UNIVERSAL BOUNDS FOR EIGENVALUES OF FOURTH-ORDER WEIGHTED POLYNOMIAL OPERATOR ON DOMAINS IN COMPLEX PROJECTIVE SPACES

Du Feng[1] & Li Yanli[2,*]
[1]School of Mathematics and Physics Science, Jingchu University of Technology,
Hubei Jingmen 448000, P.R.China
[2]School of Electronic and Information Science, Jingchu University of Technology,
Hubei Jingmen 448000, P.R.China
*E-mail: 769207397@qq.com.com

ABSTRACT

In this paper, we study the eigenvalues problem of fourth-order weighted polynomial operator and get a general inequality on compact Riemannian manifolds. By using this general inequality, we obtain universal bounds for the *k*-th eigenvalue in terms of the lower eigenvalues independently of the domains.

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1. INTRODUCTION

Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Let Δ be the Laplacian operator acting on functions on M and consider the following eigenvalue problem for the biharmonic operator

$$\begin{cases} \Delta^2 u = -\lambda u, & in\ \ \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & on\ \ \partial\Omega, \end{cases} \tag{1.1}$$

where ν denotes the outward unit normal vector field of $\partial\Omega$. It is known that this eigenvalue problem has a discrete spectrum,

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \cdots,$$

where each eigenvalue is repeated with its multiplicity. When $M = R^n, \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, Payne-Pólya-Weinberger [8] in 1956 proved

$$\lambda_{k+1} - \lambda_k \le \frac{8n+2}{n^2}\sum_{i=1}^k \lambda_i. \tag{1.2}$$

In 1984, Hile and Yeh [6] strengthened (1.2), and proved

$$\frac{n^2 k^{\frac{3}{2}}}{8n+2}\left(\sum_{i=1}^k \lambda_i\right)^{\frac{1}{2}} \le \sum_{i=1}^k \frac{\lambda_i^{\frac{1}{2}}}{\lambda_{k+1}-\lambda_i}. \tag{1.3}$$

In 2006, Cheng-Yang [11] gave the following much stronger inequality

$$\lambda_{k+1} \le \frac{1}{k}\sum_{i=1}^k \lambda_i + \left(\frac{8(n+2)}{n^2}\right)^{\frac{1}{2}} \frac{1}{k}\left(\sum_{i=1}^k \lambda_i(\lambda_{k+1}-\lambda_i)\right). \tag{1.4}$$

These inequalities are called universal inequalities because they do not involve domain dependence.
In this paper, we consider the following eigenvalue problem of fourth-order weighted polynomial operator

$$\begin{cases} \left(\Delta^2 - a\Delta + b\right) u = \lambda \rho u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where ρ is a positive and continuous function on Ω, and the constants $a, b \geq 0$.

In 2010, Sun-Chen [9] obtain universal inequalities for problem (1.5) in n-dimensional Euclidean space, they proved

$$\lambda_{k+1} \leq \frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{(1+\alpha)(2n+4)^{\frac{1}{2}}}{n\beta}\frac{1}{k}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}B_i$$
$$a\frac{\alpha}{\beta}(2n+4)^{\frac{1}{2}}\frac{1}{k}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}, \tag{1.6}$$

where

$\alpha = (inf_{\Omega}\rho)^{-1}, \beta = (sup_{\Omega}\rho)^{-1}$, and $B_i = \frac{1}{2}\left(-a\alpha + \sqrt{a^2\alpha^2 + 4\alpha(\lambda_i - b\beta)}\right)$.

In this paper, we will consider the problem (1.5) on bounded domains in the n-dimensional complex projective space $CP^n(4)$. Then we obtain

Theorem 1.1. Assume that Ω be a bounded domain in n-dimensional complex projective space $CP^n(4)$, let λ_i be the i^{th} eigenvalue of the eigenvalue problem (1.5) and $\forall x \in \Omega, \rho_1 \leq \rho(x) \leq \rho_2$,

$$\lambda_{k+1} \leq \frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{2\rho_2}{kn}\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\left((n+1)A_i + \frac{2n(n+1)+a}{\rho_1}\right)\right\}^{\frac{1}{2}}$$
$$\times\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\frac{1}{\rho_1}\left(A_i + \frac{2n(n+1)}{\rho_1}\right)\right\}^{\frac{1}{2}}. \tag{1.7}$$

where $A_i = \frac{-a+\sqrt{a^2+4\left(\lambda_i - \frac{b}{\rho_2}\right)}}{2\rho_1}$.

