\to\mathbb{R}$ *be a convex function.*
If the equality in (21) is valid, then the double inequality

$$\begin{aligned} &f\big((\alpha+\gamma\alpha_1)a+(\beta+\gamma\beta_1)b\big)\\ &\le \alpha f(a)+\beta f(b)+\gamma\frac{\int_a^b f(x)d\mu(x)}{\mu([a,b])}\\ &\le (\alpha+\gamma\alpha_1)f(a)+(\beta+\gamma\beta_1)f(b)\end{aligned} \tag{24}$$

holds for $\gamma\ge 0$, and the double inequality

$$\begin{aligned} &f\big((\alpha+\gamma\alpha_1)a+(\beta+\gamma\beta_1)b\big)\\ &\le (\alpha+\gamma\alpha_1)f(a)+(\beta+\gamma\beta_1)f(b)\\ &\le \alpha f(a)+\beta f(b)+\gamma\frac{\int_a^b f(x)d\mu(x)}{\mu([a,b])}\end{aligned} \tag{25}$$

holds for $\gamma\le 0$.

Proof. Putting $c=\alpha_1 a+\beta_1 b=\int_a^b x d\mu(x)/\mu([a,b])$ we have the representation

$$\alpha a+\beta b+\gamma c=(\alpha+\gamma\alpha_1)a+(\beta+\gamma\beta_1)b.$$

Suppose $\gamma \geq 0$. Combining the application of the formula in (8) with the above equality, it follows

$$f\big((\alpha+\gamma\alpha_1)a+(\beta+\gamma\beta_1)b\big)\leq \alpha f(a)+\beta f(b)+\gamma f\left(\frac{\int_a^b x d\mu(x)}{\mu([a,b])}\right).$$

Applying the integral form of Jenesn's inequality on the last member of the above inequality, and then the right-hand side of the inequality in (22), we get

$$\gamma f\left(\frac{\int_a^b x d\mu(x)}{\mu([a,b])}\right)\leq \gamma\frac{\int_a^b f(x)d\mu(x)}{\mu([a,b])}\leq \gamma\big(\alpha_1 f(a)+\beta_1 f(b)\big),$$

after which arises the inequality in (24).

Suppose $\gamma \leq 0$. Using the discrete form of Jensen's inequality, it follows

$$\begin{aligned} f\big((\alpha+\gamma\alpha_1)a+(\beta+\gamma\beta_1)b\big) &\leq (\alpha+\gamma\alpha_1)f(a)+(\beta+\gamma\beta_1)f(b) \\ &= \alpha f(a)+\beta f(b)+\gamma\big(\alpha_1 f(a)+\beta_1 f(b)\big). \end{aligned}$$

After using the right-hand side of the inequality in (22) multiplied by γ, we obtain

$$\gamma\big(\alpha_1 f(a)+\beta_1 f(b)\big)\leq \gamma\frac{\int_a^b f(x)d\mu(x)}{\mu([a,b])}$$

ensuring the inequality in (25).

4.3 APPLICATION TO QUASI-ARITHMETIC MEANS

Let μ be a measure on a set $\mathcal{S}\subseteq\mathbb{R}$ with $\mu(\mathcal{S})>0$, and $\varphi:\mathcal{S}\to\mathbb{R}$ be a strictly monotone continuous function that is μ-integrable on $\mathcal{S}$. The φ-quasi-barycenter of the set $\mathcal{S}$ with respect to the measure μ can be defined as the point

$$M_\varphi(\mathcal{S},\mu)=\varphi^{-1}\left(\frac{\int_{\mathcal{S}}\varphi(x)d\mu(x)}{\mu(\mathcal{S})}\right). \tag{26}$$

If $\mathcal{S}$ is an interval, then $M_\varphi(\mathcal{S},\mu)$ belongs to $\mathcal{S}$ because $\int_{\mathcal{S}}\varphi(x)d\mu(x)/\mu(\mathcal{S})$ belongs to $\varphi(\mathcal{S})$. The formula in (26) can also be applied to the φ-quasi-barycenter definition of non-negative or non-positive μ-integrable function $p:\mathcal{S}\to\mathbb{R}$ with $\int_{\mathcal{S}}p(x)d\mu(x)\neq 0$,

$$M_\varphi(p(x),\mathcal{S},\mu)=\varphi^{-1}\left(\frac{\int_{\mathcal{S}}p(x)\varphi(x)d\mu(x)}{\int_{\mathcal{S}}p(x)d\mu(x)}\right), \tag{27}$$

assuming μ-integrability of the function $p(x)\varphi(x)$. Applying the idea of â€‹â€‹the previous formula we can define

$$M_\varphi\big(p(x),[a,b],\mu;\alpha,\beta,\gamma\big)=\varphi^{-1}\left(\alpha\varphi(a)+\beta\varphi(b)+\gamma\frac{\int_a^b p(x)\varphi(x)d\mu(x)}{\int_a^b p(x)d\mu(x)}\right), \tag{28}$$

and including Corollary 4.2, get:

Corollary 4.6 *Let* μ *be a measure on the interval* $[a,b]$. *Let* $p:[a,b]\to\mathbb{R}$ *be either non-negative or non-positive* μ*-integrable function with* $\int_a^b p(x)d\mu(x)\neq 0$. *Let* $\alpha,\beta\in[0,1]$, $\gamma\in[-1,1]$ *be coefficients of the*

sum $\alpha+\beta+\gamma=1$. *Let* $\varphi,\psi:[a,b]\to\mathbb{R}$ *be strictly monotone continuous functions.*
If ψ is either φ-convex and increasing or φ-concave and decreasing, then the inequality

$$M_{\varphi}(p(x),[a,b],\mu;\alpha,\beta,\gamma)\le M_{\psi}(p(x),[a,b],\mu;\alpha,\beta,\gamma) \tag{29}$$

holds provided that the functions $p(x)\varphi(x)$ and $p(x)\psi(x))$ are μ-integrable.
If ψ is either φ-convex and decreasing or φ-concave and increasing, then the reverse inequality is valid in (29).
Proof. Let us prove the case when ψ is φ-convex and increasing. Using the inequality in (20) with the convex function $f=\psi\circ\varphi^{-1}$ and the monotone function $g=\varphi$, in which case $f(g(x))=\psi(x)$, we get

$$\psi\circ\varphi^{-1}\left(\alpha\varphi(a)+\beta\varphi(b)+\gamma\frac{\int_a^b p(x)\varphi(x)d\mu(x)}{\int_a^b p(x)d\mu(x)}\right)$$
$$\le\alpha\psi(a)+\beta\psi(b)+\gamma\frac{\int_a^b p(x)\psi(x)d\mu(x)}{\int_a^b p(x)d\mu(x)}.$$

The inequality in (29) follows after applying the increasing function ψ^{-1} on the above inequality.

More on general forms and refinements of the quasi-arithmetic means can be found in [6].

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