OPERATOR METHOD FOR A DUAL RISK MODEL WITH BARRIER STRATEGY

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ABSTRACT
In this paper, we investigate the dual of a Sparre Andersen model perturbed by diffusion under a barrier strategy, in which innovation inter-arrival times have a generalized Erlang (n) distribution. Integro-differential equations with certain boundary conditions for the expected total discounted dividends are derived. Using an algebraic operator method, we derive a defective renewal equation for the expected total discounted dividends until ruin. The special case where the innovation times are Erlang (2) distribution and the profit is exponential distribution is considered in some details.

Keywords: *Dual model, Barrier strategies, Generalized Erlang (n), The expected discounted dividends.*

1. INTRODUCTION

Recently, some interesting papers have been written on a model which is dual to the classical insurance risk model:

$$U(t) = u - ct + S_1(t), \qquad t \geq 0, \tag{1}$$

where $u\ (\geq 0)$ is the initial surplus, the constant $c\ (>0)$ is the rate of expenses. The process $\{S_1(t),\ t \geq 0\}$ is a compound Poisson process representing the aggregate gains up to time t .

The model of the form (1) seems appropriate for companies that have occasional gains whose amount and frequency is described by the process $S_1(t)$. There are many possible interpretations: pharmaceutical or petroleum companies gain from an invention or discovery; a portfolio of life annuities (the risk consists of survival and the event of death leads to gains), etc. More interpretations can be found in see Asmussen[3], Avanzi et al.[4], Avanzi and Gerber[5], Ng[15] and the references therein.

The dual of the Sparre Andersen risk model is given by

$$U(t) = u - ct + \sum_{i=1}^{N(t)} X_i + \sigma B(t), \qquad t \geq 0, \tag{2}$$

where X_i's is a sequence of independent identically distribution (i.i.d.) random variables (r.v.) with the common distribution $P\ (P(0) = 0)$, density function p and Laplace transform $\hat{p}(s) = \int_0^\infty e^{-sx} p(x)dx$.

The ordinary renewal process $\{N(t),\ t \geq 0\}$ denotes the number of innovations up to time t, with $N(t) = \sup\{k : T_1 + \cdots + T_k \leq t\}$ where the i.i.d. innovation inter-arrival times $\{T_i\}_{i=1}^{+\infty}$ have a common generalized Erlang(n) distribution. More precisely, let $T = T_1$, then T can be expressed as

$$T = W_1 + W_2 + \cdots + W_n$$

where $W_1,\ W_2,\ \cdots W_n$ are n independent exponentially distributed r.v.'s with $E[W_i] = 1/\lambda_i$.

Finally, $\{B(t),\ t \geq 0\}$ is a standard Brownian motion with $B(0) = 0$ and $\sigma > 0$ is a constant, representing the volatility parameter.

We also assume that $\{B(t), t \geq 0\}$, $\{N(t),\ t \geq 0\}$ and $\{X_i,\ i = 1, 2, \cdots\}$ are mutually independent and $EX_i > cET_i$ ensuring that ruin will not occur almost surely.

Dividend strategy was first proposed by De Finetti[6] for a binomial model. Since then, the risk model in the presence of dividend payments has become a more and more popular topic in actuarial mathematics. Some recent publications on dividend barrier strategies are [4, 5, 11, 14, 16], as well as the references therein.

In this paper, we consider the dual model (2) with a constant dividend barrier $b > 0$. That is, whenever the surplus exceeds the barrier b, the excess is paid out immediately as a dividend. We use the symbol $D(t)$ to denote the aggregate dividends paid by time t, thus the modified surplus $\tilde{U}(t)$ is denoted by

$$\tilde{U}(t) = U(t) - D(t).$$

Let D denote the present value of all dividends until ruin,

$$D = \int_0^{\tau} e^{-\delta t} \mathrm{d}D(t)$$

where $\tau = \inf\{t \geq 0, \tilde{U}(t) = 0\}$ is the time of ruin of the modified surplus process, and $\delta > 0$ is the force of interest for valuation. Let $V(u;b) = E[D|\tilde{U}(0) = u]$ be the expected total discounted dividends until ruin. Note that

$$V(u;b) = u - b + V(b;b), \qquad u > b.$$

We will mainly discuss the model for $0 \leq u \leq b$.

