CLASSIFICATION OF POSITIVE SOLUTIONS OF NONLINEAR SYSTEMS OF VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT
We give asymptotic classification of the positive solutions of a class of two-dimensional nonlinear Volterra integro-differential equations. Also, we furnish necessary and conditions for the existence of such positive solutions.

Keywords: *Classifications, Positive, Volterra, Integro-differential, Finite delay, Knaster's fixed point theorem.*

1. INTRODUCTION

In this paper we consider the two-dimensional nonlinear Volterra integro-differential equations

$$\begin{cases} x'(t) = h(t)x(t) + \int_{t-r}^{t} a(t,s)f\big(y(s)\big)ds, \\ y'(t) = p(t)y(t) + \int_{t-r}^{t} b(t,s)g\big(x(s)\big)ds, \end{cases} \tag{1.1}$$

$$where\ t_0 \leq t-r \leq s \leq t \leq \infty.$$

The functions $a(t,s) and\ b(t,s)$ are positive and continuous functions for $t_0 \leq t-r \leq s \leq t \leq \infty$. The functions $f\ and\ g$ are real-valued continuous in $y\ and\ x$, respectively, increasing on the real line R. Also, the coefficients $h\ and\ p$ are positive and continuous functions for $t \geq 0$ and satisfy the condition

$$\int_{-r}^{\infty} h(t)dt < \infty \text{ and } \int_{-r}^{\infty} p(t)dt < \infty . \tag{1.2}$$

Throughout this paper we assume that the functions $f\ and\ g$ are well behaved so that the solutions of system (1.1) exist on the interval $[0, \infty)$. Also, we assume that

$$f(x) > 0 \quad , \quad g(x) > 0 \quad for\ x \neq 0.$$

Definition 1.1. A pair of real-valued functions (x, y) is said to be
(1) a solution of system (1.1) if it satisfies (1.1) for $t \geq t_0$;
(2) Eventually positive if both x(t) and y(t) are eventually positive;
(3) nonoscillatory if both x(t) and y(t) are either eventually positive or eventually negative.

Definition 1.2. Let $(\varphi, \psi): [-r, 0] \to R$ be two given bounded continuous initial functions. We say $(x, y) = \big(x(t,\varphi), y(t,\psi)\big)$ is a solution of (1.1) if $(x(t), y(t)) = (\varphi(t_0), \psi(t_0))$ for $t_0 \in [-r, 0]\ and\ (x(t), y(t))$ satisfies (1.1) for $t \geq t_0$.

There are many papers written on the subject of oscillatory and nonoscillatory behavior of solutions in differential equations. For such topics we refer the interested reader to [2],[3],[7]-[18] and the reference therein. Recently, Li [12] and Li and Cheng [18] studied the class of two-dimensional nonlinear differential systems of the form

$$\begin{cases} x'(t) = a(t)f\big(y(t)\big), \\ y'(t) = b(t)g\big(x(t)\big). \end{cases}$$

Under similar assumptions: they provided a classification scheme for positive solutions of the above system. They also provided conditions for the existence of solutions with designated asymptotic properties. However, no study has been devoted to systems of Volterra integro-differential equations. In [19] the authors used the notion of Lyapunov functional and obtain results concerning the boundedness of solutions of Volterra integro-differential equations with unbounded perturbation.

Systems of the form of (1.1) are used for continuous risk models where the risk process is a variation / extension of the classical compound Poisson process, see for example Dickson and dos Reis (1997) [3] and Stadtmuller (1998) [6].

In this paper, we classify positive solutions of (1.1) according to their limiting behaviors and then provide sufficient and / or necessary conditions for their existence.

