FRACTAL GEOMETRY AND SUPERFORMULA TO MODEL NATURAL SHAPES

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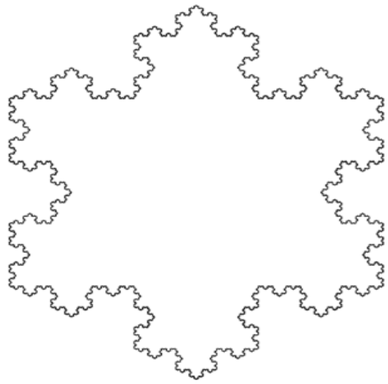
ABSTRACT
Mathematics is a branch of the science which has been used to model the natural and the biological forms, for many centuries.
Pythagoras, Aristotle, Fibonacci, Cardano, Bernoulli, Euler, Laplace, Gauss, von Helmholtz, Riemann, Einstein, Thompson, Turing, Wiener, von Neumann, Keller, and others studied important applications of mathematics to life sciences and significant developments in mathematics motivated by the life sciences.
Euclidean geometry, which dominated our mathematical thinking for centuries, has lost importance, because its primitive concepts (point, straight line and surface) and its simple constructs (squares and triangles) do not find application in the description of the natural objects.
Plato was able to reconcile the inability of classical geometry (as later formulated by Euclid) to describe the world we inhabit, and more recently, Mandelbrot has argued that fractal geometry could provide a coherent description of the design principles underlying living organisms.
This paper presents the fractal geometry, with its properties and its characteristics, as a useful tool to describe and to study the natural forms (e.g., fern, trees, seashells, bushes, basins of rivers, mountains), and the "superformula" by Gielies to model many complex shapes and curves that are found in nature.

Keywords: *Box-counting Dimension, Fractal Dimension, Fractal Geometry, Iterated Function Systems, L-Systems, Natural Shapes, Self-Similarity, Superformula.*

1. INTRODUCTION
First studies on fractals were at the beginning of the 20^{th} century by French mathematicians Pierre Fatou (1878-1929) and Gaston Julia (1893-1978). Only in the second part of the same century, the Polish-born Franco-American Benoit Mandelbrot (1924-2011) coined the word "Fractal". It derives from the Latin verb "frangere", "to break", and from the related adjective "fractus", "fragmented and irregular". This term denotes the geometry of nature, which traces inherent order in chaotic shapes and processes, and it was created to differentiate pure geometric figures from other types of figures that defy such simple classification. It also characterizes spatial or temporal phenomena that are continuous but not differentiable.
Mandelbrot (1989) defined fractal geometry as : "a workable geometric middle ground between ground between the excessive geometric order of Euclid and geometric chaos of general mathematics… Fractal geometry is conveniently viewed as a language that has proven its value by its uses" [1].
The acceptance of the word "fractal" was dated in 1975, when Mandelbrot presented the list of publications between 1951 and 1975, date when the French version of his book was published, although Mandelbrot's famous seminal paper on fractal dimension and statistical self-similarity dates back to 1967. After this book, the people were surprised by the variety of the studied fields: economy, linguistics, cosmology, noise on telephone lines, turbulence [2].
Fractal geometry replaces Euclidian geometry and it is recognized as the true geometry able to describe the Nature. Recently, the electronics evolution and the increase in the computer calculation power permitted connections between the fractal geometry and the other disciplines (for example, biology, economy, medicine, engineering, arts, architecture, computer science, industrial design), and the multiplicity of applications had an important role in the diffusion of fractal geometry [3, 4, 5, 6, 7, 8, 9].
This paper, which describes the application of fractal geometry to model natural shapes, is organized as follows: section 2 introduces the fractal objects, section 3 presents the applications of the fractal geometry to model natural forms. Section 4 describes Gielies' Superformula, and the section 5 is dedicated to the conclusions.

2. FRACTAL OBJECTS
A fractal object could be defined as a fragmented geometric shape that can be subdivided in parts, each of which is approximately a reduced-size copy of the whole [3].
Fractal objects and processes are said to display 'self-invariant' (self-similar or self-affine) properties [10].

Fractals are generally self-similar on multiple scales. So, all fractals have a built-in form of iteration or recursion. Sometimes the recursion is visible in how the fractal is constructed. For example, Koch snowflake (figure 1a), Cantor set, and the Sierpinski triangle are all generated using simple recursive rules. Self similarity is present in nature, too. Figures 1b shows a natural fractal object, the cauliflower which has the property of self similarity (it repeats its shape in different scales).



a)

b)

Figure 1. Koch snowflake is a fractal generated using simple geometric rules a). The cauliflower is a fractal present in the nature b)

Excellent summaries of basic concepts of fractal geometry can be found in Mandelbrot [1, 3], Schroeder [11], Turcotte [12], Hastings and Sugihara [10], Briggs [13].