From the theorem 1.1, we can get the following weaker but more explicit inequality

Corollary 1.2. Under the assumption of the theorem 1.1, we have

$$\lambda_{k+1} \leq \frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{(n+1)^{\frac{1}{2}}}{nk}\left(\rho_2 + \frac{\rho_2}{\rho_1}\right)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}A_i$$
$$+\frac{1}{nk}\frac{\rho_2}{\rho_1}\left(\frac{32n(n+1)^{\frac{3}{2}}+a}{\rho_1} + 4n(n+1)^{\frac{1}{2}}\right)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}. \tag{1.8}$$

where $A_i = \frac{-a+\sqrt{a^2+4\left(\lambda_i - \frac{b}{\rho_2}\right)}}{2\rho_1}$.

2. A general inequality

In this section, we will introduce a general inequality which is play a key role in the proof of the main results.

Lemma 2.1. Let(M, $\langle , \rangle$) be an n-dimensional compact Riemannian manifold with boundary ∂M (possibly empty).

Let λ_i be the i^{th} eigenvalue of the eigenvalue problem of fourth-order weighted polynomial operator with weight *p* such that

$$\begin{cases} \left(\Delta^2 - a\Delta + b\right) u = \lambda\rho u, & \text{in } M, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial M, \end{cases}$$

and u_i be the orthonormal eigenfunction corresponding to λ_i, that is,

$$\begin{cases} \left(\Delta^2 + a\Delta + b\right) u_i = \lambda_i \rho u_i, & \text{in } M, \\ u_i = 0, & \text{on } \partial\Omega, \\ \int_M \rho u_i u_j = \delta_{ij}, & \forall\, i,j = 1,2,\cdots, \end{cases}$$

Then for any $h \in C^4(\overline{M})$, we have

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \int_M u_i^2 |\nabla h|^2$$
$$\leq \sum_{i=1}^{k} \delta (\lambda_{k+1} - \lambda_i)^{\frac{1}{2}} \int_M h u_i p_i + \sum_{i=1}^{k} \frac{1}{\delta} (\lambda_{k+1} - \lambda_i)^{\frac{1}{2}} \int_M \frac{1}{\rho} \left(\langle \nabla h, \nabla u_i \rangle + \frac{u_i \Delta h}{2} \right)^2 . \tag{2.1}$$

where

$$p_i = 2\langle \nabla h, \nabla(\Delta u_i)\rangle + \Delta h \Delta u_i + 2\Delta \langle \nabla h, \nabla u_i \rangle$$
$$+\Delta\left(u_i \Delta h\right) - 2a\langle \nabla h, \nabla u_i\rangle - a u_i \Delta h.$$

proof Let $\varphi_i = h u_i - \sum_{j=1}^{k} a_{ij} u_j$ for any integer $k \geq 1$, where

$$a_{ij} = \sum_{j=1}^{k} \int_M \rho h u_i u_j = a_{ji},$$

then we have

$$\varphi_i |_{\partial M} = 0, \text{ and} \int_M \rho \varphi_i u_j = 0, \forall i, j = 1, \cdots, k,$$

from the Rayleigh-Ritz inequality, we get

$$\lambda_{k+1} \int_M \rho \varphi_i^2 \leq \int_M \varphi_i (\Delta^2 + a\Delta + b) \varphi_i . \tag{2.2}$$

By directly computation, we have

$$\Delta(h u_i) = h \Delta u_i + 2\langle \nabla h, \nabla u_i \rangle + u_i \Delta h, \tag{2.3}$$

and

$$\begin{aligned} \Delta^2(hu_i) &= \Delta\left(h\Delta u_i + 2\langle \nabla h, \nabla u_i\rangle + u_i \Delta h,\right) \\ &= h\Delta^2 u_i + 2\langle \nabla h, \nabla(\Delta u_i)\rangle + \Delta h \Delta u_i + 2\Delta\left(\langle \nabla h, \nabla u_i\rangle\right) + \Delta\left(u_i \Delta h\right) \end{aligned} \tag{2.4}$$