The dual of the classical risk model with barrier strategy was discussed in Avanzi et al.[4]. Avanzi and Gerber[5] studied a dual model perturbed by diffusion with a barrier strategy. Laplace transform is the common method in the two papers. Though the Laplace inversion can be obtained for rationally distributed gains, the process may be tedious. An algebraic operator method is used in this paper, which simplifies the calculation.

The rest of the paper is organized as follows. In section 2, we derive an integro-differential equation satisfied by $V(u;b)$. Based on algebraic operators, a defective renewal equation and its solution are given in Section 3. Finally, in Section 4, some explicit results are given for the case $n = 2$ and exponential innovation sizes.

2. INTEGRO-DIFFERENTIAL EQUATIONS

Motivated by Gerber and Shiu [9], we introduce some auxiliary functions. First define $S_j = W_1 + W_2 + \cdots + W_j \ (j = 1, 2, \cdots, n)$ with $S_0 = 0$. Define

$$V_j(u;b) = E\left[\int_t^T e^{-\delta(s-t)} \mathrm{d}D(s) | S_j < t < S_{j+1}, \tilde{U}(t) = u\right], \tag{3}$$

with the properties that $V_0(u;b) = V(u;b)$ and $V_j(0;b) = 0$ for $j = 0, 1, \cdots, n-1$.

Let $\mathcal{I}$ and $\mathcal{D}$ be the identity operator and the differentiation operator, respectively. Moreover, define $\prod_{i=1}^{0} = 1$.

Theorem 2. Suppose that $V_j(u,b)$ is twice continuously differentiable in u. For $0 < u < b$, $V(u;b)$ satisfies the following equation

$$\left\{\prod_{i=1}^{n}\left[(1 + \frac{\delta}{\lambda_i})\mathcal{I} + \frac{c}{\lambda_i}\mathcal{D} - \frac{\sigma^2}{2\lambda_i}\mathcal{D}^2\right]\right\} V(u;b) = \int_0^{+\infty} V(u+y;b)p(y)\mathrm{d}y, \tag{4}$$

with boundary conditions:

$$\left\{\prod_{i=1}^{m}\left[(1 + \frac{\delta}{\lambda_i})\mathcal{I} + \frac{c}{\lambda_i}\mathcal{D} - \frac{\sigma^2}{2\lambda_i}\mathcal{D}^2\right]\right\} V(u;b)|_{u=0} = 0 \tag{5}$$

$$\left\{\prod_{i=1}^{m}\left[(1 + \frac{\delta}{\lambda_i})\mathcal{I} + \frac{c}{\lambda_i}\mathcal{D} - \frac{\sigma^2}{2\lambda_i}\mathcal{D}^2\right]\right\} V'(u;b)|_{u=b-} = 1, \tag{6}$$

for $m = 0, 1, \cdots, n-1$.

Proof. For $j = 0, 1, \cdots, n-2$, conditioning on a infinitesimal interval $(t, t + dt)$, one obtains that

$$\begin{aligned} V_j(u;b) &= P(W_{j+1} > dt)e^{-\delta dt}E[V_j(u - cdt + \sigma B(dt);b)] \\ &\quad + P(W_{j+1} < dt)e^{-\delta dt}E[V_{j+1}(u - cdt + \sigma B(dt);b)] + o(dt). \end{aligned}$$

Using Taylor expansion and collecting all terms of order dt, we get

$$\begin{aligned} V_{j+1}(u;b) &= \left[\left(1 + \frac{\delta}{\lambda_{j+1}}\right)\mathcal{I} + \frac{c}{\lambda_{j+1}}\mathcal{D} - \frac{\sigma^2}{2\lambda_{j+1}}\mathcal{D}^2\right] V_j(u;b) \\ &= \left[\prod_{i=1}^{j+1}\left(\left(1 + \frac{\delta}{\lambda_i}\right)\mathcal{I} + \frac{c}{\lambda_i}\mathcal{D} - \frac{\sigma^2}{2\lambda_i}\mathcal{D}^2\right)\right] V(u;b). \end{aligned} \tag{7}$$

For $j = n-1$, we have

$$\left[\left(1+\frac{\delta}{\lambda_n}\right)\mathcal{I}+\frac{c}{\lambda_n}\mathcal{D}-\frac{\sigma^2}{2\lambda_n}\mathcal{D}^2\right]V_{n-1}(u;b)=\int_0^{+\infty}V(u+y;b)p(y)\mathrm{d}y,$$

which together with (7) yields (4).