3. FRACTAL GEOMETRY TO MODEL NATURAL SHAPES

Fractal properties include self-similarity and infinite length. In fractal analysis, the Euclidean concept of 'length' is conceived as a process. This process is characterized by a constant parameter D known as the fractal (or fractional) dimension.

3.1 The self-similarity

The invariance against changes in scale or size is named "self-similarity", and it is a property by which an object contains smaller copies of itself at arbitrary scales. Mandelbrot defined the self-similarity as follow: "When each piece of a shape is geometrically similar to the whole, both the shape and the cascade that generate it are called self-similar" [3]. "Similar" means that the relative proportions of the internal angles and shapes' sides remain the same.

Mandelbrot used the term "self-similar" for the first time in 1964, in an internal report at IBM, where he was doing research, and in the title of a 1965 paper. A fractal object is self-similar. This means that as viewers peer deeper into the fractal image, we can observe that the shapes seen at one scale are similar to the shapes seen in the detail at another scale.

There are three kinds of self-similarity:

- Exact self-similarity. The fractal is identical at different scales. This is the strongest kind of self-similarity.
- Quasi-self-similarity. The fractal is approximately (but not in exact way) identical at different scales. This is a less precise form of self-similarity. Quasi-self-similar fractals contain small copies of the entire fractal in degenerate and distorted shapes. This is the kind of fractals defined by recurrence relations.
- Statistical self-similarity. The fractal has statistical or numerical measures which are preserved across scales; instead of specifying exact scales, at each iteration the scale of each piece is selected randomly from a set range. This is the weakest kind of self-similarity. Most common definitions of "fractal" imply this kind of self-similarity. Random fractals are examples of fractals which are statistically self-similar.

Mandelbrot (1982) observed that self-similarity is ubiquitous in the natural world, and in the human body is possible to observe the presence of fractal geometry using two different points of view: temporal fractals and spatial fractals [3]. Temporal fractals are present in some dynamic processes, for example in the cardiac rhythm. In fact, heart rates show chaotic and self-similar patterns (as shown in figure 2) . This is not because of physical reasons, as many

might believe, but because of physiological reasons [14]. Another interesting consideration is connected to the complexity of the graph of the heart rate. A graph of a healthy heart has more complexity then a diseased heart, for example with congestive heart failure (CHF) (as shown in figure 3).
Chaotic property the complexity is associated to the healthy in many physiological aspects [15].