By (2.3) and (2.4), we have

$$(\Delta^2 + a\Delta + b)(h u_i) = \lambda_i \rho h u_i + p_i , \tag{2.5}$$

where

$$p_i = 2\langle \nabla h, \nabla(\Delta u_i)\rangle + \Delta h \Delta u_i + 2\Delta \langle \nabla h, \nabla u_i \rangle$$
$$+ \; u_i \Delta h \text{ - } 2a\langle \nabla h, \nabla u_i \rangle \text{ - } \; a u_i \nabla h$$

Because of $\int_M \rho \varphi_i u_j = 0$, we can get

$$\int_M \varphi_i (\Delta^2 + a\Delta + b) \varphi_i = \int_M \varphi_i (\Delta^2 + a\Delta + b)(h u_i)$$
$$= \lambda_i \int_M \varphi_i \rho h u_i + \int_M \varphi_i p_i$$

$$= \lambda_i \int_M \rho\varphi_i^2 + \int_M hu_i p_i - \sum_{j=1}^{k} a_{ij} b_{ij}, \tag{2.6}$$

where $b_{ij} = \int_M p_i u_j$.

By (2.2) and (2.6), we have

$$(\lambda_{k+1} - \lambda_i)\int_M \rho\varphi_i^2 \le \int_M hu_i p_i - \sum_{j=1}^{k} a_{ij} b_{ij}, \tag{2.7}$$

Using integration by parts, we have

$$\begin{aligned}
&\int_M \Delta u_j \langle \nabla h, \nabla u_i\rangle - \int_M \Delta u_i \langle \nabla h, \nabla u_j\rangle \\
&= -\int_M h div(\Delta u_j \nabla u_i) + \int_M h div(\Delta u_i \nabla u_j) \\
&= -\int_M h\langle \nabla(\Delta u_j), \nabla u_i\rangle + \int_M h\Delta u_j \Delta u_i + \int_M h\langle \nabla(\Delta u_i), \nabla u_j\rangle - \int_M h\Delta u_i \Delta u_j \\
&= -\int_M h\langle \nabla(\Delta u_j), \nabla u_i\rangle + \int_M h\langle \nabla(\Delta u_i), \nabla u_j\rangle \\
&= \int_M u_i div(h\nabla(\Delta u_j)) - \int_M u_j div(h\nabla(\Delta u_i)) \\
&= \int_M hu_i \Delta^2 u_j + \int_M u_i \langle \nabla h, \nabla(\Delta u_j)\rangle - \int_M hu_j \Delta^2 u_i - \int_M u_j \langle \nabla h, \nabla(\Delta u_i)\rangle \\
&= -\int_M hu_i \Delta^2 u_j + \int_M hu_j \Delta^2 u_i - \int_M \Delta u_j (\langle \nabla u_i, \nabla h\rangle - u_i \Delta h) + \\
&\quad \int_M \Delta ui(\langle \nabla u_j, \nabla h\rangle - u_j \Delta h),
\end{aligned} \tag{2.8}$$

which implies that

$$\begin{aligned}
&2\int_M \Delta u_j \langle \nabla h, \nabla u_i\rangle - 2\int_M \Delta u_i \langle \nabla h, \nabla u_j\rangle \\
&= -\int_M hu_i \Delta^2 u_j + \int_M hu_j \Delta^2 u_i + \int_M u_i \Delta u_j \Delta h - \int_M u_j \Delta u_i \Delta h.
\end{aligned} \tag{2.9}$$

We also have

$$\begin{aligned}
&\int_M u_j \Delta\langle \nabla h, \nabla u_i\rangle + \int_M u_j \langle \nabla h, \nabla(\Delta u_i)\rangle \\
&= \int_M \Delta u_j \langle \nabla h, \nabla u_i\rangle - \int_M \Delta u_i \langle \nabla h, \nabla u_j\rangle + \int_M u_j \Delta u_i \Delta h,
\end{aligned} \tag{2.10}$$

$$\int_M u_j \Delta(u_i \Delta h) = \int_M u_i \Delta u_j \Delta h, \tag{2.11}$$

and

$$\begin{aligned}
&\int_M u_j \{-2\langle \nabla h, \nabla u_i\rangle + u_i \Delta h\} \\
&= \int_M 2h\langle \nabla u_j, \nabla u_i\rangle - \int_M 2hu_j \Delta u_i + \int_M h\Delta(u_i u_j) \\
&= \int_M hu_i \Delta u_j - \int_M hu_j \Delta u_i.
\end{aligned} \tag{2.12}$$