If $u=0$, ruin is immediate and no dividends can be paid. Therefore, $V_j(u;b)=0$. Then the condition (5) is followed from (7). Due to oscillation, one obtains

$$V_j(b;b)=V_j(b-du;b)+du.$$

Combining the above equation with (7) yields condition (6).

Remark. For $n=1$, (4) is identical to (2.4) in Avanzi and Gerber[5]; when $n=1,\ \sigma=0$, (4) reduces to (2.3) in Avanzi et al.[4].

3. OPERATOR METHOD

When the claim sizes are some special distribution (e.g. exponential distribution or mixture of exponential distributions), $V(u;b)$ can be obtained by solving a differential equation which will be illustrated in the last section. When the claim sizes are general distributions, we will solve the integro-differential equation for $V(u;b)$ by an algebraic operator method.

Motivated by Avanzi et al. [5], we replace the random variable u by $z=b-u$, and denote W by

$$W(z;b)=V(b-z;b),\ 0\le z\le b, \tag{8}$$

with $W(0;b)=V(b;b),\ W(b;b)=V(0;b)=0$. Then (4) becomes

$$\begin{aligned}&\left[\prod_{i=1}^n\left(\left(1+\tfrac{\delta}{\lambda_i}\right)\mathcal{I}-\tfrac{c}{\lambda_i}\mathcal{D}-\tfrac{\sigma^2}{2\lambda_i}\mathcal{D}^2\right)\right]W(z;b)\\&=\int_0^z W(z-y;b)p(y)\mathrm{d}y+A(z),\quad 0\le z\le b,\end{aligned} \tag{9}$$

where

$$A(z)=\int_z^{+\infty}(y-z)p(y)\mathrm{d}y+W(0;b)(1-P(z)). \tag{10}$$

Extending the definition of $W(z;b)$ in (8) to $z\ge 0$, and denoting the resulting function by $\omega(z)$, then we have

$$\prod_{i=1}^n[(b_i\mathcal{I}-\mathcal{D})(a_i\mathcal{I}+\mathcal{D})]\,\omega(z)=\alpha\int_0^z\omega(z-y)p(y)\mathrm{d}y+\alpha A(z),\quad z\ge 0, \tag{11}$$

where

$$a_i=\frac{c+\sqrt{c^2+2\sigma^2(\lambda_i+\delta)}}{\sigma^2},\ b_i=\frac{-c+\sqrt{c^2+2\sigma^2(\lambda_i+\delta)}}{\sigma^2},\ \alpha=2^n\prod_{i=1}^n\lambda_i/\sigma^{2n},$$

and $A(z)$ is given in (10). Obviously, $\omega(b)=0$. In other words, the dividend barrier b is the zero of $\omega(z)=0$.

To derive the defective renewal equation for $\omega(z)$, the expression for $\omega^{(k)}(0)(k=0,1,\cdots,2n-1)$ will be needed. Let

$$\begin{aligned}\ell(s)&=\prod_{i=1}^n\left[1+\frac{\delta}{\lambda_i}-\frac{c}{\lambda_i}s-\frac{\sigma^2}{2\lambda_i}s^2\right]=\sum_{j=0}^{2n}\ell_js^j\\&=\alpha\prod_{i=1}^n(s+a_i)(b_i-s).\end{aligned}$$

Taking Laplace transforms on both sides of (9) leads to

$$(\ell(s)-\hat{p}(s))\hat{\omega}(s)=\hat{A}(s)+\sum_{j=1}^{2n}\ell_j\sum_{k=0}^{j-1}\omega^{(k)}(0)s^{j-1-k}.$$

Let

$$\ell(s)=\hat{p}(s), \tag{12}$$

which is the Lundberg-type equation. By Rouché Theorem, Li and Garrido [12] has shown that the characteristic equation (12) has exactly n roots with positive real parts. In fact, we can show that (12) has n positive real roots by geometric method. The positive real roots play an important role in what follows.

