APPROXIMATING COMMON ELEMENTS OF THE SET OF AMENABLE SEMIGROUP AND ZERO POINT SETS AND THE SOLUTIONS SETS OF SYSTEMS OF EQUILIBRIUM PROBLEMS

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ABSTRACT

In this paper, we introduce a new iterative scheme for finding a common element of the set of fixed point of left amenable semigroup, the zero point set of the operator which is the sum of inverse strongly monotone operators and maximal monotone operators, and the set of solutions for systems of equilibrium problems in Hilbert spaces by using a hybrid steepest descent methods. Then strong convergence of the scheme to a common element of the three sets is proved. Our results improve and generalize some well-known results in the literature.

Keywords: *Common fixed point; non-expansive mapping; amenable semigroup; system of equilibrium problem; strictly pseudo-contractive mapping; hybrid steepest descent method.*

1. INTRODUCTION

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$, respectively. When $\{x_n\}$ is a sequence in H, we denote strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. Let C be a nonempty closed convex subset of H and let $F : C \times C \to R$ be a bi-function, where R is the set of real numbers. The equilibrium problem for $F : C \times C \to R$ is to find $x^* \in C$ such that

$$F(x^*, y) = 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$. Let $\{F_i, i = 1,2,\ldots, N\}$ be a finite family of bi-functions from $C \times C$ to R, where R is the set of real numbers. The system of equilibrium problems for $\{F_1, F_2,\ldots, F_N\}$ is to find a common element $x^* \in C$ such that

$$\begin{cases} F_1(x^*, y) = 0, & \forall y \in C, \\ F_2(x^*, y) = 0, & \forall y \in C, \\ \vdots \\ F_N(x^*, y) = 0, & \forall y \in C. \end{cases} \tag{1.2}$$

We denote the set of solutions of (1.2) by $\cap_{j=1}^{M} SEP(F_j)$, where $SEP(F_j)$ is the set of solutions to the equilibrium problems, that is,

$$F_i(x^*, y) = 0, \quad \forall y \in C. \tag{1.3}$$

If $N = 1$, then the problem (1.2) is reduced to the equilibrium problems.

If $N = 1$ and $F(x^*, y) = \langle Bx^*, y - x^* \rangle$, then the problem (1.2) is reduced to the variational inequality problem of finding $x^* \in C$ such that

$$\langle Bx^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $VI(C, B)$. Many problems in applied sciences, such as monotone inclusion problems, saddle point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and equilibrium problems as special cases. Some methods have been proposed to solve $VI(C, B)$, $EP(F)$ and $SEP(F_i)$; see, for example [1,2] and references therein. The above formulations (1.2) extends this formulism to such problems, covering in particular various forms of feasibility problems.

Definition 1.1. *Let $A: C \to H$ be nonlinear mappings. Then A is called*
(1)monotone if $\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C$;
(2) $\alpha-$inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C;$$

(3) λ -strictly pseudo-contractive if there exists a constant $\lambda \in (0,1)$ such that

$$\|Ax - Ay\|^2 \leq \|x - y\|^2 + \lambda\|(I - A)x - (I - A)y\|^2, \quad \forall x, y \in C;$$

(4)For the variational inequality, the following is true:

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A), \quad \lambda > 0.$$

(5)Let $M: H \to 2^H$ be a set-valued mapping. The set $D(M)$ defined by $D(M) = \{x \in H : Mx \neq \emptyset\}$ is said to be the domain of M. The set $R(M)$ defined by $R(M) = \cup_{x\in H} Mx$ is said to be the range of M. The set $G(M)$ defined by $G(M) = \{(x, y) \in H \times H : x \in D(M), y \in R(M)\}$ is said to be the graph of M.
Recall that M is said to be monotone if

$$\langle x - y, f - g\rangle \geq 0, \quad \forall (x, f), (y, g) \in G(M).$$

M is said to be maximal monotone if it is not properly contained in any other mono-tone operator. Equivalently, M is maximal monotone if $R(I + rM) = H$ for all $r > 0$. The class of monotone mappings is one of the most important classes of mappings. With in the past several decades, many authors have been devoting to the studies on the existence and convergence of zero points for maximal monotone mappings, and the references therein. For a maximal monotone operator M on H and $r > 0$, we may define the single-valued resolvent $J_r = (I + rM)^{-1} : H \to D(M)$. It is known that J_r is firmly non-expansive and $(M)^{-1}(0) = F(J_r)$, where $F(J_r)$ denotes the fixed point set of J_r.
In 2003, Takahashi and Toyoda [3] proved the following weak convergence theorem.

Theorem 1.1. *Let C be a nonempty closed convex subset of a real Hilbert space space H. Let A be an $\alpha-$inverse strongly monotone mapping from C into H and S be a non-expansive mapping from C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$*. Let $\{x_n\}$ be a sequence generated by

$$x_0 \in C, x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 0,$$

where $\lambda_n \in [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\alpha_n \in [c, d]$ for some $c, d \in (0,1)$. Then $\{x_n\}$ converges weakly to $z \in F(S) \cap VI(C, A)$, where $z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n$.
In 2007, Tada and Takahashi [4] obtained the following weak convergence theorem.

Theorem 1.2. *Let C be a nonempty closed convex subset of a real Hilbert space space H. Let F be a bi-function from $C \times C$ to R satisfying $A(1) - A(4)$ and S be a non-expansive mapping from C into H such that $F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$ and let*

$$\begin{cases} u_n \in C \quad \text{such that} \quad F(u_n, u) + \dfrac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Su_n \end{cases} \tag{1.5}$$

for each $n \geq 1$, where $\alpha_n \in [a, b]$ for some $a, b \in (0,1)$ and $r_n \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges weakly to $w \in F(S) \cap EP(F)$, where $z = \lim_{n\to\infty} P_{F(S)\cap EP(F))} x_n$.
Very recently,Jitpeera et al.[5],introduced the iterative scheme based on viscosity and Cesro mean

$$\begin{cases} \phi(u_n, y) + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = \delta_n u_n + (1-\delta_n) P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n r f(y_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n \mu A)\dfrac{1}{n+1}\sum_{i=0}^{n} T^i y_n, \quad \forall n \ge 0, \end{cases} \tag{1.6}$$

where $B: C \to H$ a $\beta-$inverse strongly monotone, $\phi: C \to R \cup \{\infty\}$ is a proper lower semi-continuous and convex function, $T^i : C \to C$ is a non-expansive mapping for all $i = 1,2,3,\ldots,n$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\} \subset (0,1)$, $\{\lambda_n\} \subset (0,2\beta)$, and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} \delta_n = 0$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\lambda_n \subset [e, g] \subset (0,2\beta)$, $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

(v) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

They show that if $\theta := \cap_{i=1}^{n} Fix(T^i) \cap VI(C,B) \cap MEP(\phi,\varphi)$ is nonempty, then the sequence x_n converges strongly to the $z = P_\theta(I - A + rf)(z)$ which is the unique solution of the variational inequality

$$\langle (rf - A)z, x - z\rangle \le 0, \quad \forall y \in \theta.$$

In this paper, motivated and inspired by Takahashi and Toyoda [3], Tada and Takahashi [4], and Jitpeera et. al [5], we introduce a new iterative scheme for finding a common element of the set of fixed point of left amenable semigroup, the zero point set of the operator which is the sum of inverse strongly monotone operators and maximal monotone operators, and the set of solutions for systems of equilibrium problems in Hilbert spaces by using a hybrid steepest descent methods. Then strong convergence of the scheme to a common element of the three sets is proved. Our results improve and generalize some well-known results in the literature.

2. PRELIMINARIES

Lemma 2.1 ([6]) *Let S be a semigroup and C be a nonempty closed convex subset of a reflexive Banach space E. Let $\phi = \{T_t x : t \in S\}$ be a non-expansive semigroup on H such that $\{T_t x : t \in S\}$ is bounded for some $x \in C$, let X be a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \mapsto \langle T_t x, y^*\rangle$ is an element of X for each $x \in C$ and $y^* \in E^*$, and μ is a mean on X. If we write $T_\mu x$ instead of $\int T_t x d\mu(t)$, then the followings hold .*

(i) T_μ is non-expansive mapping from C into C;

(ii) $T_\mu x = x$ for each $x \in Fix(\phi)$;

(iii) $T_\mu x \in \bar{c}o\{T_t x : t \in S\}$ for each $x \in C$.

Lemma 2.2 ([7]) *Let H be a real Hilbert spaces, there hold the following identities:*

(i)for each $x \in H$ and $x^* \in C$, $x^* = P_C x \Leftrightarrow \langle x - x^*, y - x^*\rangle \le 0$ for all $y \in C$;

(ii) $P_C : H \to C$ is non-expansive,that is, $\|P_C x - P_C y\| \le \|x - y\|$ for all $x, y \in H$;

(iii) P_C is firmly non-expansive,that is, $\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle$ for all $x, y \in H$;

(iv) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \forall t \in [0,1]$, for all $x, y \in H$;

(v) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$.

Lemma 2.3 ([8]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\alpha_n\}$ be a sequence in*

$[0,1]$ *with* $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1$. *Suppose* $x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n$ *for all integers* $n \ge 0$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.4 ([9]) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-b_n)a_n + b_n c_n, \quad n \ge 0.$$

where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying the following conditions

(i) $b_n \subset (0,1), \sum_{n=0}^{\infty} b_n = \infty$,

(ii)either $\limsup_{n\to\infty} c_n \le 0$ or $\sum_{n=0}^{\infty} |b_n c_n| < \infty$.

Then, $lim_{n\to\infty} a_n = 0$.

Lemma 2.5 ([10]) *Assume* A *be a strongly positive linear bounded operator on* H *with coefficient* $\bar{\gamma} > 0$ *and* $0 < \rho \le \|A\|^{-1}$. *Then* $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$.