To simplify our notation we let

$$A(t) = \int_{t_0}^{t} \left(\int_{u-r}^{u} a(u,s)ds \right) \; du \tag{1.3}$$

And

$$B(t) = \int_{t_0}^{t} \left(\int_{u-r}^{u} b(u,s)ds \right) \; du \tag{1.4}$$

Let

$$A(\infty) = \lim_{t\to\infty} A(t) \text{ and } B(\infty) = \lim_{t\to\infty} B(t) \, .$$

Note that if $a(t,s) = e^{-t-s}$, then

$$A(t) = \int_{t_0}^{t} \left(\int_{u-r}^{u} e^{-u-s} ds \right) du$$

$$= \int_{t_0}^{t} e^{-u} \left(-e^{u} + e^{-u+r} \right) du$$

$$= \int_{t_0}^{t} \left(-e^{-2u} + e^{-2u+r} \right) du$$

$$= \int_{t_0}^{t} e^{-2u} \left(-1 + e^{r} \right) du$$

$$= \frac{1}{2}(1 - e^{r})(e^{-2t} - e^{-2t_0})$$

Since $(e^{-2t} - e^{-2t_0}) \leq 1$, for $t \geq t_0$, we have

$A(\infty) \leq \frac{1}{2}(e^{r} - 1) \, , r \geq 0.$

We will discuss each of the following cases:

(i) $A(\infty) = \infty, and\ B(\infty) = \infty;$
(ii) $A(\infty) = \infty, and\ B(\infty) < \infty;$
(iii) $A(\infty) < \infty, and\ B(\infty) = \infty;$
(iv) $A(\infty) < \infty, and\ B(\infty) < \infty.$

Let C be the set of all continuous functions and define
$\Omega = \{(x, y) \in C([0, \infty), R) \times C([0, \infty), R): x, y \textit{ are eventually positive}\}$. For $t_0 \in [-r, 0]$, we let

$x(t_0) = \varphi \textit{ and } y(t_0) = \psi$. Consider

$$x'(t) = h(t)x(t) + \int_{t-r}^{t} a(t, s)f\big(y(s)\big)ds \quad (1.1)$$

And let $k(t) = \int_{t-r}^{t} a(t, s)f\big(y(s)\big)ds$. Consider the nonlinear 1st order differential equation,

$$x'(t) = h(t)x(t) + k(t).$$

By multiplying both sides of the above equation by $e^{-\int_{t_0}^{t} h(s)ds}$. We get;

$$\frac{d}{dt}\left(x(t)e^{-\int_{t_0}^{t} h(s)ds}\right) = e^{-\int_{t_0}^{t} h(s)ds}k(t). \qquad (*)$$

Then integrating (*) from $u = t_0 \textit{ to } u = t \textit{ yield};$

$$\int_{t_0}^{t} \frac{d}{du}\left(x(u)e^{-\int_{t_0}^{u} h(s)ds}\right) dt = \int_{t_0}^{t} e^{-\int_{t_0}^{u} h(s)ds}k(u)\, du,$$

Or $$x(t)e^{-\int_{t_0}^{t} h(s)ds} - x(t_0) = \int_{t_0}^{t} e^{-\int_{t_0}^{u} h(s)ds}k(u)\, du,$$

$$x(t)e^{-\int_{t_0}^{t} h(s)ds} = x(t_0) + \int_{t_0}^{t} e^{-\int_{t_0}^{u} h(s)ds}k(u)\, du.$$

Hence, $$\mathrm{x}(t) = x(t_0)e^{\int_{t_0}^{t} h(s)ds} + \int_{t_0}^{t} e^{-\int_{t_0}^{u} h(s)ds} + \int_{t_0}^{t} e^{\int_{t_0}^{t} h(s)ds}k(u)\, du,$$

$$x(t) = x(t_0)e^{\int_{t_0}^{t} h(s)ds} + \int_{t_0}^{t} k(u)e^{\int_{u}^{t} h(s)ds}\, du.$$

Since, $k(u) = \int_{u-r}^{u} a(u, s)f\big(y(s)\big)ds$, we arrive at the variation of parameters formulas

$$x(t) = \varphi(t_0)e^{\int_{t_0}^{t} h(s)ds} + \int_{t_0}^{t}\left(\int_{u-r}^{u} a(u, s)\, f(y(s))ds\right) e^{\int_{u}^{t} h(s)ds}\, du. \qquad (1.5)$$

And by similar steps we get

$$y(t) = \psi(t_0)e^{\int_{t_0}^{t} p(s)ds} + \int_{t_0}^{t}\left(\int_{u-r}^{u} b(u, s)\, g(x(s))ds\right) e^{\int_{u}^{t} p(s)ds}\, du. \qquad (1.6)$$

It is clear from (1.5) and (1.6) that x and y are positive provided that $\varphi(t_0), \psi(t_0) \geq 0$. Moreover, since $h, p > 0$, then from (1.1) we have $x', y' > 0$. Now, for some positive constants $\alpha \textit{ and } \beta$, we define the set

$$K(\alpha, \beta) = \{(x, y) \in \Omega, \lim_{t\to\infty} x(t) = \alpha\, , \lim_{t\to\infty} y(t) = \beta\}.$$

Note that $\alpha \textit{ and } \beta$ maybe considered infinite.