Theorem 3. The character equation $\ell(s) = \hat{q}(s)$ has exactly n positive real roots say $\rho_1(\delta,\sigma), \cdots, \rho_n(\delta,\sigma)$.
Proof. It is obvious that the zeros of

$$\frac{1}{\ell(s)} = \frac{1}{\hat{p}(s)}$$

are also zeros of (12).
Without loss of generality, we assume that $0 = b_0 < b_1 < \cdots < b_n < b_{n+1} = \infty$. For $s \geq 0,\ 1/\ell(s)$ satisfies the following properties:
(1) Except for the singularities $b_i (i = 1, 2, \cdots, n),\ 1/\ell(s)$ is a continuous function on $[0, \infty)$.
(2) When n is a even number, $1/\ell(s)$ is concave function on (b_{2i-1}, b_{2i}) and convex function on $(b_{2(i-1)}, b_{2i-1})$ and

$$\lim_{s\uparrow b_{2i}} \frac{1}{\ell(s)} = -\infty, \quad \lim_{s\downarrow b_{2i}} \frac{1}{\ell(s)} = \infty,$$
$$\lim_{s\uparrow b_{2i-1}} \frac{1}{\ell(s)} = \infty, \quad \lim_{s\downarrow b_{2i-1}} \frac{1}{\ell(s)} = -\infty,$$

for $i = 1, 2, \cdots, n/2$.
When n is an odd number, $1/\ell(s)$ is convex function on (b_{2i-1}, b_{2i}) and concave function on (b_{2i}, b_{2i+1}) and

$$\lim_{s\uparrow b_{2i}} \frac{1}{\ell(s)} = \infty, \quad \lim_{s\downarrow b_{2i}} \frac{1}{\ell(s)} = -\infty,$$
$$\lim_{s\uparrow b_{2i-1}} \frac{1}{\ell(s)} = -\infty, \quad \lim_{s\downarrow b_{2i-1}} \frac{1}{\ell(s)} = \infty,$$

for $i = 1, 2, \cdots, (n+1)/2$.
While $1/\hat{p}(s)$ is a monotone increasing convex function on $[0, \infty)$. Moreover,
$\hat{p}(0) = 1 > 1/\ell(0)$, $\lim_{s\to\infty} 1/\hat{p}(s) = \infty$ and $\lim_{s\to\infty} 1/\ell(s) = 0$. Therefore, $1/\ell(s)$ and $1/\hat{p}(s)$ must have n cross points on $[0, \infty)$. It is clear that b_i are different to $\rho_i(\delta, \sigma)$, since $\ell(b_i) - \hat{p}(b_i) < 0$.
Remarks. 1) For notational simplicity, we write the roots as $\rho_1, \rho_2, \cdots, \rho_n$ in what follows. Furthermore, we assume that $\rho_1, \rho_2, \cdots, \rho_n$ are distinct in this sequel.
2) We illustrate the method in Figure 1 with $\hat{p}(s) = \beta/(s+\beta)$. The parameters are
$c = 1.1,\ \delta = 0.1,\ \beta = \sigma = 1,\ \lambda_1 = 2,\ \lambda_2 = 3.$