Throughout this article, we assume that a bi-function $F : C \times C \to R$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \le 0$ for all $x, y \in C$;

(A3)for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \le F(x,y)$;

(A4)for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semi-continuous.

Lemma 2.6 ([11]) *Let* C *be a nonempty closed convex subset of a real Hilbert space space* H *and let* F *be a bi-function from* $C \times C$ *to* R *satisfying* $A(1) - A(4)$. *Let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in C$ *such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C,$$

Lemma 2.7 ([12]) *Let* C *be a nonempty closed convex subset of a real Hilbert space space* H, $A : C \to H$ *be a mapping and* $M : H \to 2^H$ *be a maximal monotone mapping. Then*

$$F(J_r(I - rA)) = (A + M)^{-1}(0), \quad \forall r > 0.$$

Lemma 2.8 ([13]) *Let* H *be a real Hilbert space space. For* q *which solves the variational inequality* $\langle (rf - \mu F)q, x_{n_k} - q\rangle \le 0$, $f \in \prod_H$, $p \in F(T)$, *the following statement is true :*

$$\langle (rf - \mu F)q, x_{n_k} - q\rangle \le 0 \quad \Leftrightarrow \quad P_\Theta(I - \mu F + rf)q = q,$$

where $\Theta := F(S) \cap (A+M)^{-1}(0) \cap (B+W)^{-1}(0) \cap \cap_{j=1}^{M} SEP(F_j) \ne \varnothing$.

Let I_C be the indicator function of C,i.e.,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Lemma 2.9 ([14]) *Let* C *be a nonempty closed convex subset of a real Hilbert space space* H. *Let* P_C *be the metric projection from* H *onto* C, ∂I_C *be the sub-differential of* I_C, *where* I_C *is as defined in (2.1) and* $J_r = (I + r\partial I_C)^{-1}$. *Then*

$$y = J_r x \quad \Leftrightarrow \quad y = P_C x, x \in H, y \in C.$$

3. MAIN RESULTS

Theorem 3.1 *Let* C *be a nonempty closed convex subset of a real Hilbert space space* H, $A : C \to H$ *be an*

$\alpha-$inverse strongly monotone, $B:C\to H$ be an $\beta-$inverse strongly monotone, f be a contraction of C into itself with coefficient $\eta\in(0,1)$, F is a strongly positive linear bounded operator on H with coefficient $\bar{\gamma}>0$ and $0<\gamma<\frac{\bar{\gamma}}{\eta}$. Let $\{F_k, k=1,2,\dots,M\}$ be a finite family of bi-functions from $C\times C$ to R which satisfy $A(1)-A(4)$, S a semigroup and $\phi=\{T_t x: t\in S\}$ be a non-expansive semigroup from C into C such that $Fix(\phi)=\cap_{t\in S}Fix(T_t)\neq\varnothing$. Let X be a left invariant subspace of $B(S)$ such that $1\in X$, and the function $t\mapsto\langle T_t x, y\rangle$ is an element of X for each $x\in C$ and $y\in H$, $\{\mu_n\}$ a left regular sequence of means on X such that $\lim_{n\to\infty}\|\mu_{n+1}-\mu_n\|=0$. $M:H\to 2^H$ and $N:H\to 2^H$ be maximal monotone operators such that $D(M)\subset C$ and $D(W)\subset C$. Assume that $\Omega:=F(S)\cap(A+M)^{-1}(0)\cap(B+W)^{-1}(0)\cap\cap_{j=1}^{M}SEP(F_j)\neq\varnothing$. If the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are generated iteratively by $x_1\in C$ and

$$\begin{cases} u_n = T_{r_{M,n}}^{F_M}\cdots T_{r_{2,n}}^{F_2}T_{r_{1,n}}^{F_1}x_n \\ y_n=\delta_n J_{\xi_n}(u_n-\xi_n Au_n)+(1-\delta_n)J_{s_n}(u_n-s_nBu_n) \\ x_{n+1}=\eta_n rf(y_n)+\beta_n x_n+((1-\beta_n)I-\eta_n\mu F)T_{\mu_n}y_n \end{cases}$$

where $J_{\xi_n}=(I+\xi_n M)^{-1}$, $J_{s_n}=(I+s_nW)^{-1}$, $\{\xi_n\}$ is a sequence in $(0,2\alpha)$, $\{s_n\}$ is a sequence in $(0,2\beta)$, $\{r_{k,n}\}_{k=1}^{M}$ are a real sequence in $(0,\infty)$ and $\{\beta_n\}$, $\{\eta_n\}$, $\{\delta_n\}$ are three sequences in $(0,1)$. Assume that the following restrictions are satisfied

(C1) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;

(C2) $\lim_{n\to\infty}\eta_n=0$ and $\Sigma_{n=1}^{\infty}\eta_n=\infty$;

(C3) $0<a\leq\xi_n\leq b<2\alpha$, $0<c\leq s_n\leq d<2\beta$ and $0<k\leq\delta_n\leq e<1$;

(C4) $\lim_{n\to\infty}|\eta_{n+1}-\eta_n|=\lim_{n\to\infty}|s_{n+1}-s_n|=\lim_{n\to\infty}|\delta_{n+1}-\delta_n|=0$;

(C5) $\liminf_{n\to\infty}r_{k,n}>0$ and $\lim_{n\to\infty}|r_{k,n+1}-r_{k,n}|=0$ for each $k\in\{1,2,3,\dots,M\}$, where a, b, c, d, k, e are real numbers. Then the sequence $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converges strongly to a point $q\in\Omega$, which is the unique solution of the variational inequality

$$\langle(\mu F-rf)q, p-q\rangle\geq 0,\quad \forall p\in\Omega. \tag{3.2}$$

Equivalently, we have $q=P_\Omega(I-\mu F+rf)(q)$.

Proof *We shall divide the proof into several steps.*

By taking $v_n=J_{\xi_n}(u_n-\xi_n Au_n)$, $w_n=J_{s_n}(u_n-s_nBu_n)$ and for $k\in\{1,2,3,\dots,M\}$ and for all $n\in N$, we shall equivalently write scheme (3.1) as follows:

$$\begin{cases} u_n=\mathfrak{T}_n^M x_n \\ y_n=\delta_n v_n+(1-\delta_n)w_n \\ x_{n+1}=\eta_n rf(y_n)+\beta_n x_n+((1-\beta_n)I-\eta_n\mu F)T_{\mu_n}y_n \end{cases}$$

Step 1. We show that the mapping $P_\Omega(I-\mu F+rf)$ has a unique fixed point.

Since f be a contraction of C into itself with coefficient $\eta\in(0,1)$. Then, we have

$$\begin{aligned}\|P_\Omega(rf+(I-\mu F))(x)-P_\Omega(rf+(I-\mu F))(y)\| &\le \|(rf+(I-\mu F))(x)-(rf+(I-\mu F)(y)\|\\ &\le r\|f(x)-f(y)\|+\|I-\mu F\|\|x-y\|\\ &\le r\eta\|x-y\|+(1-\mu\bar{\gamma})\|x-y\|\\ &=(1-(\mu\bar{\gamma}-r\eta))\|x-y\|,\quad \forall x,y\in C.\end{aligned}$$

Since $0<1-(\mu\bar{\gamma}-r\eta)<1$*, it follows that* $P_\Omega(I-\mu F+rf)$ *is a contraction of* C *into itself. Therefore, by the Banach Contraction Mapping Principle, has a unique fixed point, say* $q\in C$ *that is,*

$$q=P_\Omega(I-\mu F+rf)(q).$$

Step 2. *We show that* $(I-\xi_n A)$, $(I-s_n B)$ *is non-expansive.*

For all $x,y\in C$, we have

$$\begin{aligned}\|(I-\xi_n A)x-(I-\xi_n A)y\|^2 &= \|x-y\|^2-2\xi_n\langle x-y,Ax-Ay\rangle+\xi_n^2\|Ax-Ay\|^2\\ &\le \|x-y\|^2-\xi_n(2\alpha-\xi_n)\|Ax-Ay\|^2\\ &\le \|x-y\|^2.\end{aligned}$$

which implies that $(I-\xi_n A)$ *is a non-expansive, so is* $(I-s_n B)$.

Step 3. *We show that the sequence* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *are bounded.*

Let $\mathfrak{T}_n^k=T_{r_{k,n}}^{F_k}T_{r_{k-1,n}}^{F_{k-1}}\cdots T_{r_{2,n}}^{F_2}T_{r_{1,n}}^{F_1}$ for $k\in\{1,2,3,\ldots,M\}$ and $\mathfrak{T}_n^0=I$. Since for each $k\in\{1,2,3,\ldots,M\}$, $T_{r_{k,n}}^{F_k}$ is non-expansive, $\forall x^*\in\Omega$, we note that $u_n=\mathfrak{T}_n^M x_n$. It follows that

$$\|u_n-x^*\|=\|\mathfrak{T}_n^M x_n-\mathfrak{T}_n^M x^*\|\le\|x_n-x^*\|.3.3$$

$$\|v_n-x^*\|=\|J_{\xi_n}(u_n-\xi_n Au_n)-J_{\xi_n}(x^*-\xi_n Ax^*)\|\le\|u_n-x^*\|\le\|x_n-x^*\|.3.4$$

$$\|w_n-x^*\|=\|J_{s_n}(u_n-s_n Bu_n)-J_{s_n}(x^*-s_n Bx^*)\|\le\|u_n-x^*\|\le\|x_n-x^*\|. \tag{3.5}$$

$$\begin{aligned}\|y_n-x^*\| &\le \delta_n\|v_n-x^*\|+(1-\delta_n)\|w_n-x^*\|\\ &\le \delta_n\|x_n-x^*\|+(1-\delta_n)\|x_n-x^*\|\\ &=\|x_n-x^*\|.\end{aligned} \tag{3.6}$$