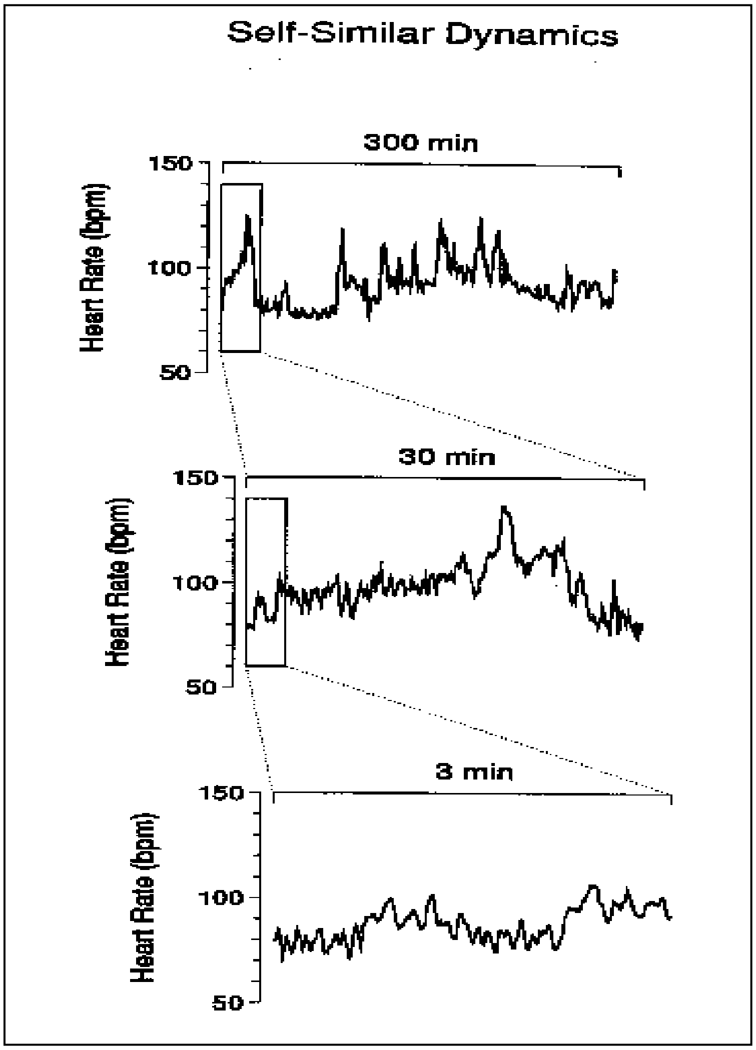


Figure 2. Heart rates show chaotic and self-similar patterns

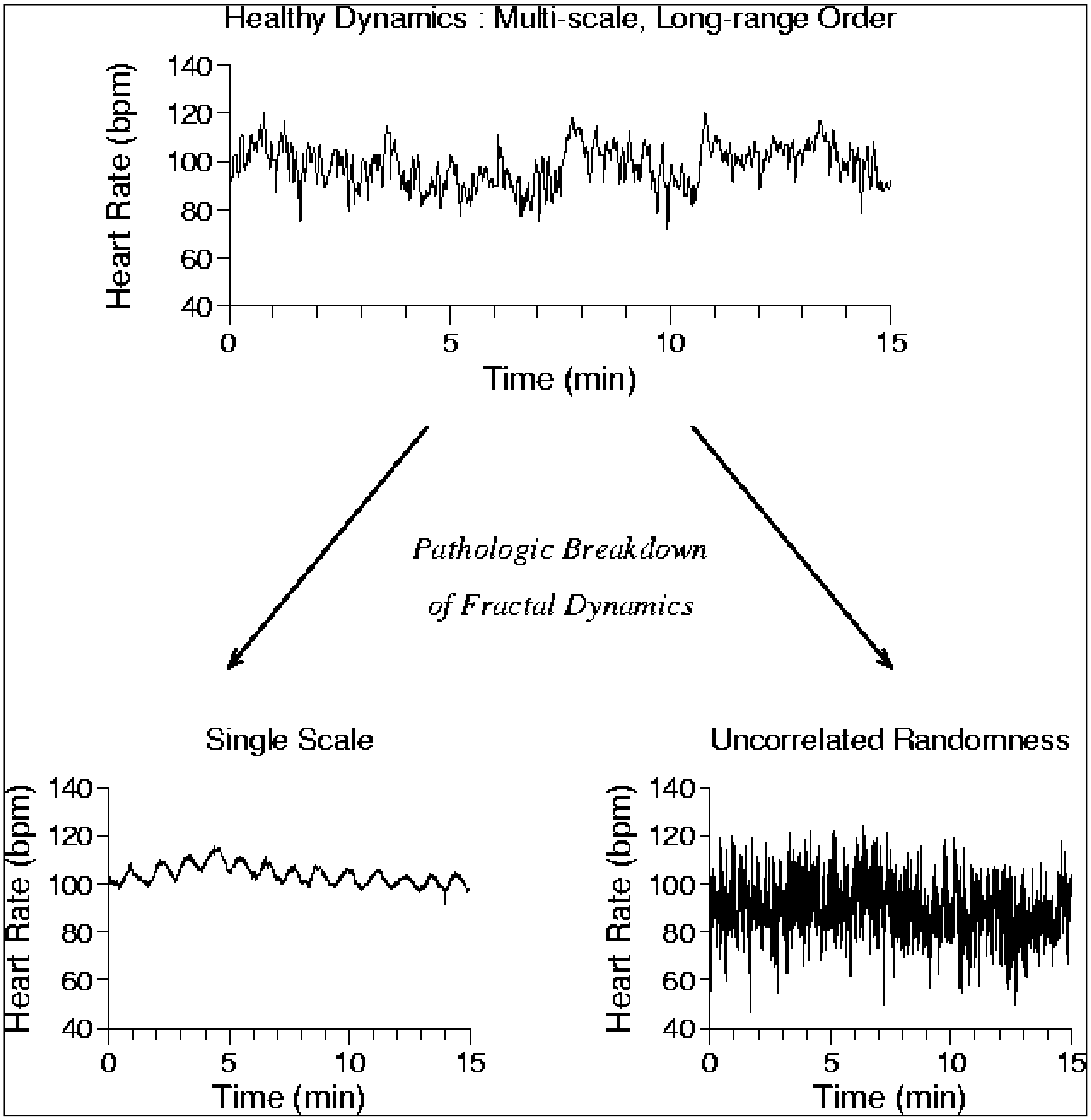


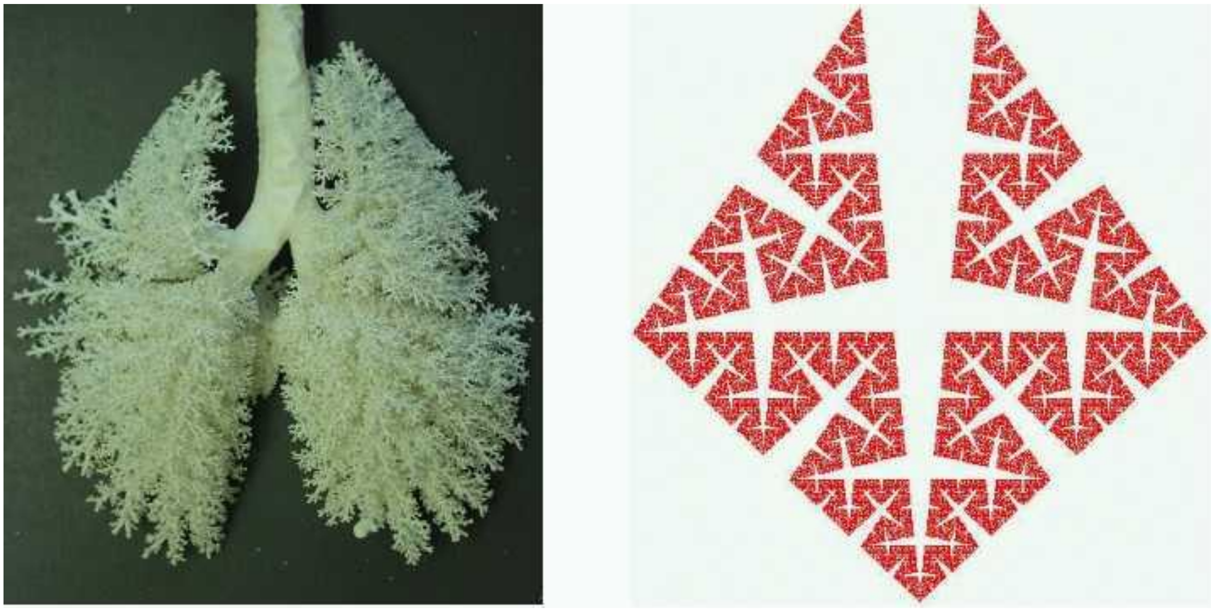
Figure 3. The top heart rate time series is from a healthy subject; bottom left is from a subject with heart failure, and bottom right from a subject with atrial fibrillation

Spatial fractals refer to the presence of self-similarity observed to various enlargements, for instance, the small intestine repeats its form on different scales. Spatial fractals also refer to the branched patterns that are present inside the human body for enlarging the available surface for the absorption of the substances, and the distribution and the collection of the solutes occupying a relative small fraction of the body. The lungs are an excellent example of a natural fractal organ, they have the branching pattern as the trees. [16]

Figure 4a shows the frontal view of dog's lungs. It is an example of self-similarity in the nature. In figure 4b there is an example of a fractal object which represents an attempt to reproduce complex shapes of the lungs, using few simple geometric rules.
Figure 5 shows a cast of human airway tree (a) compared with Koch tree model for airways (b) [17].

Fractal patterns are recognised at various spatial scales, as shown in figure 6.

Fractal geometry could be a unifying theme in biology, it permits a generalization of the concepts of dimension and length measurement [18].



a) b)

Figure 4. Frontal view of dog's lungs a) it is a natural fractal object. "Fractal lungs" b) is a fractal generated using simple geometric rules, starting from two triangles.