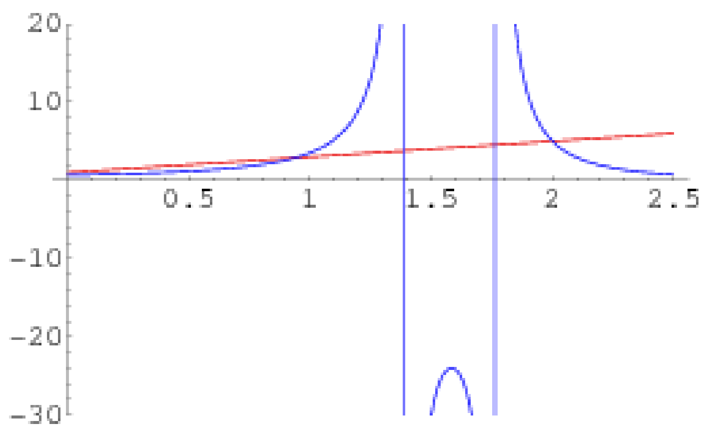

Figure 1 roots for n=2

Since $\ell(\rho_i) - \hat{p}(\rho_i) = 0$, then

$$\hat{A}(\rho_i) + \sum_{j=1}^{2n} \ell_j \sum_{k=0}^{j-1} \omega^{(k)}(0) \rho_i^{j-1-k} = 0, \qquad i = 1, 2, \cdots, n. \tag{13}$$

By the definition of $\omega(z)$, we have

$$\omega^{(k)}(0) = (-1)^k \frac{\partial V(u;b)}{\partial u}|_{u=b}. \tag{14}$$

Solving the linear system composed by (6), (13) and (14) leads to the exact expressions for $\omega^{(k)}(0), k = 0, 1, \cdots, 2n-1$.

Now we introduce two operators that play crucial roles in what follows. Define for $s \in \{s : \mathbf{Re}(s) \geq 0\}$ and function for which such integral exist,

$$\begin{aligned} \mathcal{T}_s f(x) &= \int_x^\infty \mathrm{e}^{-s(u-x)} f(u)\mathrm{d}u, \quad x > 0, \\ \mathcal{E}_s f(x) &= \int_0^x e^{-s(x-y)} f(y)\mathrm{d}y, \quad x > 0. \end{aligned}$$

The operator $\mathcal{T}$ is known as the Dickson-Hipp operators (see Dickson and Hipp[7] and Li and Garrido[10]) in ruin literature and operator $\mathcal{T}$ and $\mathcal{E}$ also known as Green operators (see Albrecher et al.[1]). More properties for $\mathcal{T}$ and $\mathcal{E}$ are given in Albrecher et al.[1] and Feng[8].

From the definition of $\mathcal{E}$, we have the following properties:

Lemma 2. (1) If $s > 0$ and $g(x) = se^{-sx}, (x > 0)$ then $s\mathcal{E}_s f(x) = g * f(x)$.

(2) If $s_i > 0 (i = 1, 2, \cdots, m)$ and $g_i(x) = s_i e^{-s_i x}(x > 0)$, then

$\prod_{i=1}^m s_i \mathcal{E}_{s_i} f(x) = g_1 * g_2 * \cdots * g_m * f(x)$.

Proof. The result can be obtained directly from the definition of $\mathcal{E}_s$.

For notational simplicity, we

$$\begin{aligned} \beta_i &= a_i - b_i + \rho_i, \qquad i = 1, 2, \cdots, n; \\ p_i(x) &= \mathcal{E}_{\beta_i} \mathcal{T}_{\rho_i} \mathcal{E}_{\beta_{i+1}} \mathcal{T}_{\rho_{i+1}} \cdots \mathcal{E}_{\beta_n} \mathcal{T}_{\rho_n} p(x), \quad x > 0, \, i = 1, 2, \cdots, n-1; \\ F_i(x) &= \prod_{j=1}^i (a_j \mathcal{I} + \mathcal{D})(b_j \mathcal{I} - \mathcal{D})\omega(x), \\ &= (a_i \mathcal{I} + \mathcal{D})(b_i \mathcal{I} - \mathcal{D}) F_{i-1}(x), \qquad x > 0, \, i = 1, 2, \cdots, n-1; \\ F_0(x) &= \omega(x), \end{aligned}$$

define $A_i(x) = \mathcal{E}_{\beta_{n+1-i}} \mathcal{T}_{\rho_{n-i+1}} A_{i-1}(x) + F_{n-i}(0) \mathcal{E}_{\beta_{n+1-i}} \delta_0 / \alpha, \quad x > 0, \, i = 1, 2, \cdots, n-1$, with $A_0 = A$.