which yields that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\eta_n(rf(y_n)-\mu Fx^*)+\beta_n(x_n-x^*)+((1-\beta_n)I-\eta_n\mu F)(T_{\mu_n}y_n-x^*)\|\\ &\le \eta_n\|rf(y_n)-\mu Fx^*\|+\beta_n\|x_n-x^*\|+\|(1-\beta_n)I-\eta_n\mu F\|\|T_{\mu_n}y_n-x^*\|\\ &\le \eta_n r\|f(y_n)-f(x^*)\|+\eta_n\|rf(x^*)-\mu Fx^*\|+\beta_n\|x_n-x^*\|+\|1-\beta_n-\eta_n\mu\bar{\gamma}\|\|y_n-x^*\|\\ &\le \eta_n r\eta\|u_n-x^*\|+\eta_n\|rf(x^*)-\mu Fx^*\|+\beta_n\|x_n-x^*\|+\|1-\beta_n-\eta_n\mu\bar{\gamma}\|\|x_n-x^*\|\\ &\le \eta_n r\eta\|x_n-x^*\|+\eta_n\|rf(x^*)-\mu Fx^*\|+\beta_n\|x_n-x^*\|+\|1-\beta_n-\eta_n\mu\bar{\gamma}\|\|x_n-x^*\|\\ &=(1-(\mu\bar{\gamma}-r\eta)\eta_n)\|x_n-x^*\|+\frac{(\mu\bar{\gamma}-r\eta)\eta_n)}{(\mu\bar{\gamma}-r\eta)}\|rf(x^*)-\mu Fx^*\|\\ &\le max\{\|x_n-x^*\|,\frac{\|rf(x^*)-\mu Fx^*\|}{(\mu\bar{\gamma}-r\eta)}\}.\end{aligned}$$

By induction,

$$\|x_n - x^*\| \le max\{\|x_1 - x^*\|, \frac{\|rf(x^*) - \mu Fx^*\|}{(\mu\bar{\gamma} - r\eta)}\} \quad \forall n \in N. \tag{3.7}$$

and we obtain $\{x_n\}$ *is bounded. So are* $\{y_n\}$, $\{u_n\}$ *and* $\{f(y_n)\}$.

Step 4. *We claim that if* $\{x_n\}$ *is a bounded sequence in* C, *then* $\lim_{n\to\infty}\|T_{\mu_{n+1}}y_n - T_{\mu_n}y_n\| = 0$ *and* $\lim_{n\to\infty}\|\mathfrak{T}_n^k x_n - \mathfrak{T}_{n+1}^k x_n\| = 0$, *for each* $k \in \{1,2,3,\ldots,M\}$.

These two assertion are proved in $[15, step3]$ and $[16, step2]$.

Step 5. *We show that* $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$ *and* $\lim_{n\to\infty}\|y_{n+1} - y_n\| = 0$.

$$\begin{aligned}\|u_{n+1} - u_n\| &= \|\mathfrak{T}_n^k x_n - \mathfrak{T}_{n+1}^k x_{n+1}\| \\ &\le \|\mathfrak{T}_n^k x_n - \mathfrak{T}_n^k x_{n+1}\| + \|\mathfrak{T}_n^k x_{n+1} - \mathfrak{T}_{n+1}^k x_{n+1}\| \\ &\le \|x_{n+1} - x_n\| + \|\mathfrak{T}_n^k x_{n+1} - \mathfrak{T}_{n+1}^k x_{n+1}\| \\ &\le \|x_{n+1} - x_n\|.\end{aligned} \tag{3.8}$$

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|\delta_{n+1}v_{n+1} + (1-\delta_{n+1})w_{n+1} - \delta_n v_n - (1-\delta_n)w_n\| \\ &= \|\delta_{n+1}(v_{n+1} - v_n) + (\delta_{n+1} - \delta_n)v_n + (1-\delta_{n+1})w_{n+1} - (1-\delta_{n+1})w_n + (\delta_{n+1} - \delta_n)w_n\| \\ &\le \delta_{n+1}\|v_{n+1} - v_n\| + |\delta_{n+1} - \delta_n|(\|v_n\| + \|w_n\|) + (1-\delta_{n+1})\|w_{n+1} - w_n\| \\ &= \delta_{n+1}\|J_{\xi_{n+1}}(u_{n+1} - \xi_{n+1}Au_{n+1}) - J_{\xi_n}(u_n - \xi_n Au_n)\| + |\delta_{n+1} - \delta_n|(\|v_n\| + \|w_n\|) \\ &\quad + (1-\delta_{n+1})\|J_{s_{n+1}}(u_{n+1} - s_{n+1}Bu_{n+1}) - J_{s_n}(u_n - s_n Bu_n)\| \\ &= \delta_{n+1}\|J_{\xi_{n+1}}(u_{n+1} - \xi_{n+1}Au_{n+1}) - J_{\xi_{n+1}}(u_n - \xi_{n+1}Au_n) + J_{\xi_{n+1}}(u_n - \xi_{n+1}Au_n) \\ &\quad - J_{\xi_{n+1}}(u_n - \xi_n Au_n) + J_{\xi_{n+1}}(u_n - \xi_n Au_n) - J_{\xi_n}(u_n - \xi_n Au_n)\| \\ &\quad + |\delta_{n+1} - \delta_n|(\|v_n\| + \|w_n\|) + (1-\delta_{n+1})\|J_{s_{n+1}}(u_{n+1} - s_{n+1}Bu_{n+1}) \\ &\quad - J_{s_{n+1}}(u_n - s_{n+1}Bu_n) + J_{s_{n+1}}(u_n - s_{n+1}Bu_n) - J_{s_{n+1}}(u_n - s_n Bu_n) \\ &\quad + J_{s_{n+1}}(u_n - s_n Bu_n) - J_{s_n}(u_n - s_n Bu_n)\| \\ &\le \delta_{n+1}\|u_{n+1} - u_n\| + \delta_{n+1}|\xi_{n+1} - \xi_n|\|Au_n\| + \delta_{n+1}\left|\frac{\xi_{n+1} - \xi_n}{\xi_{n+1}}\right|\|J_{\xi_{n+1}}(u_n - \xi_n Au_n) \\ &\quad - (u_n - \xi_n Au_n)\| + |\delta_{n+1} - \delta_n|(\|v_n\| + \|w_n\|) + (1-\delta_{n+1})\|u_{n+1} - u_n\| + (1-\delta_{n+1}) \\ &\quad |s_{n+1} - s_n|\|Bu_n\| + (1-\delta_{n+1})\left|\frac{s_{n+1} - s_n}{s_{n+1}}\right|\|J_{s_{n+1}}(u_n - s_n Bu_n) - (u_n - s_n Bu_n)\| \\ &\le \|u_{n+1} - u_n\| + \delta_{n+1}|\xi_{n+1} - \xi_n|\|Au_n\| + \delta_{n+1}\left|\frac{\xi_{n+1} - \xi_n}{\xi_{n+1}}\right|\|J_{\xi_{n+1}}(u_n - \xi_n Au_n) \\ &\quad - (u_n - \xi_n Au_n)\| + |\delta_{n+1} - \delta_n|(\|v_n\| + \|w_n\|) + (1-\delta_{n+1})|s_{n+1} - s_n|\|Bu_n\| \\ &\quad + (1-\delta_{n+1})\left|\frac{s_{n+1} - s_n}{s_{n+1}}\right|\|J_{s_{n+1}}(u_n - s_n Bu_n) - (u_n - s_n Bu_n)\|.\end{aligned} \tag{3.9}$$

We define a sequence $\{z_n\}$ *by* $z_n = (x_{n+1} - \beta_n x_n)/(1-\beta_n)$, *so that* $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$.

We now observe that

$$\begin{aligned}\|z_{n+1}-z_n\| &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\ &= \frac{\eta_{n+1}rf(y_{n+1})-((1-\beta_{n+1})I-\eta_{n+1}\mu F)T_{\mu_{n+1}}y_{n+1}}{1-\beta_{n+1}} \\ &\quad -\frac{\eta_n rf(y_n)-((1-\beta_n)I-\eta_n\mu F)T_{\mu_n}y_n}{1-\beta_n} \\ &= \frac{\eta_{n+1}}{1-\beta_{n+1}}(rf(y_{n+1})-\mu FT_{\mu_{n+1}}y_{n+1})+\frac{\eta_n}{1-\beta_n}(\mu FT_{\mu_n}y_n-rf(y_n)) \\ &\quad +(T_{\mu_{n+1}}y_{n+1}-T_{\mu_n}y_n).\end{aligned} \tag{3.10}$$

$$\begin{aligned}\|T_{\mu_{n+1}}y_{n+1}-T_{\mu_n}y_n\| &\le \|T_{\mu_{n+1}}y_{n+1}-T_{\mu_{n+1}}y_n\|+\|T_{\mu_{n+1}}y_n-T_{\mu_n}y_n\| \\ &\le \|T_{\mu_{n+1}}y_n-T_{\mu_n}y_n\|+\|y_{n+1}-y_n\|.\end{aligned} \tag{3.11}$$