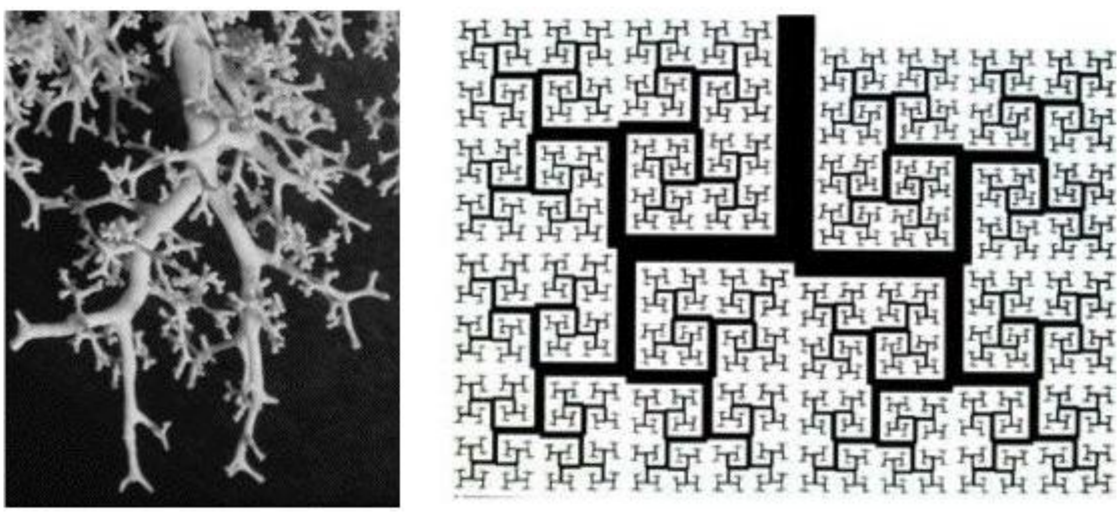


Figure 5. Cast of human airway tree (a) compared with Koch tree model for airways (b) [17]

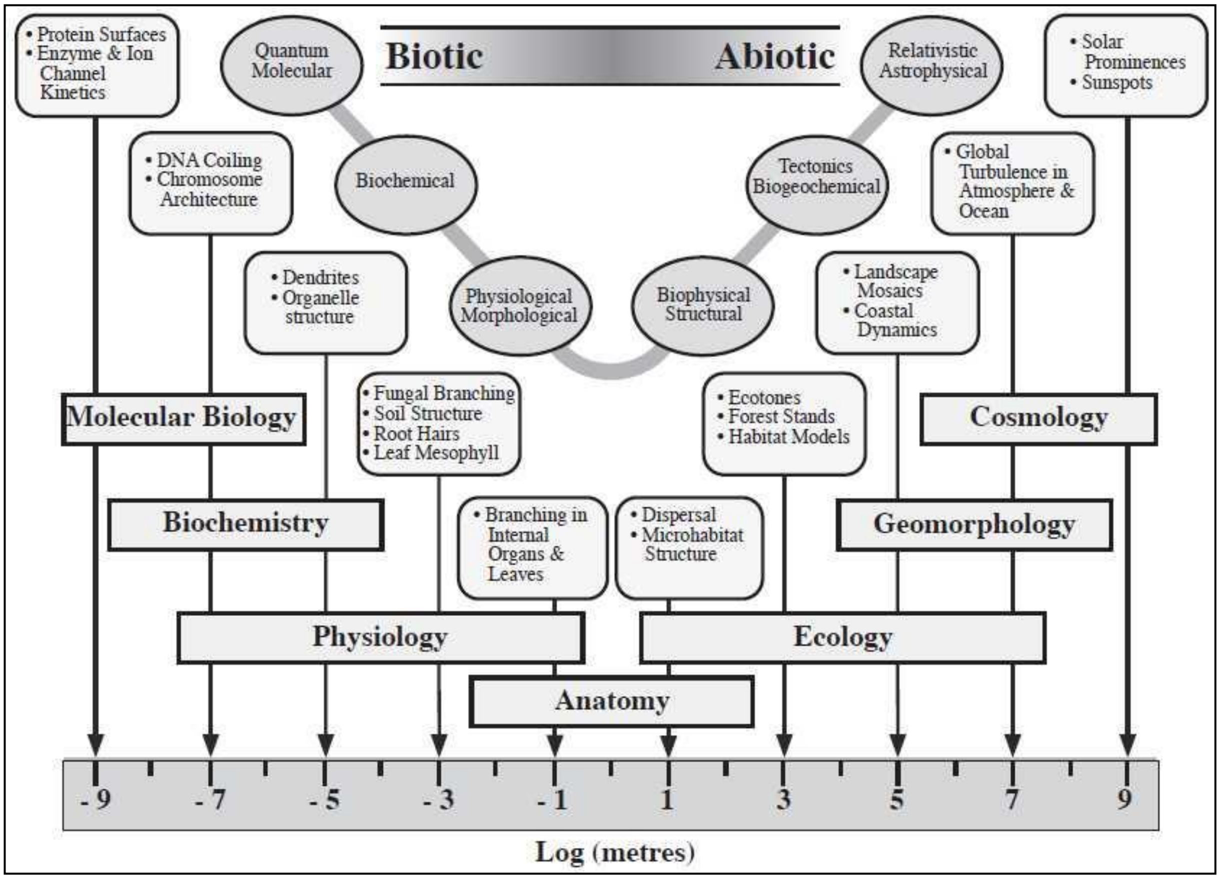


Figure 6. Fractal patterns are observed various spatial scales. The ovals are general processes operating at each scale: biotic processes predominate at inner spatial scales, abiotic processes at coarser scales. Rectangles represent the scientific disciplines [18].