Theorem 4. For $z \geq 0$, $\omega(z)$ satisfies a defective equation representation

$$\omega(z) = \alpha(\omega * h(z) + \mathcal{E}_{\beta_1} \mathcal{T}_{\rho_1} A_1(z)) + \omega(0) \mathcal{E}_{\beta_1} \delta_0 \tag{15}$$

where $h(z) = \mathcal{E}_{\beta_1} \mathcal{T}_{\rho_1} \mathcal{E}_{\beta_2} \mathcal{T}_{\rho_2} \cdots \mathcal{E}_{\beta_n} \mathcal{T}_{\rho_n} p(z)$.

Proof. It follows from (11) and the notations given above, we have

$$(b_n \mathcal{I} - \mathcal{D})(a_n \mathcal{I} + \mathcal{D}) F_{n-1}(z) = \alpha(\omega * p(z) + A(z)).$$

Let $R_1 = \mathcal{T}_{b_n}(\omega * p + A) + F_{n-1}(0)\delta_0/\alpha$, then

$$F_{n-1} = \alpha \mathcal{E}_{a_n} R_1.$$

Similar to Feng[8], we have

$$F_{n-1} = \alpha(\omega * \mathcal{E}_{\beta_n} \mathcal{T}_n p + \mathcal{E}_{\beta_n} \mathcal{T}_{\rho_n} A) + F_{n-1}(0) \mathcal{E}_{\beta_n} \delta_0. \tag{16}$$

According to the definition of F_i, (16) can be rewritten as

$$\begin{aligned} (a_{n-1} \mathcal{I} + \mathcal{D})(b_{n-1} \mathcal{I} - \mathcal{D}) F_{n-2} &= \alpha(\omega * \mathcal{E}_{\beta_n} \mathcal{T}_n p + \mathcal{E}_{\beta_n} \mathcal{T}_{\rho_n} A) + F_{n-1}(0) \mathcal{E}_{\beta_n} \delta_0 \\ &= \alpha(\omega * p_n + A_1). \end{aligned}$$

Therefore,

$$F_{n-2} = \alpha(\omega * \mathcal{E}_{\beta_{n-1}} \mathcal{T}_{n-1} p_n + \mathcal{E}_{\beta_{n-1}} \mathcal{T}_{\rho_{n-1}} A_1) + F_{n-2}(0) \mathcal{E}_{\beta_{n-1}} \delta_0.$$

Recursively, we have

$$\begin{aligned} \omega &= F_0 = \alpha(\omega * \mathcal{E}_{\beta_1} \mathcal{T}_{\rho_1} p_2 + \mathcal{E}_{\beta_1} \mathcal{T}_{\rho_1} A_1) + \omega(0) \mathcal{E}_{\beta_1} \delta_0 \\ &= \alpha\omega * \mathcal{E}_{\beta_1} \mathcal{T}_{\rho_1} \mathcal{E}_{\beta_2} \mathcal{T}_{\rho_2} \cdots \mathcal{E}_{\beta_n} \mathcal{T}_{\rho_n} p + \alpha \mathcal{E}_{\beta_1} \mathcal{T}_{\rho_1} A_1 + \omega(0) \mathcal{E}_{\beta_1} \delta_0, \end{aligned}$$

which is (15).
For (15) to be a defective renewal equation, it remains to show that $\alpha\int_0^\infty h(y)\mathrm{d}y<1$. Since $\mathcal{T}_{\rho_i}\mathcal{E}_{\beta_j}\geq\mathcal{E}_{\beta_j}\mathcal{T}_{\rho_i}$, then