Substituting (3.9) and (3.11) into (3.10), we can obtain

$$\begin{aligned}\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| &\le \frac{\eta_{n+1}}{1-\beta_{n+1}}\|rf(y_{n+1})-\mu FT_{\mu_{n+1}}y_{n+1}\|+\frac{\eta_n}{1-\beta_n}\|\mu FT_{\mu_n}y_n-rf(y_n)\| \\ &\quad +\|T_{\mu_{n+1}}y_n-T_{\mu_n}y_n\|+\delta_{n+1}|\xi_{n+1}-\xi_n|\,\|Au_n\|+\delta_{n+1}\left|\frac{\xi_{n+1}-\xi_n}{\xi_{n+1}}\right| \\ &\quad \|J_{\xi_{n+1}}(u_n-\xi_n Au_n)-(u_n-\xi_n Au_n)\|+|\delta_{n+1}-\delta_n|(\|v_n\|+\|w_n\|) \\ &\quad +(1-\delta_{n+1})|s_{n+1}-s_n|\,\|Bu_n\|+(1-\delta_{n+1})\left|\frac{s_{n+1}-s_n}{s_{n+1}}\right| \\ &\quad \|J_{s_{n+1}}(u_n-s_n Bu_n)-(u_n-s_n Bu_n)\| \\ &\le \frac{\eta_{n+1}}{1-\beta_{n+1}}(\|rf(y_{n+1})\|+\|\mu FT_{\mu_{n+1}}y_{n+1}\|)+\frac{\eta_n}{1-\beta_n}(\|rf(y_n)\|+\|\mu FT_{\mu_n}y_n)\| \\ &\quad +\|T_{\mu_{n+1}}y_n-T_{\mu_n}y_n\|+\delta_{n+1}|\xi_{n+1}-\xi_n|\,\|Au_n\|+\delta_{n+1}\left|\frac{\xi_{n+1}-\xi_n}{\xi_{n+1}}\right| \\ &\quad \|J_{\xi_{n+1}}(u_n-\xi_n Au_n)-(u_n-\xi_n Au_n)\|+|\delta_{n+1}-\delta_n|(\|v_n\|+\|w_n\|) \\ &\quad +(1-\delta_{n+1})|s_{n+1}-s_n|\,\|Bu_n\|+(1-\delta_{n+1})\left|\frac{s_{n+1}-s_n}{s_{n+1}}\right| \\ &\quad \|J_{s_{n+1}}(u_n-s_n Bu_n)-(u_n-s_n Bu_n)\|.\end{aligned} \tag{3.12}$$

From (3.12), Step4 and conditions $(C2)-(C4)$, it follows that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0.$$

By Lemma 2.4, we obtain

$$\lim_{n\to\infty}\|x_n-z_n\|=0.$$

Hence,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|x_n-z_n\|=0. \tag{3.13}$$

Therefore from (3.8), (3.9), (3.13) and the condition (C4), we get

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=\lim_{n\to\infty}\|y_{n+1}-y_n\|=0. \tag{3.14}$$

Step 6. *We claim that*

$$\lim_{n\to\infty}\|Au_n-Ax^*\|=\lim_{n\to\infty}\|Bu_n-Bx^*\|=\lim_{n\to\infty}\|u_n-v_n\|=\lim_{n\to\infty}\|u_n-w_n\|=0.$$

For any $x^* \in \Omega$ and (3.4), we obtain

$$\begin{aligned}\|v_n - x^*\|^2 &= \|J_{\xi_n}(u_n - \xi_n Au_n) - J_{\xi_n}(x^* - \xi_n Ax^*)\|^2 \\ &\le \|(u_n - x^*) - \xi_n(Au_n - Ax^*)\|^2 \\ &\le \|u_n - x^*\|^2 - \xi_n(2\alpha - \xi_n)\|Au_n - Ax^*\|^2.\end{aligned} \tag{3.15}$$

By using the same method as (3.15), we have

$$\|w_n - x^*\|^2 \le \|u_n - x^*\|^2 - s_n(2\beta - s_n)\|Bu_n - Bx^*\|^2.$$

By Lemma 2.2(iii) and (3.15), we have

$$\begin{aligned}\|y_n - x^*\|^2 &\le \delta_n\|v_n - x^*\|^2 + (1-\delta_n)\|w_n - x^*\|^2 \\ &\le \delta_n(\|u_n - x^*\|^2 - \xi_n(2\alpha - \xi_n)\|Au_n - Ax^*\|^2) + (1-\delta_n)(\|u_n - x^*\|^2 \\ &\quad - s_n(2\beta - s_n)\|Bu_n - Bx^*\|^2) \\ &\le \|u_n - x^*\|^2 - \delta_n\xi_n(2\alpha - \xi_n)\|Au_n - Ax^*\|^2 - s_n(1-\delta_n)(2\beta - s_n)\|Bu_n - Bx^*\|^2.\end{aligned} \tag{3.16}$$

From (3.1), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\eta_n(rf(y_n) - \mu Fx^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \eta_n\mu F)(T_{\mu_n}y_n - x^*)\|^2 \\ &= \|((1-\beta_n)I - \eta_n\mu F)(T_{\mu_n}y_n - x^*) + \beta_n(x_n - x^*)\|^2 + \eta_n^2\|rf(y_n) - \mu Fx^*\|^2 \\ &\quad + 2\beta_n\eta_n\langle x_n - x^*, rf(y_n) - \mu Fx^*\rangle + 2\eta_n\langle((1-\beta_n)I - \eta_n\mu F)(y_n - x^*), rf(y_n) - \mu Fx^*\rangle \\ &\le ((1-\beta_n - \eta_n\mu\bar{\gamma})\|y_n - x^*\| + \beta_n\|x_n - x^*\|)^2 + \eta_n L_n \\ &\le (1-\beta_n - \eta_n\mu\bar{\gamma})^2\|y_n - x^*\|^2 + \beta_n^2\|x_n - x^*\|^2 + \eta_n L_n \\ &\quad + 2\beta_n(1-\beta_n - \eta_n\mu\bar{\gamma})\|x_n - x^*\|\|y_n - x^*\| \\ &\le \{(1-\eta_n\mu\bar{\gamma})^2 - 2\beta_n(1-\eta_n\mu\bar{\gamma}) + \beta_n^2\}\|y_n - x^*\|^2 + \beta_n(1-\beta_n - \eta_n\mu\bar{\gamma})\{\|x_n - x^*\|^2 \\ &\quad + \|y_n - x^*\|^2\} + \beta_n^2\|x_n - x^*\|^2 + \eta_n L_n \\ &= (1-\eta_n\mu\bar{\gamma})(1-\beta_n - \eta_n\mu\bar{\gamma})\|y_n - x^*\|^2 + (1-\eta_n\mu\bar{\gamma})\beta_n\|x_n - x^*\|^2 + \eta_n L_n \\ &\le (1-\eta_n\mu\bar{\gamma})(1-\beta_n - \eta_n\mu\bar{\gamma})\{\|u_n - x^*\|^2 - \delta_n\xi_n(2\alpha - \xi_n)\|Au_n - Ax^*\|^2 \\ &\quad - s_n(1-\delta_n)(2\beta - s_n)\|Bu_n - Bx^*\|^2\} + (1-\eta_n\mu\bar{\gamma})\beta_n\|x_n - x^*\|^2 + \eta_n L_n \\ &\le (1-\eta_n\mu\bar{\gamma})(1-\beta_n - \eta_n\mu\bar{\gamma})\{\|x_n - x^*\|^2 - \delta_n\xi_n(2\alpha - \xi_n)\|Au_n - Ax^*\|^2 \\ &\quad - s_n(1-\delta_n)(2\beta - s_n)\|Bu_n - Bx^*\|^2\} + (1-\eta_n\mu\bar{\gamma})\beta_n\|x_n - x^*\|^2 + \eta_n L_n \\ &= (1-\eta_n\mu\bar{\gamma})^2\|x_n - x^*\|^2 + \eta_n L_n - (1-\eta_n\mu\bar{\gamma})(1-\beta_n - \eta_n\mu\bar{\gamma})\delta_n\xi_n(2\alpha - \xi_n)\|Au_n \\ &\quad - Ax^*\|^2 - (1-\eta_n\mu\bar{\gamma})(1-\beta_n - \eta_n\mu\bar{\gamma})s_n(1-\delta_n)(2\beta - s_n)\|Bu_n - Bx^*\|^2 \\ &\le \|x_n - x^*\|^2 + \eta_n L_n - (1-\eta_n\mu\bar{\gamma})(1-\beta_n - \eta_n\mu\bar{\gamma})\delta_n\xi_n(2\alpha - \xi_n)\|Au_n - Ax^*\|^2 \\ &\quad - (1-\eta_n\mu\bar{\gamma})(1-\beta_n - \eta_n\mu\bar{\gamma})s_n(1-\delta_n)(2\beta - s_n)\|Bu_n - Bx^*\|^2.\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}L_n &= \eta_n\|rf(y_n) - \mu Fx^*\|^2 + 2\beta_n\langle x_n - x^*, rf(y_n) - \mu Fx^*\rangle \\ &\quad + 2\langle((1-\beta_n)I - \eta_n\mu F)(y_n - x^*), rf(y_n) - \mu Fx^*\rangle.\end{aligned}$$

It follows from (3.17) that

$$\begin{aligned}&(1-\eta_n\mu\bar{\gamma})(1-\beta_n - \eta_n\mu\bar{\gamma})ka(2\alpha - b)\|Au_n - Ax^*\|^2 \\ &\le (1-\eta_n\mu\bar{\gamma})(1-\beta_n - \eta_n\mu\bar{\gamma})\delta_n\xi_n(2\alpha - \xi_n)\|Au_n - Ax^*\|^2 \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \eta_n L_n \\ &\le \|x_{n+1} - x_n\|(\|x_{n+1} - x^*\| + \|x_n - x^*\|) + \eta_n L_n.\end{aligned}$$

and

$$\begin{aligned}&(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})c(1-e)(2\beta-d)\|Bu_n-Bx^*\|^2\\ &\le(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})s_n(1-\delta_n)(2\beta-s_n)\|Bu_n-Bx^*\|^2\\ &\le\|x_{n+1}-x_n\|(\|x_{n+1}-x^*\|+\|x_n-x^*\|)+\eta_nL_n.\end{aligned}$$