Combining (2.9)-(2.12), we get

$$\begin{aligned} b_{ij} &= \int_M p_i u_j \\ &= \int_M u_j \left\{2\langle\nabla h, \nabla(\Delta u_i)\rangle + \Delta h \Delta u_i + 2\Delta\left(\langle\nabla h, \nabla u_i\rangle\right)\right\} \\ &\quad + \int_M u_j \left\{\Delta\left(u_i \Delta h\right) - 2a\langle\nabla h, \nabla u_i\rangle - a u_i \Delta h\right\} \\ &= -\int_M h u_i \Delta^2 u_j - \int_M h u_j \Delta^2 u_i + \int_M a h u_i \Delta u_j - \int_M a h u_j \Delta u_i \\ &= \int_M h u_i (\Delta^2 + a\Delta + b)(u_j) - \int_M h u_j (\Delta^2 + a\Delta + b)(u_i) \\ &= (\lambda_j - \lambda_i) a_{ij}. \end{aligned} \tag{2.13}$$

It follows from (2.7) and (2.13) that

$$(\lambda_{k+1} - \lambda_i)\int_M \rho\varphi_i^2 \le \int_M h u_i p_i - \sum_{j=1}^k (\lambda_j - \lambda_i) a_{ij}^2. \tag{2.14}$$

Setting $t_{ij} = \int_M u_j \langle\nabla h, \nabla u_i\rangle + \frac{u_i \Delta h}{2}$, then t_{ij}=- $-t_{ji}$ and

$$\begin{aligned} &\int_M -2\varphi_i(\langle\nabla h, \nabla u_i\rangle + \frac{u_i \Delta h}{2}) \\ &= \int_M \left(-2h u_i \langle\nabla h, \nabla u_i\rangle - h u_i^2 \Delta h\right) + 2\sum_{j=1}^k a_{ij} t_{ij} \\ &= \int_M u_i^2 |\nabla h|^2 + 2\sum_{j=1}^k a_{ij} t_{ij}. \end{aligned} \tag{2.15}$$

By (2.14), (2.15) and Schwarz inequality, we get

$$\begin{aligned} &(\lambda_{k+1} - \lambda_i)(\int_M u_i^2 |\nabla h|^2 + 2\sum_{j=1}^k a_{ij} t_{ij}) \\ &= (\lambda_{k+1} - \lambda_i)\int_M -2\sqrt{\rho}\varphi_i \left(\frac{1}{\sqrt{\rho}}\left(\langle\nabla h, \nabla u_i\rangle + \frac{u_i \Delta h}{2}\right) - \sum_{j=1}^k t_{ij}\sqrt{\rho} u_j\right) \\ &\le \delta(\lambda_{k+1} - \lambda_i)^{\frac{3}{2}} \int_M \rho\varphi_i^2 + \frac{1}{\delta}(\lambda_{k+1} - \lambda_i)^{\frac{1}{2}} \int_M \left(\frac{1}{\sqrt{\rho}}\left(\langle\nabla h, \nabla u_i\rangle + \frac{u_i \Delta h}{2}\right) - \sum_{j=1}^k t_{ij}\sqrt{\rho} u_j\right)^2 \\ &\le \delta(\lambda_{k+1} - \lambda_i)^{\frac{1}{2}} \left(\int_M h u_i p_i - \sum_{j=1}^k (\lambda_j - \lambda_i) a_{ij}^2\right) \\ &+ \frac{1}{\delta}(\lambda_{k+1} - \lambda_i)^{\frac{1}{2}} \left(\int_M \frac{1}{\rho}\left(\langle\nabla h, \nabla u_i\rangle + \frac{u_i \Delta h}{2}\right)^2 - \sum_{j=1}^k t_{ij}^2\right), \end{aligned} \tag{2.16}$$

where δ is any positive constant. Summing over i from 1 to k in (2.16), we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_M u_i^2 |\nabla h|^2 + 2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)a_{ij}t_{ij}$$
$$\leq \sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\int_M hu_i p_i + \sum_{i=1}^{k}\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\int_M \frac{1}{\rho}\left(\langle\nabla h,\nabla u_i\rangle + \frac{u_i\Delta h}{2}\right)^2$$
$$+\sum_{i,j=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}(\lambda_i-\lambda_j)a_{ij}^2 - \sum_{i,j=1}^{k}\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}t_{ij}^2. \tag{2.17}$$