$$\begin{aligned}
&\alpha\int_0^\infty h(y)\mathrm{d}y\\
&=\alpha\int_0^\infty \mathcal{E}_{\beta_1}\mathcal{T}_{\rho_1}\mathcal{E}_{\beta_2}\mathcal{T}_{\rho_2}\cdots\mathcal{E}_{\beta_n}\mathcal{T}_{\rho_n}p(y)\mathrm{d}y\\
&\leq\alpha\int_0^\infty \mathcal{T}_{\rho_1}\mathcal{E}_{\beta_1}\mathcal{T}_{\rho_2}\mathcal{E}_{\beta_2}\cdots\mathcal{T}_{\rho_{n-1}}\mathcal{E}_{\beta_{n-1}}\mathcal{T}_{\rho_n}\mathcal{E}_{\beta_n}p(y)\mathrm{d}y\\
&\leq\alpha\int_0^\infty \mathcal{T}_{\rho_1}\mathcal{T}_{\rho_2}\cdots\mathcal{T}_{\rho_n}\mathcal{E}_{\beta_1}\cdots\mathcal{E}_{\beta_n}p(y)\mathrm{d}y\\
&=\frac{\alpha}{\prod_{i=1}^n\beta_i}\int_0^\infty \mathcal{T}_{\rho_1}\mathcal{T}_{\rho_2}\cdots\mathcal{T}_{\rho_n}(f_n*p)(y)\mathrm{d}y,\\
&\leq\frac{\prod_{i=1}^n\lambda_i}{c^n}\int_0^\infty \mathcal{T}_{\rho_1}\mathcal{T}_{\rho_2}\cdots\mathcal{T}_{\rho_n}(f_n*p)(y)\mathrm{d}y,
\end{aligned}$$

where f_n is the density function of a generalized Erlang distribution with parameters $\beta_1,\cdots,\beta_n$(see Lemma 2). Since $\int_0^\infty f_n*p(y)\mathrm{d}y=1$, then f_n*p is also a density function. Similar to Theorem 4 in Li and Garrido[10], we have

$$\frac{\prod_{i=1}^n\lambda_i}{c^n}\int_0^\infty \mathcal{T}_{\rho_1}\mathcal{T}_{\rho_2}\cdots\mathcal{T}_{\rho_n}(f_n*p)(y)\mathrm{d}y<1,$$

which means (15) is a defective renewal equation.

Remark. According to Theorem 2.1 of Lin and Willmot [13], the solution to (15) is given by

$$\omega(z)=\frac{\int_0^z(\alpha\mathcal{E}_{\beta_1}T_{\rho_1}A_1(z-y)+\omega(0)e^{-\beta_1(z-y)})h(y)\mathrm{d}y}{(1-\alpha\int_0^\infty h(z)\mathrm{d}z)\int_0^\infty h(z)\mathrm{d}z}.$$

4. AN EXAMPLE

In this section, we consider the special case for $n=2$ and $p(x)=\beta e^{-\beta x},\ x>0$. Then

$$\ell(s)=\frac{1}{\lambda_1\lambda_2}\left[(\lambda_1+\delta)-cs-\frac{\sigma^2}{2}s^2\right]\left[(\lambda_2+\delta)-cs-\frac{\sigma^2}{2}s^2\right]=\sum_{k=0}^4\ell_k s^k,$$

where

$$\begin{aligned}
\ell_0&=\frac{(\lambda_1+\delta)(\lambda_2+\delta)}{\lambda_2\lambda_2},\quad \ell_1=\frac{-c(\lambda_1+\lambda_2+2\delta)}{\lambda_1\lambda_2},\\
\ell_2&=\frac{2c^2-\sigma^2(\lambda_1+\lambda_2+2\delta)}{2\lambda_1\lambda_2},\quad \ell_3=\frac{c\sigma^2}{\lambda_1\lambda_2},\quad \ell_4=\frac{\sigma^4}{4\lambda_1\lambda_2}.
\end{aligned}$$

Then (4) can be rewritten as

$$\sum_{i=0}^4(-1)^i\ell_iV^{(i)}(u;b)=\beta\int_u^b V(y;b)e^{-\beta(y-u)}\mathrm{d}y+\beta\int_b^\infty(y-b)e^{-\beta(y-u)}y. \tag{17}$$

Differentiating on both sides of (17), one obtains

$$\begin{aligned}
&\ell_4V^{(5)}(u;b)-(\ell_3+\beta\ell_4)V^{(4)}(u;b)+(\ell_2+\beta\ell_3)V'''(u;b)\\
&-(\ell_1+\beta\ell_2)V''(u;b)+(\ell_0+\beta\ell_1)V'(u;b)-\beta(\ell_0-1)V(u;b)=0,
\end{aligned}$$

from which it follows that

$$V(u;b)=\sum_{i=1}^2C_ie^{-\rho_iu}+\sum_{i=3}^5C_ie^{R_iu},\qquad 0\leq u\leq b,$$

where ρ_i with $\mathbf{Re}(\rho_i) > 0$ (for $i = 1, 2$) and $-R_i$ with $\mathbf{Re}(R_i) > 0$ (for i=3,4,5) are the solutions of $\ell(s) - \hat{p}(s) = 0$.