By the condition (C2) and (3.13), we obtain

$$\lim_{n\to\infty}\|Au_n-Ax^*\|=\lim_{n\to\infty}\|Bu_n-Bx^*\|=0. \tag{3.18}$$

Since J_{ξ_n} , J_{s_n} are firmly non-expansive mappings, we have

$$\begin{aligned}\|v_n-x^*\|^2 &=\|J_{\xi_n}(u_n-\xi_nAu_n)-J_{\xi_n}(x^*-\xi_nAx^*)\|^2\\ &\le\langle(u_n-\xi_nAu_n)-(x^*-\xi_nAx^*),v_n-x^*\rangle\\ &=\frac{1}{2}\{\|(u_n-\xi_nAu_n)-(x^*-\xi_nAx^*)\|^2+\|v_n-x^*\|^2-\|(u_n-\xi_nAu_n)\\ &\quad-(x^*-\xi_nAx^*)-(v_n-x^*)\|^2\\ &\le\frac{1}{2}\{\|u_n-x^*\|^2+\|v_n-x^*\|^2-\|(u_n-v_n)-\xi_n(Au_n-Ax^*)\|^2\}\\ &\le\frac{1}{2}\{\|u_n-x^*\|^2+\|v_n-x^*\|^2-\|u_n-v_n\|^2-\xi_n^2\|Au_n-Ax^*\|^2\\ &\quad+2\xi_n\|u_n-v_n\|\|Au_n-Ax^*\|\}.\end{aligned}$$

Hence, we have

$$\|v_n-x^*\|^2\le\|u_n-x^*\|^2-\|u_n-v_n\|^2+2\xi_n\|u_n-v_n\|\|Au_n-Ax^*\|.$$

By using the same method, we have

$$\|w_n-x^*\|^2\le\|u_n-x^*\|^2-\|u_n-w_n\|^2+2s_n\|u_n-w_n\|\|Bu_n-Bx^*\|.$$

and so

$$\begin{aligned}\|y_n-x^*\|^2 &\le\delta_n\|v_n-x^*\|^2+(1-\delta_n)\|w_n-x^*\|^2\\ &\le\delta_n(\|u_n-x^*\|^2-\|u_n-v_n\|^2+2\xi_n\|u_n-v_n\|\|Au_n-Ax^*\|)\\ &\quad+(1-\delta_n)(\|u_n-x^*\|^2-\|u_n-w_n\|^2+2s_n\|u_n-w_n\|\|Bu_n-Bx^*\|)\\ &=\|u_n-x^*\|^2-\delta_n\|u_n-v_n\|^2+2\delta_n\xi_n\|u_n-v_n\|\|Au_n-Ax^*\|\\ &\quad-(1-\delta_n)\|u_n-w_n\|^2+2s_n(1-\delta_n)\|u_n-w_n\|\|Bu_n-Bx^*\|\\ &\le\|x_n-x^*\|^2-\delta_n\|u_n-v_n\|^2+2\delta_n\xi_n\|u_n-v_n\|\|Au_n-Ax^*\|\\ &\quad-(1-\delta_n)\|u_n-w_n\|^2+2s_n(1-\delta_n)\|u_n-w_n\|\|Bu_n-Bx^*\|.\end{aligned} \tag{3.19}$$

Using (3.17) and (3.19), we also have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})\|y_n-x^*\|^2+(1-\eta_n\mu\bar{\gamma})\beta_n\|x_n-x^*\|^2+\eta_nL_n\\ &\le(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})\{\|x_n-x^*\|^2-\delta_n\|u_n-v_n\|^2\\ &\quad+2\delta_n\xi_n\|u_n-v_n\|\|Au_n-Ax^*\|-(1-\delta_n)\|u_n-w_n\|^2+2s_n(1-\delta_n)\|u_n-w_n\|\|Bu_n-Bx^*\|\}\\ &\quad+\eta_nL_n+(1-\eta_n\mu\bar{\gamma})\beta_n\|x_n-x^*\|^2\\ &\le\|x_n-x^*\|^2+2(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})\{\delta_n\xi_n\|u_n-v_n\|\|Au_n-Ax^*\|\\ &\quad+s_n(1-\delta_n)\|u_n-w_n\|\|Bu_n-Bx^*\|\}-\delta_n(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})\|u_n-v_n\|^2\\ &\quad-(1-\delta_n)(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})\|u_n-w_n\|^2+\eta_nL_n.\end{aligned}$$

It follows that

$$\delta_n(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})\|u_n-v_n\|^2$$

$$\begin{aligned}&\leq \|x_{n+1}-x_n\|(\|x_{n+1}-x^*\|+\|x_n-x^*\|)+\eta_n L_n\\&\quad +2(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})\{\delta_n\xi_n\|u_n-v_n\|\|Au_n-Ax^*\|\\&\quad +s_n(1-\delta_n)\|u_n-w_n\|\|Bu_n-Bx^*\|\}.\end{aligned}$$

and

$$\begin{aligned}&(1-\delta_n)(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})\|u_n-w_n\|^2\\&\leq \|x_{n+1}-x_n\|(\|x_{n+1}-x^*\|+\|x_n-x^*\|)+\eta_n L_n\\&\quad +2(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})\{\delta_n\xi_n\|u_n-v_n\|\|Au_n-Ax^*\|\\&\quad +s_n(1-\delta_n)\|u_n-w_n\|\|Bu_n-Bx^*\|\}.\end{aligned}$$

By the condition (C2) and (3.13), (3.18), we obtain

$$\lim_{n\to\infty}\|u_n-v_n\|=\lim_{n\to\infty}\|u_n-w_n\|=0. \tag{3.20}$$

From (3.1) and (3.20) we have

$$\lim_{n\to\infty}\|y_n-u_n\|=\lim_{n\to\infty}\delta_n\|v_n-u_n\|+\lim_{n\to\infty}(1-\delta_n)\|w_n-u_n\|=0. \tag{3.21}$$

Step 7. We show that $\lim_{n\to\infty}\|\mathfrak{T}_n^k x_n-\mathfrak{T}_n^{k+1}x_n\|=0$, for each $k\in\{1,2,3,\ldots,M-1\}$ and $\lim_{n\to\infty}\|u_n-x_n\|=0.$

For any $x^*\in\Omega$, note that $T_{r_{k,n}}^{F_k}$ is non-expansive, for $k\in\{1,2,3,\ldots,M-1\}$, then we have

$$\begin{aligned}\|\mathfrak{T}_n^{k+1}x_n-x^*\|^2 &=\|T_{r_{k+1,n}}^{F_{k+1}}\mathfrak{T}_n^k x_n-T_{r_{k+1,n}}^{F_{k+1}}x^*\|^2\\&\leq\langle T_{r_{k+1,n}}^{F_{k+1}}\mathfrak{T}_n^k x_n-T_{r_{k+1,n}}^{F_{k+1}}x^*,\mathfrak{T}_n^k x_n-x^*\rangle\\&=\langle\mathfrak{T}_n^{k+1}x_n-x^*,\mathfrak{T}_n^k x_n-x^*\rangle\\&=\frac{1}{2}(\|\mathfrak{T}_n^{k+1}x_n-x^*\|^2+\|\mathfrak{T}_n^k x_n-x^*\|^2-\|\mathfrak{T}_n^{k+1}x_n-\mathfrak{T}_n^k x_n\|^2).\end{aligned}$$

So, we obtain

$$\begin{aligned}\|\mathfrak{T}_n^{k+1}x_n-x^*\|^2 &\leq\|\mathfrak{T}_n^k x_n-x^*\|^2-\|\mathfrak{T}_n^{k+1}x_n-\mathfrak{T}_n^k x_n\|^2\\&\leq\|x_n-x^*\|^2-\|\mathfrak{T}_n^{k+1}x_n-\mathfrak{T}_n^k x_n\|^2.\end{aligned}$$

Using (3.6) and (3.17), we also have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})\|y_n-x^*\|^2+(1-\eta_n\mu\bar{\gamma})\beta_n\|x_n-x^*\|^2+\eta_n L_n\\&\leq(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})\|u_n-x^*\|^2+(1-\eta_n\mu\bar{\gamma})\beta_n\|x_n-x^*\|^2+\eta_n L_n\\&=(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})\|\mathfrak{T}_n^k x_n-x^*\|^2+(1-\eta_n\mu\bar{\gamma})\beta_n\|x_n-x^*\|^2+\eta_n L_n\\&\leq(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})(\|x_n-x^*\|^2-\|\mathfrak{T}_n^{k+1}x_n-\mathfrak{T}_n^k x_n\|^2)+\eta_n L_n\\&\quad+(1-\eta_n\mu\bar{\gamma})\beta_n\|x_n-x^*\|^2\\&\leq\|x_n-x^*\|^2-(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})\|\mathfrak{T}_n^{k+1}x_n-\mathfrak{T}_n^k x_n\|^2+\eta_n L_n.\end{aligned}$$

Thus, we have

$$(1-\eta_n\mu\bar{\gamma})(1-\beta_n-\eta_n\mu\bar{\gamma})\|\mathfrak{T}_n^{k+1}x_n-\mathfrak{T}_n^k x_n\|^2\leq\|x_{n+1}-x_n\|(\|x_{n+1}-x^*\|+\|x_n-x^*\|)+\eta_n L_n.$$

By the condition (C2) and (3.13), so we deduce that

$$\lim_{n\to\infty}\|\mathfrak{T}_n^k x_n-\mathfrak{T}_n^{k+1}x_n\|=0,\quad\forall k\in\{1,2,3,\ldots,M-1\}. \tag{3.22}$$

Therefore, we have

$$\begin{aligned}\|u_n-x_n\| &=\|\mathfrak{T}_n^k x_n-\mathfrak{T}_n^0 x_n\|\\&\leq\|\mathfrak{T}_n^M x_n-\mathfrak{T}_n^{M-1}x_n\|+\cdots+\|\mathfrak{T}_n^2 x_n-\mathfrak{T}_n^1 x_n\|+\|\mathfrak{T}_n^1 x_n-\mathfrak{T}_n^0 x_n\|.\end{aligned}$$

From (3.22) we have

$$\lim_{n\to\infty}\|\mu_n - x_n\| = 0. \tag{3.23}$$

Step 8. We claim that $\lim_{n\to\infty}\|x_n - T_t x_n\| = 0, \quad \forall t \in S$.