2. CLASSIFICATION OF SOLUTIONS

In this section, we should classify positive solutions of (1.1) according to their limiting behaviors and then provide necessary and sufficient conditions for their existence in the cases (ii), (iii) and (iv). Our results are based on the application of Knaster's fixed point theorem, which we state below.

Knaster's Fixed Point Theorem Let X be a partially ordered Banach space with ordering $\leq$. Let M be a subset of X with the following properties: The infimum of M belongs to M and every nonempty subset of M has a supremum which belongs to M. let $T\colon M \to M$ be an increasing mapping, $i.e.$, $x \leq y$ implies $Tx \leq Ty$. Then T has a fixed point in M .

Theorem 2.1. Any solution $(x, y) \in \Omega$ of $(1{,}1)$ belongs to one of the following subsets

$$K(\alpha,\beta), K(\alpha,\infty), K(\infty,\beta) \text{ and } K(\infty,\infty).$$

Proof. Since $(x, y) \in \Omega$, we have $x', y' > 0$ for $t \geq t_0$. Thus x and y are increasing. Hence,

$$\lim_{t\to\infty} x(t) = \alpha > 0 \ or \ \lim_{t\to\infty} x(t) = \infty,$$

And

$$\lim_{t\to\infty} y(t) = \beta > 0 \ or \ \lim_{t\to\infty} y(t) = \infty,$$

In the following we state four theorems. Each theorem is related to one of the above mentioned cases.

Theorem 2.2. Suppose (1.2), $A\,(\infty) = \infty$ and $B(\infty) = \infty$ $hold$. Then any solution $(x, y) \in \Omega$ of (1.1) belongs to the set $K\,(\infty, \infty)$.

Proof. Let $(x, y) \in \Omega$ be a solution of system (1.1). Then $x'(t), y'(t) > 0$ and $\varphi(t_0), \psi(t_0) \geq 0$ for $t \geq t_0$. Thus x and y are increasing. As a consequence of this and the fact that f is increasing an integration of (1.1) yields

$$x(t) = \varphi(t_0) + \int_{t_0}^{t} h(s)x(s)ds + \int_{t_0}^{t}\left(\int_{u-r}^{u} a(u,s)\, f\big(y(s)\big)ds\right) du$$

$$\geq f(\psi(t_0))\int_{t_0}^{t}\left(\int_{u-r}^{u} a(u,s)\, ds\right) du \to \infty \quad for\ t\to\infty$$

And

$$y(t) = \psi(t_0) + \int_{t_0}^{t} h(s)y(s)ds + \int_{t_0}^{t}\left(\int_{u-r}^{u} b(u,s)\, g\big(x(s)\big)ds\right) du$$

$$\geq g(\varphi(t_0))\int_{t_0}^{t}\left(\int_{u-r}^{u} a(u,s)\, ds\right) du \to \infty \ \ for\ t\to\infty$$

Showing that $x(t) \to \infty$ and $y(t) \to \infty, as\ t \to \infty$.

Theorem 2.3. Suppose (1.2), $A(\infty) = \infty$ and $B(\infty) < \infty$ $hold$. Then there exists a solution $(x, y) \in \Omega$ of (1.1) that belongs to the set $K(\infty, \beta)$ if and only if

$$\lim_{t\to\infty}\int_{t_0}^{t}\left[\int_{u-r}^{u} b(u,s)g\left(\int_{s-r}^{s}\left(e^{\int_v^s h(\tau)d\tau}\int_{v-r}^{v} a(v,k)f(c)dk\right)dv\right)ds\right]du \ < \infty, \quad (2.1)$$

for some positive constant c.