Let us compute

$$\sum_{i,j=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}(\lambda_j-\lambda_i)a_{ij}^2$$
$$= \frac{\delta}{2}\sum_{i,j=1}^{k}\left((\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}-(\lambda_{k+1}-\lambda_j)^{\frac{1}{2}}\right)(\lambda_i-\lambda_j)^{\frac{1}{2}}a_{ij}^2$$
$$= -\frac{\delta}{2}\sum_{i,j=1}^{k}\frac{1}{(\lambda_{k+1}-\lambda_j)^{\frac{1}{2}}-(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}}(\lambda_i-\lambda_j)^2a_{ij}^2, \tag{2.18}$$

and

$$-\frac{1}{\delta}\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_j)^{\frac{1}{2}}t_{ij}^2 = -\frac{1}{2\delta}\sum_{i,j=1}^{k}((\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}-(\lambda_{k+1}-\lambda_j)^{\frac{1}{2}})t_{ij}^2 \tag{2.19}$$

Since $a_{ij}=a_{ji}, t_{ij}=-t_{ji}$, we have

$$2\sum_{i,j=1}^{k}(\lambda_{k+1}-\lambda_i)a_{ij}t_{ij} = -2\sum_{i,j=1}^{k}(\lambda_i-\lambda_j)a_{ij}t_{ij}$$
$$\leq -\frac{\delta}{2}\sum_{i,j=1}^{k}\frac{1}{(\lambda_{k+1}-\lambda_j)^{\frac{1}{2}}-(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}}(\lambda_i-\lambda_j)^2a_{ij}^2$$
$$-\frac{1}{2\delta}\sum_{i,j=1}^{k}((\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}-(\lambda_{k+1}-\lambda_j)^{\frac{1}{2}})t_{ij}^2 \tag{2.20}$$

Introducing (2.18)-(2.20) into (2.17), we get

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\int_M u_i^2 |\nabla h|^2$$
$$\leq \sum_{i=1}^{k}\delta(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\int_M hu_i p_i + \sum_{i=1}^{k}\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\int_M \frac{1}{\rho}\left(\langle\nabla h,\nabla u_i\rangle + \frac{u_i\Delta h}{2}\right)^2.$$

Hence (2.1) is true, this completes the proof of Lemma 2.1.

3. PROOF OF THE MAIN RESULTS

In this section, we will give the proof of Theorem 1.1 and Corollary 1.2. Firstly, we introduce a lemma.

Lemma 3.1[4] Let Ω be a bounded domain in n-dimensional complex projective space $CP^n(4)$. $\forall P\in\Omega$, there are some functions $\{g_\alpha, \alpha=1\cdots,2(n+1)^2\}$ satisfy

$$\begin{cases} \sum_{\alpha=1}^{2(n+1)^2} |\nabla g_\alpha|^2 = 4n, \\ \sum_{\alpha=1}^{2(n+1)^2} |\Delta g_\alpha|^2 = 16n(n+1), \\ \sum_{\alpha=1}^{2(n+1)^2} \nabla g_\alpha \Delta g_\alpha = 0, \\ \sum_{\alpha=1}^{2(n+1)^2} |\langle \nabla g_\alpha, \nabla u_i \rangle|^2 = 2|\nabla u_i|^2 . \end{cases} \tag{3.1}$$

proof of Theorem 1.1. Taking $h = g_\alpha$ in (2.1) and summing over α from 1 to $2(n+1)^2$, we can get

$$\begin{aligned} &\sum_{i=1}^{k}\sum_{\alpha=1}^{2(n+1)^2}(\lambda_{k+1}-\lambda_i)\int_\Omega u_i^2|\nabla g_\alpha|^2 \\ &\le \sum_{i=1}^{k}\sum_{\alpha=1}^{2(n+1)^2}\delta(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\int_\Omega g_\alpha u_i p_i \\ &+\sum_{i=1}^{k}\sum_{\alpha=1}^{2(n+1)^2}\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\int_\Omega \frac{1}{\rho}\left(\langle\nabla g_\alpha,\nabla u_i\rangle+\frac{u_i\Delta g_\alpha}{2}\right)^2. \end{aligned} \tag{3.2}$$