Put

$$K = max\{\|x_1 - x^*\|, \frac{\|rf(x^*) - \mu F x^*\|}{(\mu\bar{\gamma} - r\eta)}\}.$$

Set $D = \{y \in H, \|y - x^*\| \leq H\}$. Then D is a nonempty bounded closed convex set and it is invariant under $\{T_{r_{k,n}}^{F_k} : k = 1,2,\ldots,M, \forall n \in N\}$. Moreover, $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{w_n\}$ and $\{u_n\}$ are in D. From[17], there exists $\delta > 0$ such that

$$\bar{c}oF_\delta(T_t, D) + B_\delta \subseteq \bar{c}oF_\varepsilon(T_t, D), \quad \forall t \in S. \tag{3.24}$$

From Corollary 1.1 in [17], there exists a natural number N such that

$$\|\frac{1}{N+1}\sum_{i=o}^{N} T_{t^i s} y - T_t \frac{1}{N+1}\sum_{i=o}^{N} T_{t^i s} y\| \leq \delta. \tag{3.25}$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$. Since $\{\mu_n\}$ is left regular, there exists $n_0 \in N$ such that

$$\|\mu_n - l_{t^i}^* \mu_n\| \leq \frac{\delta}{3(K + \|x^*\|)}.$$

for all $n \geq n_0$ and $i = 1,2,3,\ldots,N$. So we have for all $n \geq n_0$

$$\begin{aligned}\sup_{y\in D}\|T_{\mu_n} y - \int \frac{1}{N+1}\sum_{i=o}^{N} T_{t^i s} y d\mu_n(s)\| &= \sup_{y\in D}\sup_{\|z\|=1} |(\mu_n)_s\langle T_s y, z\rangle - (\mu_n)_s\langle \frac{1}{N+1}\sum_{i=o}^{N} T_{t^i s} y, z\rangle| \\ &\leq \frac{1}{N+1}\sum_{i=o}^{N}\sup_{y\in D}\sup_{\|z\|=1} |(\mu_n)_s\langle T_s y, z\rangle - (l_{t^i}\mu_n)_s\langle T_s y, z\rangle| \\ &\leq \max_{i=1,2,3,..,N}\|\mu_n - l_{t^i}^*\mu_n\|(K + \|x^*\|) \leq \frac{\delta}{3}.\end{aligned} \tag{3.26}$$

Observe, by Lemma 2.2

$$\int \frac{1}{N+1}\sum_{i=o}^{N} T_{t^i s} y d\mu_n(s) \in \bar{c}o\{\frac{1}{N+1}\sum_{i=o}^{N} T_{t^i}(T_s y) : s \in S\}. \tag{3.27}$$

Combining (3.24)-(3.27), we derive

$$\begin{aligned}T_{\mu_n} y &= \int \frac{1}{N+1}\sum_{i=o}^{N} T_{t^i s} y d\mu_n(s) + (T_{\mu_n} y - \int \frac{1}{N+1}\sum_{i=o}^{N} T_{t^i s} y d\mu_n(s)) \\ &\in \bar{c}o\{\frac{1}{N+1}\sum_{i=o}^{N} T_{t^i}(T_s y) : s \in S\} + B_{\frac{\delta}{3}} \\ &\subseteq \bar{c}oF_\delta(T_t, D) + B_{\frac{\delta}{3}}.\end{aligned} \tag{3.28}$$

for all $y \in D$ and $n \geq n_0$. Let $t \in S$ and $\varepsilon > 0$. Then there exists $\delta > 0$ which satisfies (3.24).

Observe

$$x_{n+1} = T_{\mu_n} y_n + \frac{\beta_n}{1-\beta_n}(x_{n+1} - x_n) + \frac{\eta_n}{1-\beta_n}(rf(y_n) - \mu F T_{\mu_n} y_n).$$

Since $\|x_{n+1} - x_n\| \to 0$ and $\alpha_n \to 0$, there exists $k \in N$ such that

$$\begin{aligned} x_{n+1} &= T_{\mu_n} y_n + \frac{\beta_n}{1-\beta_n}(x_{n+1}-x_n) + \frac{\eta_n}{1-\beta_n}(rf(y_n) - \mu F T_{\mu_n} y_n) \\ &\in \bar{c}oF_{\delta}(T_t, D) + B_{\frac{\delta}{3}} + B_{\frac{\delta}{3}} + B_{\frac{\delta}{3}} \\ &\subseteq \bar{c}oF_{\delta}(T_t, D) + B_{\delta} \subseteq \bar{c}oF_{\varepsilon}(T_t, D). \end{aligned}$$

for all $n > k$. Hence, $\limsup_{n\to\infty}\|x_n - T_t x_n\| \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary,

$$\lim_{n\to\infty}\|x_n - T_t x_n\| = 0. \tag{3.29}$$

Step 9. We show that $\lim_{n\to\infty}\|x_n - T_{\mu_n} y_n\| = 0$.

From the definition of (3.1), we have

$$\begin{aligned} \|x_n - T_{\mu_n} y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_{\mu_n} y_n\| \\ &= \|x_n - x_{n+1}\| + \|\eta_n rf(y_n) + \beta_n x_n + ((1-\beta_n)I - \eta_n \mu F)T_{\mu_n} y_n - T_{\mu_n} y_n\| \\ &= \|x_n - x_{n+1}\| + \|\eta_n (rf(y_n) - \mu F T_{\mu_n} y_n) - \beta_n (x_n - T_{\mu_n} y_n)\| \\ &\le \|x_n - x_{n+1}\| + \eta_n (\|rf(y_n)\| + \|\mu F T_{\mu_n} y_n\|) + \beta_n \|x_n - T_{\mu_n} y_n\|. \end{aligned}$$

It follows that

$$\|x_n - T_{\mu_n} y_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\eta_n}{1-\beta_n}(\|rf(y_n)\| + \|\mu F T_{\mu_n} y_n\|).$$

By the condition (C2) and (3.13), so we have

$$\lim_{n\to\infty}\|x_n - T_{\mu_n} y_n\| = 0. \tag{3.30}$$

Therefore

$$\lim_{n\to\infty}\|x_n - y_n\| \le \lim_{n\to\infty}\|x_n - T_{\mu_n} y_n\| + \lim_{n\to\infty}\|T_{\mu_n} y_n - y_n\| = 0. \tag{3.31}$$

Step 10. We show that $\limsup_{n\to\infty}\langle (rf - \mu F)q, x_n - q\rangle \le 0$, where $q = P_{\Omega}(I - \mu F + rf)(q)$.

The Banach contraction principle guarantees that $P_{\Omega}(I - \mu F + rf)$ has a unique fixed point q which is the unique solution of (3.2). Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{k\to\infty}\langle (rf - \mu F)q, x_{n_k} - q\rangle = \limsup_{n\to\infty}\langle (rf - \mu F)q, x_n - q\rangle.$$

Since $\{x_{n_k}\}$ is bounded, then there exists a subsequence $\{x_{n_{k_i}}\}$ which converges weakly to $z \in H$. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup z$. We claim that $z \in \Omega$. Next, we need to show that $z \in \cap_{j=1}^{M} SEP(F_j)$.

First, by lemma 2.6 and given $y \in C$ and $k \in \{1,2,3,\ldots,M-1\}$. we have

$$\frac{1}{r_{k+1,n}}\langle y - \mathfrak{T}_n^{k+1}x_n, \mathfrak{T}_n^{k+1}x_n - \mathfrak{T}_n^{k}x_n\rangle \ge F_{k+1}(y, \mathfrak{T}_n^{k+1}x_n).$$

Thus,

$$\langle y - \mathfrak{T}_{n_m}^{k+1}x_{n_m}, \frac{\mathfrak{T}_{n_m}^{k+1}x_{n_m} - \mathfrak{T}_{n_m}^{k}x_{n_m}}{r_{k+1,n_m}}\rangle \ge F_{k+1}(y, \mathfrak{T}_{n_m}^{k+1}x_{n_m}). \tag{3.32}$$

From $(A4)$, $F(y,\cdot)$ is a lower semi-continuous and convex, and thus weakly semi-continuous. The condition (C3) and (3.32)imply that

$$\frac{\mathfrak{T}_{n_m}^{k+1}x_{n_m} - \mathfrak{T}_{n_m}^{k}x_{n_m}}{r_{k+1,n_m}} \to 0. \tag{3.33}$$

in norm. Therefore, letting $m \to \infty$ in (3.32) yields

$$F_{k+1}(y,z) \le \limsup_{m\to\infty} F_{k+1}(y, \mathfrak{T}_{n_m}^{k+1} x_{n_m}) \le 0.$$

for all $y \in H$ and $k \in \{1,2,3,\dots,M-1\}$. Replacing y with $z_t = ty + (1-t)z$ with $t \in (0,1)$ and using (A1) and (A4), we obtain

$$0 = F_{k+1}(z_t, z_t) \le tF_{k+1}(z_t, y) + (1-t)F_{k+1}(z_t, z) \le tF_{k+1}(z_t, y).$$

Hence, $F_{k+1}(ty + (1-t)z, y) \ge 0$, for all $t \in (0,1)$ and $y \in H$. Letting $t \to 0^+$ and using (A3), we conclude that $F_{k+1}(y,z) \ge 0$ for all $y \in H$ and $k \in \{1,2,3,\dots,M-1\}$. Therefore,

$$z \in EP(F_j), \quad \forall j = 1,2,3,\dots,M.$$

that is, $z \in \cap_{j=1}^{M} SEP(F_j)$.