proof. Let $(x, y) \in \Omega$ be a solution of system (1.1). Then $x'(t), y'(t) > 0$ and $\varphi(t_0), \psi(t_0) \geq 0$ for $t \geq t_0$. Thus $x \; and \; y$ are increasing and there exists a positive constant $\beta > 0$ such that $\psi(t_0) \leq y(t) \leq \beta$ for $t \geq t_0$. From (1.5) we have

$$x(t) = \varphi(t_0)\; e^{\int_{t_0}^{t} h(s)ds} + \int_{t_0}^{t}\left(\int_{u-r}^{u} a(u,s)\, f(y(s))ds\right) e^{\int_{u}^{t} h(s)ds}\; du$$

$$\geq \int_{t_0}^{t}\left(\int_{u-r}^{u} a(u,s)\, f(\psi(t_0))ds\right) e^{\int_{u}^{t} h(s)ds}\; du \to \infty \; as \; t \to \infty.$$

And

$$\beta \geq y(t) = \psi(t_0) \;+\; \int_{t_0}^{t} p(s)y(s)ds \;+ \int_{t_0}^{t}\left[\int_{u-r}^{u} b(u,s)g\big(x(s)\big)ds\right] du$$

$$\geq \int_{t_0}^{t}\left[\int_{u-r}^{u} b(u,s)g\left(\int_{s-r}^{s}\left(e^{\int_{v}^{s} h(\tau)d\tau}\int_{v-r}^{v} a(v,k)f(y_0)dk\right)dv\right)ds\right]du.$$

By taking the limit at infinity in the above inequality we obtain (2.1). Conversely, suppose that (2.1) holds. First notice that, for $t \geq t - r$, the second equation of (1.1) can be written as

$$y(t) = \psi(t_0) + \int_{t_0}^{t} p(s)y(s)ds + \int_{t_0}^{t}\left(\int_{u-r}^{u} b(u,s)g\big(x(s)\big)ds\right)du.$$

Next, we can choose a large enough T so that

$$\int_{T}^{\infty}\left[\int_{u-r}^{u} b(u,s)g\left(\int_{s-r}^{s}\left(e^{\int_{v}^{s} h(\tau)d\tau}\int_{v-r}^{v} a(v,k)dk\right)dv\right)ds\right]du \;\leq \frac{c}{4f(c)}\;\;,$$

And

$$\int_{T}^{\infty} p(s)ds \;\leq \frac{1}{4}.$$

Let χ be the Banach space of all bounded real-valued functions $y(t) on\; [T, \infty)$ with the norm

$$\|y\| = \sup_{t \geq T} |y(t)|$$

and with the usual pointwise ordering $\leq$. Define a subset ω of χ by

$$\omega = \left\{y(t) \in \chi : \frac{c}{2} \leq y(t) \leq c, t \geq T\right\}.$$

It is clear that for any subset B of ω, inf B $\in \omega$ and sup B $\in \omega$. Define the

operator E : $\omega \to \chi$ by

$$(Ey)(t) = \frac{c}{2} + \int_T^t p(s)y(s)ds + \int_T^t \left[\int_{u-r}^u b(u,s)g\left(\int_{s-r}^s \left(e^{\int_v^s h(\tau)d\tau} \int_{v-r}^v a(v,k)f(y(k))dk\right)dv\right)ds\right]du,\ y \in \omega.$$

We claim that E maps $\omega\ into\ \omega$. To see this we let $y \in \omega$. Then

$$c/_2\,(Ey)(t) = c/_2 + \int_T^t p(s)y(s)ds + \int_T^t \left[\int_{u-r}^u b(u,s)g\left(\int_{s-r}^s \left(e^{\int_v^s h(\tau)d\tau} \int_{v-r}^v a(v,k)f(y(k))dk\right)dv\right)ds\right]du$$

$$\le c/_2 + c\int_T^\infty p(s)ds$$

$$+f(c)\int_T^\infty \left[\int_{u-r}^u b(u,s)g\left(\int_{s-r}^s \left(e^{\int_v^s h(\tau)d\tau} \int_{v-r}^v a(v,k)dk\right)dv\right)ds\right]du$$

$$\le c/_2 + \frac{c}{4} + \frac{c}{4} = c.$$

Since E is increasing, the mapping E satisfies the hypothesis of Knaster's fixed point theorem and hence we conclude that there exists $y\ in\ \omega$ such that y=Ey.