Now we have to determine the five constants C_i. Substituting back the solution for $V(u;b)$ into (16) and comparing the coefficients of $e^{-\beta(b-u)}$, we have

$$\sum_{i=1}^{2} C_i \frac{\rho_i}{\beta+\rho_i} e^{-\rho_i b} - \sum_{i=3}^{5} C_i \frac{R_i}{\beta - R_i} e^{R_i b} + \frac{1}{\beta} = 0. \tag{18}$$

Using the boundary conditions (5) and (6), we have

$$\sum_{i=1}^{5} C_i = 0, \tag{19}$$

$$\sum_{i=1}^{2} C_i \rho_i e^{-\rho_i b} - \sum_{i=3}^{5} C_i R_i e^{R_i b} = -1, \tag{20}$$

$$\sum_{i=1}^{2} (2C + \sigma^2 \rho_i) C_i \rho_i - \sum_{i=3}^{5} (2C - \sigma^2 R_i) C_i R_i = 0, \tag{21}$$

$$\sum_{i=1}^{2} (2C + \sigma^2 \rho_i) C_i \rho_i^2 e^{-\rho_i b} + \sum_{i=3}^{5} (2C - \sigma^2 R_i) C_i R_i^2 e^{R_i b} = -2\delta. \tag{22}$$

Write M as the coefficient matrix of the system (18)-(22). Define the column vector $\zeta = (-\frac{1}{\beta}, 0, -1, 0, -2\delta)^T$. Let M_i be the matrix obtained from M by replace its ith column by column vector ζ for $i = 1, 2, \cdots, 5$. Denote the determinant of the matrix M by $\det M$. Then, we have

$$C_i = \frac{\det M_i}{\det M}, \qquad i = 1, 2, \cdots, 5,$$

given that $\det M \neq 0$.

Consider the numerical illustration. Setting $c = 0.5,\ \delta = 0.01,\ \lambda_1 = 1,\ \lambda_2 = 2,$
$\sigma = 1,\ \beta = 1$, then the equation $\ell(s) = \hat{p}(s)$ has five roots in the whole complex plane:
$\rho_1 = 0.225373,\ \rho_2 = 1.78844,\ R_1 = 0.0465075,\ R_2 = 2.48365 - 0.504265i,$
$R_3 = 2.48365 + 0.504265i$. Figure 2 shows that $V(u;b)$ increase very quickly for smaller u and tends to stability for larger n.

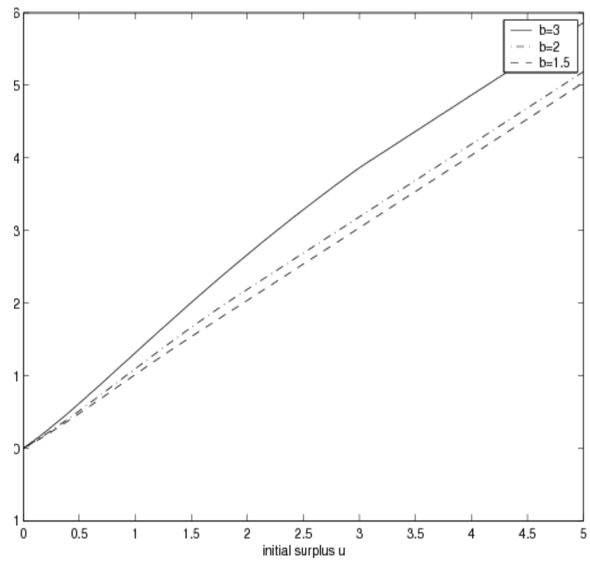

Figure 2 dividends

ACKNOWLEDGEMENT

This paper was supported by the Tianyuan Fund for Mathematics (No. 11226251), the Natural Science Foundation of Qufu Normal University (No. 2012ZRB01473) and the Science Foundation of Qufu Normal University for Doctor (No. BSQD2012039).

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