Next, we show that $z \in (A+M)^{-1}(0)$. Notice that

$$u_n - \xi_n A u_n \in v_n + \xi_n M v_n.$$

Let $\mu \in Mv$. Since M is monotone, we have

$$\langle \frac{u_n - v_n}{\xi_n} - Au_n - \mu, v_n - v \rangle \ge 0.$$

In view of the restriction (C3), we see from (3.18) that

$$\langle -Az - \mu, z - v \rangle \ge 0.$$

This implies that $-Az \in Mz$, that is, $z \in (A+M)^{-1}(0)$. In a similar way, we can obtain that $z \in (B+W)^{-1}(0)$.

Next, by lemma 2.9 and (3.29)we directly obtain that $z \in F(S)$. This proves that $z \in \Omega$.

Hence, by Lemma 2.8, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \langle (rf - \mu F)q, x_n - q \rangle &= \limsup_{k\to\infty} \langle (rf - \mu F)q, x_{n_k} - q \rangle \\ &= \langle (rf - \mu F)q, z - q \rangle \le 0. \end{aligned} \tag{3.34}$$

Step 11. Finally, we claim that $\{x_n\}$ converges strongly to $q = P_\Omega(I - \mu F + rf)(q)$.

We observe that

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &= \|((1-\beta_n)I-\eta_n\mu F)(T_{\mu_n}y_n-q)+\beta_n(x_n-q)+\eta_n(rf(y_n)-\mu Fq)\|^2 \\
&\le \|((1-\beta_n)I-\eta_n\mu F)(T_{\mu_n}y_n-q)+\beta_n(x_n-q)\|^2+2\eta_n\langle rf(y_n)-\mu Fq, x_{n+1}-q\rangle \\
&= \|(1-\beta_n)\frac{(1-\beta_n)I-\eta_n\mu F}{(1-\beta_n)}(T_{\mu_n}y_n-q)+\beta_n(x_n-q)\|^2 \\
&\quad +2\eta_n r\langle f(y_n)-f(q), x_{n+1}-q\rangle+2\eta_n\langle rf(q)-\mu Fq, x_{n+1}-q\rangle \\
&\le (1-\beta_n)\|\frac{(1-\beta_n)I-\eta_n\mu F}{(1-\beta_n)}(T_{\mu_n}y_n-q)\|^2+\beta_n\|x_n-q\|^2 \\
&\quad +2\eta_n r\eta\|y_n-q\|\|x_{n+1}-q\|+2\eta_n\langle rf(q)-\mu Fq, x_{n+1}-q\rangle \\
&\le (1-\beta_n)\|\frac{(1-\beta_n)I-\eta_n\mu F}{(1-\beta_n)}(T_{\mu_n}y_n-q)\|^2+\beta_n\|x_n-q\|^2 \\
&\quad +2\eta_n r\eta\|x_n-q\|\|x_{n+1}-q\|+2\eta_n\langle rf(q)-\mu Fq, x_{n+1}-q\rangle \\
&\le \frac{((1-\beta_n)I-\eta_n\mu F)^2}{(1-\beta_n)}\|x_n-q\|^2+\beta_n\|x_n-q\|^2+\eta_n r\eta(\|x_{n+1}-q\|^2+\|x_n-q\|^2) \\
&\quad +2\eta_n\langle rf(q)-\mu Fq, x_{n+1}-q\rangle \\
&\le (\frac{((1-\beta_n)I-\eta_n\mu\bar{\gamma})^2}{(1-\beta_n)}+\beta_n+\eta_n r\eta)\|x_n-q\|^2+\eta_n r\eta\|x_{n+1}-q\|^2 \\
&\quad +2\eta_n\langle rf(q)-\mu Fq, x_{n+1}-q\rangle \\
&= (1-\eta_n(2\mu\bar{\gamma}-r\eta)+\frac{\eta_n^2\mu^2\bar{\gamma}^2}{1-\beta_n})+\eta_n r\eta\|x_{n+1}-q\|^2+2\eta_n\langle rf(q)-\mu Fq, x_{n+1}-q\rangle.
\end{aligned}
$$

So we have,

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &\le (1-\frac{2\eta_n(\mu\bar{\gamma}-r\eta)}{1-\eta_n r\eta})\|x_n-q\|^2+\frac{2\eta_n(\mu\bar{\gamma}-r\eta)}{1-\eta_n r\eta}(\frac{1}{\mu\bar{\gamma}-r\eta}\langle rf(q) \\
&\quad -\mu Fq, x_{n+1}-q\rangle+\frac{\eta_n\mu^2\bar{\gamma}^2}{2(1-\beta_n)(\mu\bar{\gamma}-r\eta)}\|x_n-q\|^2).
\end{aligned} \tag{3.35}
$$

where $\tau_n=\frac{1}{\mu\bar{\gamma}-r\eta}\langle rf(q)-\mu Fq, x_{n+1}-q\rangle+\frac{\eta_n\mu^2\bar{\gamma}^2}{2(1-\beta_n)(\mu\bar{\gamma}-r\eta)}\|x_n-q\|^2$, $\varepsilon_n=\frac{2\eta_n(\mu\bar{\gamma}-r\eta)}{1-\eta_n r\eta}$. From the condition (C4) and (3.34), we see that

$$
\sum_{n=1}^{\infty}\varepsilon_n=\infty, \limsup_{n\to\infty}\frac{\tau_n}{\varepsilon_n}\le 0.
$$

By Lemma 2.4 we obtain the sequence $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converges strongly to a point $q\in\Omega$, This completes the proof.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space space H, L be a bi-function from $C\times C$ to R satisfying $A(1)-A(4)$ and $\{T_i\}_{i=1}^{\infty}$ an infinite family of non-expansive mapping of C into C such that $\Omega:=\cap_{i=1}^{\infty}Fix(T_i)\cap EP(L)\neq\varnothing$, f be a contraction of C into itself with coefficient $\eta\in(0,1)$, F is a strongly positive linear bounded operator on H with coefficient $\bar{\gamma}>0$ and $0<\gamma<\frac{\bar{\gamma}}{\eta}$. If the sequences $\{x_n\}$ is generated iteratively by $x_1\in C$ and*

$$
x_{n+1}=\eta_n rf(x_n)+\beta_n x_n+((1-\beta_n)I-\eta_n\mu F)W_nT_{r_n}^{L}x_n
$$

where $\{r_n\}$ are a real sequence in $(0,\infty)$ and $\{\beta_n\}$, $\{\eta_n\}$ are two sequences in $(0,1)$. Assume that the following restrictions are satisfied

(C1) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(C2) $\lim_{n\to\infty} \eta_n = 0$ and $\Sigma_{n=1}^{\infty} \eta_n = \infty$;

(C3) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} | r_{n+1} - r_n | = 0$.

Then the sequence $\{x_n\}$ converges strongly to a point $q \in \Omega$, which is the unique solution of the variational inequality

$$\langle (\mu F - rf)q, p - q\rangle \ge 0, \quad \forall p \in \Omega.$$

Equivalently, we have $q = P_{\Omega}(I - \mu F + rf)(q)$.

Proof Put $A = B = 0$, $\phi = \{I\}$, $F_1 = L$ and $F_k = 0$ for $k = \{2,3,\ldots,M\}$ in Theorem 3.1, then we have $T_{\mu_n} = I$ and $y_n = u_n = T_{r_n}^k x_n$. In view of Theorem 3.1, we can obtain the desired result immediately.

Corollary 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space space H, f be a contraction of C into itself with coefficient $\eta \in (0,1)$, F is a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\eta}$. Let $\{F_k, k = 1,2,\ldots,M\}$ be a finite family of bi-functions from $C \times C$ to R which satisfy $A(1) - A(4)$, $J : C \to C$ be a non-expansive mapping. Assume that $\Omega := Fix(J) \cap \cap_{j=1}^{M} SEP(F_j) \ne \varnothing$. If the sequences $\{x_n\}$ is generated iteratively by $x_1 \in C$ and*

$$x_{n+1} = \eta_n rf(y_n) + \beta_n x_n + ((1-\beta_n)I - \eta_n \mu F)\frac{1}{n}\sum_{k=0}^{\infty}(\frac{n-1}{n})^k J^k T_{r_{M,n}}^{F_M} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} x_n$$

where $\{r_{k,n}\}_{k=1}^{M}$ are a real sequence in $(0,\infty)$ and $\{\beta_n\}$, $\{\eta_n\}$ are two sequences in $(0,1)$. Assume that the following restrictions are satisfied

(C1) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(C2) $\lim_{n\to\infty} \eta_n = 0$ and $\Sigma_{n=1}^{\infty} \eta_n = \infty$;

(C3) $\liminf_{n\to\infty} r_{k,n} > 0$ and $\lim_{n\to\infty} | r_{k,n+1} - r_{k,n} | = 0$ for each $k \in \{1,2,3,\ldots,M\}$.

Then the sequence $\{x_n\}$ converges strongly to a point $q \in \Omega$, which is the unique solution of the variational inequality

$$\langle (\mu F - rf)q, p - q\rangle \ge 0, \quad \forall p \in \Omega.$$

Equivalently, we have $q = P_{\Omega}(I - \mu F + rf)(q)$.

Proof Take $S = \{0,1,\ldots\}$, $\phi = \{T^i : i \in S\}$ and $T^0 = I$. For $f = (x_0, x_1, \ldots) \in B(S)$, define

$$\mu_n(f) = \frac{1}{n}\sum_{k=0}^{\infty}(\frac{n-1}{n})^k x_k, \quad \forall n \in N.$$

Then $\{\mu_n\}$ a left regular sequence of means on $B(S)$ such that $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$; for more details, see [7]. Next for each $x \in H$ and $n \in N$, we have

$$T_{\mu_n} x = \frac{1}{n}\sum_{k=0}^{\infty}(\frac{n-1}{n})^k J^k x.$$

Put $A = B = 0$, $T_i = I$ for all $i \in N$ in Theorem 3.1 then we have $y_n = z_n$ and $W_n = I$ for all $n \in N$. Therefore, it follows from (3.1) that the sequence $\{x_n\}$ converges strongly to a point $q \in \Omega$.