Set

$$x(t) = \int_{t_0}^t \left(\int_{u-r}^u a(s)f(y(s))ds\right)e^{\int_u^t h(s)ds}du,$$

then

$$x'(t) = h(t)x(t) + \int_{t_0}^t a(s)f(y(s))ds,$$

and

$$x(t) = \int_{t_0}^t \left(\int_{u-r}^u a(u,s)f(y(s))ds\right)e^{\int_u^t h(s)ds}du$$

$$\ge f(\psi(t_0))\int_{t_0}^t \left(\int_{u-r}^u a(u,s)ds\right)du.$$

In view of $A(\infty) = \infty$, we have

$$\lim_{t\to\infty} x(t) = \infty.$$

On the other hand,

$$y(t) = \frac{c}{2} + \int_T^t p(s)y(s)ds + \int_T^t \left[\int_{u-r}^u b(u,s)g\left(\int_{s-r}^s \left(e^{\int_v^s h(\tau)d\tau} \int_{v-r}^v a(v,k)f(y(k))dk\right)dv\right)ds\right]du,$$

from which we obtain,

$$\lim_{t\to\infty} y(t) = \beta,$$

Where β is constant. Hence, (x, y) is a positive solution of (1.1) which belongs to $K(\infty, \beta)$.

The proof of the next theorem follows along the lines of the proof of theorem 2.3 and hence we omit.

Theorem 2.4. Suppose (1.2), $A\,(\infty) < \infty$ and B (∞) hold. Then there exists a solution

$(x, y) \in \Omega$ of (1.1) that belongs to the set K (α, ∞) if and only if

$$\lim_{t\to\infty} \int_{t_0}^{t} \left[\int_{u-r}^{u} a(u,s) f\left(\int_{s-r}^{s} \left(e^{\int_v^s p(\tau)d\tau} \int_{v-r}^{v} b(v,k) g(c) dk\right) dv\right) ds\right] du < \infty,$$

for some positive constant c.

Theorem 2.5. Suppose (1.2) hold. Then any solution $(x, y) \in \Omega$ of (1.1) that belongs to the set K (α, β) if and only if A $(\infty) < \infty$ and B $(\infty) < \infty$.

Proof. Let (x, y) be a solution in Ω with $\lim_{t\to\infty} x(t) \to \alpha > 0$ *and* $\lim_{t\to\infty} y(t) \to \beta > 0$. Then, there exists $T \geq t_0$ and two positive constants; namly, c_1 *and* c_2 such that $c_1 \leq x(t) \leq \alpha, c_2 \leq y(t) \leq \beta$ *for* $t \geq T \geq t_0$. From system (1.1) we have for $t \geq T$ that

$$x(t) = x(T) + \int_T^t h(s)x(s)ds + \int_T^t \left(\int_{u-r}^{u} a(u,s) f\big(y(s)\big) ds\right) du, \qquad (2.2)$$

and

$$y(t) = y(T) + \int_T^t p(s)y(s)ds + \int_T^t \left(\int_{u-r}^{u} b(u,s) g\big(x(s)\big) ds\right) du, \qquad (2.3)$$

Thus

$$c_1 \leq x(t) = x(T) + \int_T^t h(s)x(s)ds + \int_T^t \left(\int_{u-r}^{u} a(u,s) f\big(y(s)\big) ds\right) du,$$

$$\leq x(T) + \alpha \int_T^\infty h(s)ds + \int_T^\infty \left(\int_{u-r}^{u} a(u,s) f(\beta) ds\right) du < \infty,$$

And

$$c_2 \leq y(t) = y(T) + \int_T^t p(s)y(s)ds + \int_T^t \left(\int_{u-r}^{u} b(u,s) g\big(x(s)\big) ds\right) du,$$

$$\leq y(T) + \beta \int_T^\infty p(s)ds + \int_T^\infty \left(\int_{u-r}^{u} b(u,s) g(\alpha) ds\right) du < \infty,$$