3.2 Fractal dimension

Euclidean geometric forms are regular and have integer dimensions (1, 2, and 3, for line, surface, and volume respectively). A fractal line has a dimension between 1 and 2. Fractal dimension characterizes fractal sets and fractal patterns by quantifying their complexity as a ratio of the change in detail to the change in scale [19].

The basic idea of a "fractured" dimension has been described by Benoit Mandelbrot in his 1967 paper on self-similarity, where he cited a previous work written by English mathematician and meteorologist Lewis Fry Richardson (1881-1953) describing the counter-intuitive notion that a coastline's measured length changes with the length of the measuring ruler used. The estimated length, L, equals the length of the ruler, s, multiplied by the N, the number of such rulers needed to cover the measured object.

Richardson demonstrated that the measured length of coastlines appears to increase without limit as the unit of measurement is made smaller.

This is called “Richardson effect” (also known as “Coastlines Paradox”): the length of the coast of Britain depends on the scale of measurement [19] (figure 7).

Richardson pointed out his attention on the regularity between the length of national boundaries, and coastlines, and the scale size; observing that the relation between length estimate and length of scale is linear on a log-log plot (Figure 8).

Figure 7. The coastline of the United Kingdom as measured with measuring rulers of different dimensions.

Mandelbrot indicated the terms (1-D) as the slope, finding the following relationship: log[L(s)] = (1-D)log(s) + b, where D is the Fractal Dimension.

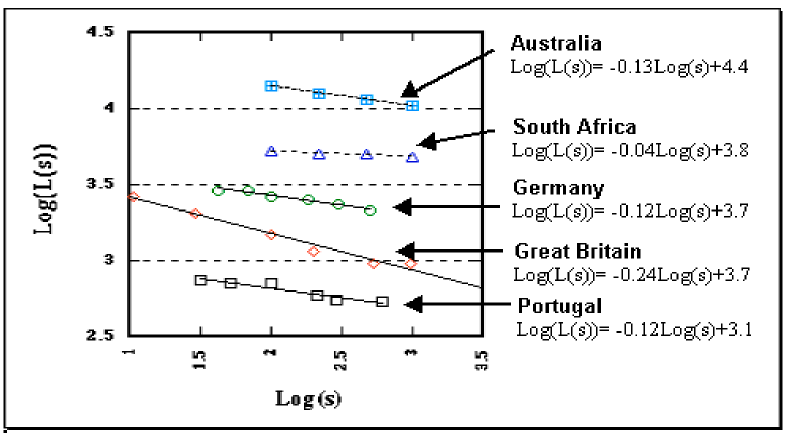


Figure 8. The Richardson Effect. Observing the plot, it is easy to determine the fractal dimension. For example, for Portugal: 1-D = -0.12, D= 1+0.12= 1.12.

The box-counting dimension is connected to the problem of determining the fractal dimension of a complex two-dimensional image. It is defined as the exponent D_b in the relationship:

$$N(d) \approx \frac{1}{d^{D_b}} \qquad (1)$$

where N(d) is the number of boxes of linear size d, necessary to cover a data set of points distributed in a two-dimensional plane. The basis of this method is that, for objects that are Euclidean, equation (1) defines their dimension. One needs a number of boxes proportional to 1/d to cover a set of points lying on a smooth line, proportional to $1/d^2$ to cover a set of points evenly distributed on a plane, and so on. Applying the logarithms to the equation (1) we obtain: $N(d) \approx -D_b \log(d)$.

The box-counting dimension can be produced using this iterative procedure:

- superimpose a grid of square boxes over the image (the grid size as given as s_1);
- count the number of boxes that contain some of the image ($N(s_1)$);
- repeat this procedure, changing (s_1), to smaller grid size (s_2);
- count the resulting number of boxes that contain the image ($N(s_2)$);
- repeat this procedures changing s to smaller and smaller grid sizes.