Corollary 3.4 *Let C be a nonempty closed convex subset of a real Hilbert space space H, P_C be the metric projection from H onto C, $A : C \to H$ be an $\alpha-$inverse strongly monotone, $B : C \to H$ be an $\beta-$inverse strongly monotone, f be a contraction of C into itself with coefficient $\eta \in (0,1)$, F is a strongly positive linear* *bounded operator on* H *with coefficient* $\bar{\gamma} > 0$ *and* $0 < \gamma < \frac{\bar{\gamma}}{\eta}$. *Let* $\{F_k, k = 1,2,\ldots,M\}$ *be a finite family of bi-functions from* $C \times C$ *to* R *which satisfy* $A(1) - A(4)$, S *a semigroup and* $\phi = \{T_t x : t \in S\}$ *be a non-expansive semigroup from* C *into* C *such that* $Fix(\phi) = \cap_{t\in S} Fix(T_t) \neq \emptyset$. *Let* X *be a left invariant subspace of* $B(S)$ *such that* $1 \in X$, *and the function* $t \mapsto \langle T_t x, y\rangle$ *is an element of* X *for each* $x \in C$ *and* $y \in H$, $\{\mu_n\}$ *a left regular sequence of means on* X *such that* $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$. $M : H \to 2^H$ *and* $N : H \to 2^H$ *be maximal monotone operators such that* $D(M) \subset C$ *and* $D(W) \subset C$. *Assume that* $\Omega := F(S) \cap VI(C, A) \cap VI(C, B) \cap \cap_{j=1}^{M} SEP(F_j) \neq \emptyset$. *If the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *are generated iteratively by* $x_1 \in C$ *and*

$$\begin{cases} u_n = T_{r_{M,n}}^{F_M} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} x_n \\ y_n = \delta_n P_C(u_n - \xi_n A u_n) + (1-\delta_n) P_C(u_n - s_n B u_n) \\ x_{n+1} = \eta_n r f(y_n) + \beta_n x_n + ((1-\beta_n) I - \eta_n \mu F) T_{\mu_n} y_n \end{cases}$$

where $\{\xi_n\}$ is a sequence in $(0,2\alpha)$, $\{s_n\}$ is a sequence in $(0,2\beta)$, $\{r_{k,n}\}_{k=1}^{M}$ are a real sequence in $(0,\infty)$ and $\{\beta_n\}$, $\{\eta_n\}$, $\{\delta_n\}$ are three sequences in $(0,1)$. Assume that the following restrictions are satisfied

(C1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(C2) $\lim_{n\to\infty} \eta_n = 0$ and $\Sigma_{n=1}^{\infty} \eta_n = \infty$;

(C3) $0 < a \leq \xi_n \leq b < 2\alpha$, $0 < c \leq s_n \leq d < 2\beta$ and $0 < k \leq \delta_n \leq e < 1$;

(C4) $\lim_{n\to\infty} |\eta_{n+1} - \eta_n| = \lim_{n\to\infty} |s_{n+1} - s_n| = \lim_{n\to\infty} |\delta_{n+1} - \delta_n| = 0$;

(C5) $\liminf_{n\to\infty} r_{k,n} > 0$ and $\lim_{n\to\infty} |r_{k,n+1} - r_{k,n}| = 0$ for each $k \in \{1,2,3,\ldots,M\}$, where a, b, c, d, k, e are real numbers. Then the sequence $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converges strongly to a point $q \in \Omega$, which is the unique solution of the variational inequality

$$\langle (\mu F - rf) q, p - q\rangle \geq 0, \quad \forall p \in \Omega.$$

Equivalently, we have $q = P_\Omega(I - \mu F + rf)(q)$.

Proof Take $M = W = \partial I_C$. Next, we show that $VI(C, A) = (A + \partial I_C)^{-1}(0)$ and $VI(C, B) = (B + \partial I_C)^{-1}(0)$. Notice that

$$\begin{aligned} x \in (A + \partial I_C)^{-1}(0) \quad &\Leftrightarrow \quad 0 \in Ax + \partial I_C x \quad \Leftrightarrow \quad -Ax \in \partial I_C x \\ &\Leftrightarrow \quad \langle Ax, y - x\rangle \geq 0 \quad \Leftrightarrow \quad x \in VI(C, A). \end{aligned}$$

In the same way, we can obtain that $x \in (B + \partial I_C)^{-1}(0) = x \in VI(C, B)$. From Lemma 2.9, we can conclude the desired conclusion immediately.

Corollary 3.5 *Let* C *be a nonempty closed convex subset of a real Hilbert space space* H, $J : C \to H$ *be an* $\alpha-$*strictly pseudo-contractive,* $R : C \to H$ *be an* $\beta-$*strictly pseudo-contractive,* f *be a contraction of* C *into itself with coefficient* $\eta \in (0,1)$, F *is a strongly positive linear bounded operator on* H *with coefficient* $\bar{\gamma} > 0$ *and* $0 < \gamma < \frac{\bar{\gamma}}{\eta}$. *Let* $\{F_k, k = 1,2,\ldots,M\}$ *be a finite family of bi-functions from* $C \times C$ *to* R *which satisfy* $A(1) - A(4)$, S *a semigroup and* $\phi = \{T_t x : t \in S\}$ *be a non-expansive semigroup from* C *into* C *such* *that* $Fix(\phi)=\cap_{t\in S}Fix(T_t)\neq\varnothing$. *Let* X *be a left invariant subspace of* $B(S)$ *such that* $1\in X$, *and the function* $t\mapsto\langle T_t x,y\rangle$ *is an element of* X *for each* $x\in C$ *and* $y\in H$, $\{\mu_n\}$ *a left regular sequence of means on* X *such that* $\lim_{n\to\infty}\|\mu_{n+1}-\mu_n\|=0$. *Assume that* $\Omega:=F(S)\cap F(J)\cap F(R)\cap\cap_{j=1}^{M}SEP(F_j)\neq\varnothing$. *If the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *are generated iteratively by* $x_1\in C$ *and*

$$\begin{cases} u_n=T_{r_{M,n}}^{F_M}\cdots T_{r_{2,n}}^{F_2}T_{r_{1,n}}^{F_1}x_n \\ y_n=\delta_n((1-\xi_n)u_n+\xi_n Ju_n)+(1-\delta_n)((1-s_n)u_n+s_n Ru_n) \\ x_{n+1}=\eta_n rf(y_n)+\beta_n x_n+((1-\beta_n)I-\eta_n\mu F)T_{\mu_n}y_n \end{cases}$$

where $J_{\xi_n}=(I+\xi_n M)^{-1}$, $J_{s_n}=(I+s_n W)^{-1}$, $\{\xi_n\}$ is a sequence in $(0,2\alpha)$, $\{s_n\}$ is a sequence in $(0,2\beta)$, $\{r_{k,n}\}_{k=1}^{M}$ are a real sequence in $(0,\infty)$ and $\{\beta_n\}$, $\{\eta_n\}$, $\{\delta_n\}$ are three sequences in $(0,1)$. Assume that the following restrictions are satisfied

(C1) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;

(C2) $\lim_{n\to\infty}\eta_n=0$ and $\Sigma_{n=1}^{\infty}\eta_n=\infty$;

(C3) $0<a\leq\xi_n\leq b<2\alpha$, $0<c\leq s_n\leq d<2\beta$ and $0<k\leq\delta_n\leq e<1$;

(C4) $\lim_{n\to\infty}|\eta_{n+1}-\eta_n|=\lim_{n\to\infty}|s_{n+1}-s_n|=\lim_{n\to\infty}|\delta_{n+1}-\delta_n|=0$;

(C5) $\liminf_{n\to\infty}r_{k,n}>0$ and $\lim_{n\to\infty}|r_{k,n+1}-r_{k,n}|=0$ for each $k\in\{1,2,3,\ldots,M\}$; where a, b, c, d, k, e are real numbers. Then the sequence $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converges strongly to a point $q\in\Omega$, which is the unique solution of the variational inequality

$$\langle(\mu F-rf)q,p-q\rangle\geq 0,\quad\forall p\in\Omega.$$

Equivalently, we have $q=P_\Omega(I-\mu F+rf)(q)$.

Proof *Put* $A=I-J$, *we see that* A *is* $\frac{1-\alpha}{2}-$ *inverse strongly monotone. We also have* $F(J)=VI(C,A)$ *and* $P_C(u_n-\xi_n Au_n)=(1-\xi_n)u_n+\xi_n Ju_n$. *Put* $B=I-R$, *we see that* B *is* $\frac{1-\beta}{2}-$ *inverse strongly monotone. We also have* $F(R)=VI(C,B)$ *and* $P_C(u_n-s_n Au_n)=(1-s_n)u_n+s_n Ju_n$. *In view of Theorem 3.1, we can obtain the desired result immediately.*

Remark 3.6 *Theorem 3.1 improve in the following aspects.*

(a) Our iterative process (3.1) is more general than [3],[4] because it can be applied to solving the problem of finding a common element of the set of solutions of systems of equilibrium problems and obtain the strong convergence.

(b) Our iterative process (3.1) is very diffident from [3],[4]because there are left amenable semigroup of non-expansive mappings.

(c) Our method of proof is very different from [3],[4] and our Corollary extents and improves Theorem 3.1 from a pair of inverse strongly monotone to an infinite family of non-expansive mappings and strictly pseudo-contractive mappings.

ACKNOWLEDGMENT

This project is supported by the National Natural Science Foundation of China (Grant No.11371015), the Key Project of Chinese Ministry of Education(Grant No.211163), Sichuan Youth Science and Technology Foundation(Grant No.2012JQ0032), the Foundation of China West Normal University(Grant Nos.11A028;11A029) and the Fundamental Research Funds of China West Normal University(Grant No.13D016).

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