Conversely, suppose that $A(\infty) < \infty$ *and* $B(\infty) < \infty$. First notice that for $t \geq t_0$, the first equation of (1.1) can be written as

$$x(t) = \varphi(t_0) + \int_{t_0}^{t} h(s)x(s)ds + \int_{t_0}^{t} \left(\int_{u-r}^{u} a(u,s) f\big(y(s)\big) ds\right) du.$$

In a similar fashion, we obtain from the second equation of (1.1),

$$y(t) = \psi(t_0) + \int_{t_0}^{t} p(s)y(s)ds + \int_{t_0}^{t}\left(\int_{u-r}^{u} b(u,s)g\big(x(s)\big)ds\right)du.$$

Next, we can choose a T large enough so that

$$\int_{T}^{\infty}\left[\int_{u-r}^{u} a(u,s)ds\right] du \ \le \frac{d}{4f(c)},$$

and

$$\int_{T}^{\infty} p(s)ds \ \le \ \frac{1}{4d}.$$

Let χ be the Banach space of all bounded real-valued functions (x, y) on $[T, \infty)$ with the norm

$$\|(x, y)\| = \max\ \{\sup_{t\ge T}|x(t)|,\ \sup_{t\ge T}|y(t)|\}$$

and with the usual pointwise ordering $\le$. Define a subset ω of χ by

$$\omega = \left\{(x, y) \in \chi : \frac{\mathrm{d}}{2} \le x(\mathrm{t}) \le \mathrm{d}, \frac{\mathrm{c}}{2} \le \ y(\mathrm{t}) \le \mathrm{c}, \mathrm{t} \ge \mathrm{T}\right\}.$$

It is clear that any subset B of ω, $\inf \mathrm{B} \in \omega$ and $\sup \mathrm{B} \in \omega$. Define the operator

$\mathrm{E}:\omega \to \chi$ by

$$E\begin{pmatrix} x \\ y \end{pmatrix}(\mathrm{t}) = \begin{bmatrix} \frac{d}{2} \\ \frac{c}{2} \end{bmatrix} + \begin{bmatrix} \int_T^t h(s)x(s)ds + \int_T^t\left(\int_{u-r}^{u} a(u,s)f\big(y(s)\big)ds\right)du \\ \int_T^t p(s)y(s)ds + \int_T^t\left(\int_{u-r}^{u} b(u,s)g\big(x(s)\big)ds\right)du \end{bmatrix}, \quad (x, y) \in \omega.$$

We claim that E maps ω into ω. To see this we let $\mathrm{x} \in \omega$. Then

$$d/2 \le (Ex)(t) = d/2 + \int_{T}^{t} p(s)x(s)ds + \int_{T}^{t}\left(\int_{u-r}^{u} a(u,s)f\big(y(s)\big)ds\right)du$$

$$\le d/2 + d\int_{T}^{\infty} p(s)ds + f(c)\int_{T}^{\infty}\left(\int_{u-r}^{u} a(u,s)ds\right)du$$

$$\le d/2 + \frac{d}{4} + \frac{d}{4} = d.$$

Showing that, for $y \in \omega$, $c/2 \le E\ y \le c$ is similar and hence we omit it. Since E is increasing, the mapping E satisfies the hypothesis of Knaster's fixed point theorem and hence we conclude that there exists (x, y) in ω such that $(x, y) = E(x, y)$. that is,

$$x(t) = \frac{d}{2} + \int_{T}^{t} h(s)x(s)ds + \int_{T}^{t}\left(\int_{u-r}^{u} a(u,s)f\big(y(s)\big)ds\right)du,$$

$$y(t) = \frac{c}{2} + \int_{T}^{t} p(s)y(s)ds \ + \int_{T}^{t}\left(\int_{u-r}^{u} b(u,s)g\big(x(s)\big)ds\right)du$$

From which we obtain,

$$\lim_{t\to\infty} x(t) = \alpha \; and \; \lim_{t\to\infty} y(t) = \beta,$$

Where $\alpha \; and \; \beta$ are positive constant. Hence, (x, y) is a positive solution of (1.1) which belongs to $K(\alpha, \beta)$.

Remark it is easy to extended the results of this paper to systems of more than two-dimensional volterra integro-differential equations.

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