Because of $\rho_1 \le \rho(x) \le \rho_2$ and $\int_\Omega \rho u_i^2 = 1$, we have

$$\int_M u_i^2 \ge \frac{1}{\rho_2} \tag{3.3}$$

Taking (3.3) into (3.2) and noticing $\sum_{\alpha=1}^{2(n+1)^2}|\nabla g_\alpha|^2 = 4n$, we have

$$\begin{aligned} &\frac{4n}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i) \\ &\le \sum_{i=1}^{k}\sum_{\alpha=1}^{2(n+1)^2}\delta(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\int_\Omega g_\alpha u_i p_i \\ &+\sum_{i=1}^{k}\sum_{\alpha=1}^{2(n+1)^2}\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\int_\Omega \frac{1}{\rho}\left(\langle\nabla g_\alpha,\nabla u_i\rangle+\frac{u_i\Delta g_\alpha}{2}\right)^2. \end{aligned} \tag{3.4}$$

Since $(\Delta^2 - a\Delta + b)(u_i) = \lambda_i \rho u_i$, then we have

$$\int_\Omega u_i\Delta^2 u_i - a\int_\Omega u_i\Delta u_i + b\int_\Omega u_i^2 = \int_\Omega u_i(\Delta^2 - a\Delta + b)(u_i) = \lambda_i\int_\Omega \rho u_i^2 = \lambda_i, \tag{3.5}$$

and by Schwarz inequality, we have

$$\int_\Omega u_i\Delta u_i \le \int_\Omega((\Delta u_i)^2\int_\Omega u_i^2)^{\frac{1}{2}} \le \int_\Omega(\frac{1}{\rho_1}(\Delta u_i)^2)^{\frac{1}{2}} = \int_\Omega(\frac{1}{\rho_1}u_i\Delta^2 u_i)^{\frac{1}{2}} \tag{3.6}$$

From (3.5) and (3.6), we can get

$$\lambda_i \ge \rho_1\left(\int_\Omega u_i\Delta u_i\right)^2 - a\int_\Omega u_i\Delta u_i + \frac{b}{\rho_2},$$

this is a quadratic inequality of $\int_\Omega u_i\Delta u_i$, solving it, we obtain

$$\frac{a-\sqrt{a^2+4\left(\lambda_i-\frac{b}{\rho_2}\right)}}{2\rho_1}\le\int_\Omega u_i\Delta u_i\le\frac{a+\sqrt{a^2+4\left(\lambda_i-\frac{b}{\rho_2}\right)}}{2\rho_1},$$

Setting

$$A_i=\frac{-a+\sqrt{a^2+4\left(\lambda_i-\frac{b}{\rho_2}\right)}}{2\rho_1},$$

which imply that

$$-\int_\Omega u_i\Delta u_i\le A_i. \tag{3.7}$$

Since $\int_\Omega u_i\Delta u_i=-\int_\Omega|\nabla u_i|^2$, we have

$$\int_\Omega|\nabla u_i|^2\le A_i. \tag{3.8}$$

From (3.1), (3.6) and (3.7), we can get

$$\begin{aligned}&\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega\frac{1}{\rho}\left(\langle\nabla g_\alpha,\nabla u_i\rangle^2+\frac{1}{2}u_i\Delta g_\alpha\right)^2\\&=\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega\frac{1}{\rho}\left(\langle\nabla g_\alpha,\nabla u_i\rangle+u_i\Delta g_\alpha\langle\nabla g_\alpha,\nabla u_i\rangle+\frac{1}{4}u_i^2(\Delta g_\alpha^2)^2\right)\\&=\frac{1}{\rho_1}\int_\Omega|2\nabla u_i|^2+4n(n+1)\int_\Omega u_i^2\end{aligned} \tag{3.9}$$

and by the definition of p_i, we can get

$$\begin{aligned}&\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega g_\alpha u_i p_i\\=&\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega g_\alpha u_i\{2\langle\nabla g_\alpha,\nabla(\Delta u_i)\rangle+\Delta g_\alpha\Delta u_i+2\Delta\left(\langle\nabla g_\alpha,\nabla u_i\rangle\right)\\&+\Delta\left(u_i\Delta g_\alpha\right)-2a\langle\nabla g_\alpha,\nabla u_i\rangle-au_i\Delta g_\alpha\}\\=&\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega-2\{u_i\Delta u_i\langle\nabla g_\alpha,\nabla g_\alpha\rangle+g_\alpha\Delta u_i\langle\nabla u_i,\nabla g_\alpha\rangle+g_\alpha u_i\Delta g_\alpha\Delta u_i\}\\&+\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega g_\alpha u_i\Delta g_\alpha\Delta u_i+\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega\{\Delta g_\alpha u_i+g_\alpha\Delta u_i+2\langle\nabla g_\alpha,\nabla u_i\rangle\}\end{aligned}$$

$$\begin{aligned}
&\times\{2\langle\nabla g_\alpha,\nabla u_i\rangle+u_i\Delta f\}+\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega -2ag_\alpha u_i\langle\nabla g_\alpha,\nabla u_i\rangle\\
&+\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega 2ag_\alpha u_i\langle\nabla g_\alpha,\nabla u_i\rangle+\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega au_i^2\langle\nabla g_\alpha,\nabla g_\alpha\rangle\\
=&-\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega 2u_i\Delta u_i\langle\nabla g_\alpha,\nabla g_\alpha\rangle+\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega 4\langle\nabla g_\alpha,\nabla u_i\rangle^2\\
&+\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega 4u_i\Delta g_\alpha\langle\nabla g_\alpha,\nabla u_i\rangle+\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega (u_i\Delta g_\alpha)^2+\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega au_i^2\langle\nabla g_\alpha,\nabla g_\alpha\rangle\\
=&-\sum_{\alpha=1}^{2(n+1)^2}\int_\Omega 8nu_i\Delta u_i+\int_\Omega 8|\nabla u_i|^2+\int_\Omega 16n(n+1)u_i^2+\int_\Omega au_i^2\\
\leq&\ 8(n+1)A_i+\frac{16n(n+1)+a}{\rho_1}.
\end{aligned}\tag{3.10}$$

Introducing (3.9) and (3.10) into (3.4), we obtain

$$\begin{aligned}
&\frac{4n}{\rho_2}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\\
&\leq\sum_{i=1}^k\delta(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}(8(n+1)A_i+\frac{16n(n+1)+a}{\rho_1})\\
&\quad+\sum_{i=1}^k\frac{1}{\delta}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\frac{1}{\rho_1}\left(2A_i+\frac{4n(n+1)}{\rho_1}\right).
\end{aligned}\tag{3.11}$$

In (3.10), taking

$$\delta=\frac{\left((\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\frac{1}{\rho_1}\left(2A_i+\frac{4n(n+1)}{\rho_1}\right)\right)^{\frac{1}{2}}}{\left((\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\left(8(n+1)A_i+\frac{16n(n+1)+a}{\rho_1}\right)\right)^{\frac{1}{2}}},\tag{3.12}$$

we can get

$$\begin{aligned}
&\frac{n}{\rho_2}\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\\
\leq\ &2\left\{\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\left((n+1)A_i+\frac{2n(n+1)+a}{\rho_1}\right)\right\}^{\frac{1}{2}}\\
&\times\left\{\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\frac{1}{\rho_1}\left(A_i+\frac{2n(n+1)}{\rho_1}\right)\right\}^{\frac{1}{2}},
\end{aligned}\tag{3.13}$$

which implies that

$$\lambda_{k+1} \leq \frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{2\rho_2}{kn}\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\left((n+1)A_i + \frac{2n(n+1)+a}{\rho_1}\right)\right\}^{\frac{1}{2}} \times\left\{\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\frac{1}{\rho_1}\left(A_i+\frac{2n(n+1)}{\rho_1}\right)\right\}^{\frac{1}{2}}. \tag{3.14}$$

This completes the proof of Theorem 1.1.

proof of Corollary 1.2. By (3.11), we obtain

$$\frac{4n}{\rho_2}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i) \leq \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}(8(n+1)\delta+\frac{2}{\delta\rho_1})A_i + \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}\frac{1}{\rho_1}\left(\frac{16n(n+1)+a}{\delta\rho_1}+4\delta n(n+1)\right). \tag{3.15}$$

Taking $\delta=\frac{1}{2}(n+1)^{\frac{1}{2}}$ in (3.15), we have

$$\lambda_{k+1} \leq \frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{(n+1)^{\frac{1}{2}}}{nk}\left(\rho_2+\frac{\rho_2}{\rho_1}\right)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}A_i + \frac{1}{nk}\frac{\rho_2}{\rho_1}\left(\frac{32n(n+1)^{\frac{3}{2}}+a}{\rho_1}+4n(n+1)^{\frac{1}{2}}\right)\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^{\frac{1}{2}}. \tag{3.16}$$

This completes the proof of Corollary 1.1.

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