The box-counting dimension is defined by:

$$D_b = \frac{[\log(N(s_2)) - \log(N(s_1))]}{\left[\log\left(N\left(\frac{1}{s_2}\right)\right) - \log\left(N\left(\frac{1}{s_1}\right)\right)\right]} \qquad (2)$$

where 1/s is the number of boxes across the bottom of the grid. We can apply the box-counting dimension in the industrial design, too. It is calculated by counting the number of boxes that contain lines from the drawing inside them. Next figure 9 illustrates the procedure to determine the complexity of a fern using the box count [20].

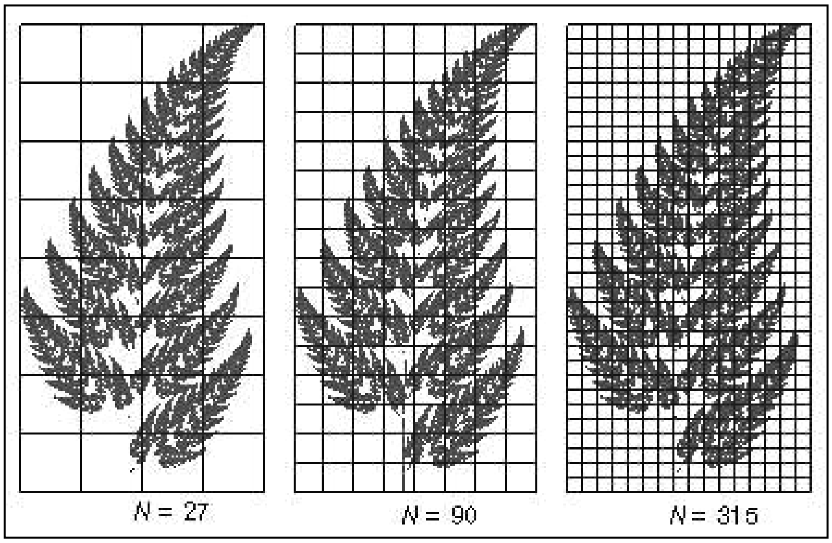


Figure 9. Box counting dimension applied to a fern.

Using (3) to determine the box counting dimension of the fern shown in figure 11, we have:

$$D_B = \frac{[\log(N(s_2)) - \log(N(s_1))]}{\left[\log\left(N\left(\frac{1}{s_2}\right)\right) - \log\left(N\left(\frac{1}{s_1}\right)\right)\right]} = \frac{[\log(90) - \log(27)]}{[\log(8) - \log(4)]} = 1.736\ldots$$

$$D_B = \frac{[\log(N(s_2)) - \log(N(s_1))]}{\left[\log\left(N\left(\frac{1}{s_2}\right)\right) - \log\left(N\left(\frac{1}{s_1}\right)\right)\right]} = \frac{[\log(315) - \log(90)]}{[\log(16) - \log(8)]} = 1.807\ldots$$

The box-counting dimension of the fern, calculated using (2), is an irrational value between 1.736 and 1.807.
Fractal dimension plays an important role in biology. Losa affirmed: " Fractal dimension is also a numerical descriptor that measures qualitative morphological traits and self-similar properties of biological elements. Recourse to the principles of fractal geometry has revealed that most biological elements, whether at cellular, tissue, or organ level, have self-similar structures within a defined scaling domain that can be characterized by means of the fractal dimension." [8]
The fractal dimension has been used as a characterization parameter of premalignant and malignant epithelial lesions of the floor of the mouth in humans [21].

3.3 The Iterated Function System

Barnsley [20] defined the Iterated Function System (IFS) as follow: "A (hyperbolic) iterated function system consists of a complete metric space (**X**, d) together with a finite set of contraction mappings w_n: **X**→ **X** with respective contractivity factor s_n, for n = 1, 2,.., N. The abbreviation "IFS" is used for "iterated function system". The notation for the IFS just announced is { **X**, w_n, n = 1, 2,.., N} and its contractivity factor is s = max {s_n : n = 1, 2, …, N}." Barnsley put the word "hyperbolic" in parentheses because it is sometimes dropped in practice. He also defined the following theorem : "Let {**X**, w_n, n = 1, 2, …, N} be a hyperbolic iterated function system with contractivity factor s [20]. Then the transformation W: H(**X**) → H(**X**) defined by:

$$W(B) = \cup_{n=1}^{n} w_n(B) \tag{3}$$

For all B∈ H(**X**), is a contraction mapping on the complete metric space (H(**X**), h(d)) with contractivity factor s. That is:

$$H(W(B), W(C)) \leq s \cdot h(B,C) \tag{4}$$

for all B, C ∈ H(**X**). Its unique fixed point, A ∈ H(**X**), obeys

$$A = W(A) = \cup_{n=1}^{n} w_n(A) \tag{5}$$

and is given by $A = \lim_{n\to\infty} W^{on}(B)$ for any B ∈ H(**X**)."
The fixed point A ∈ H(**X**), described in the theorem by Barnsley is called the "attractor of the IFS" or "invariant set".
An affine map of the plane is given by form:

$$w\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

and it is determined by six number, a, b, c, d, e, and f. The affine maps are combinations of translations, rotations and scalings in the plane. If the scaling factor is less than 1, we have contractive affine maps.
Bogomolny (1998) affirms that two problems arise. One is to determine the fixed point of a given IFS, and it is solved by what is known as the "deterministic algorithm".
The second problem is the inverse of the first: for a given set A∈H(**X**), find an iterated function system that has A as its fixed point [22]. This is solved approximately by the Collage Theorem [20].
The Collage Theorem states: "Let (**X,** d), be a complete metric space. Let L∈H(**X**) be given, and let ε ≥ o be given. Choose an IFS (or IFS with condensation) {**X**, (w_n), w_1, w_2,…, w_n} with contractivity factor 0 ≤ s ≤ 1, so that

$$h(L, \cup_{\substack{n=1 \\ (n=0)}}^{n} w_n(L)) \leq \varepsilon \tag{6}$$

Where h(d) is the Hausdorff metric. Then

$$h(L, A) \leq \frac{\varepsilon}{1-s} \tag{7}$$

Where A is the attractor of the IFS. Equivalently,

$$h(L, A) \leq (1-s)^{-1} h(L, \cup_{\substack{n=1 \\ (n=0)}} w_n(L)) \tag{8}$$

for all L∈H(**X**)."
The Collage Theorem describes how to find an Iterated Function System whose attractor is "close to" a given set, one must endeavour to find a set of transformations such that the union, or collage, of the images of the given set under transformations is near to the given set.
The IFS is produced by polygons, in this case: squares, that are put in one another. The final step of this iterative process shows a fern which has high degree of similarity to real one (figure 10).

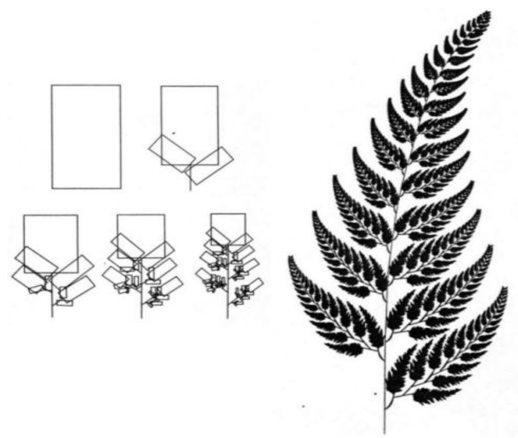


Figure 10. Fern leaf created using the IFS

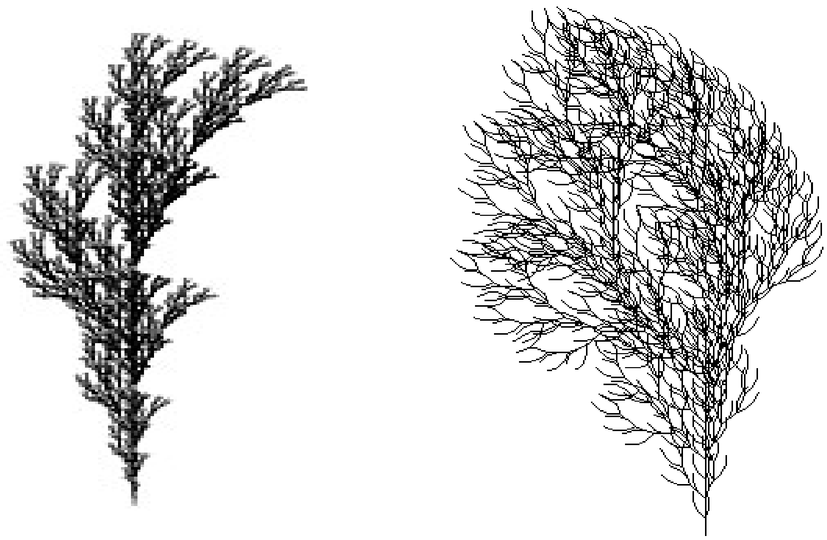
3.4 L -systems

Hungarian biologist Aristid Lindenmayer (1925-1989) introduced a kind of fractals, called L-systems, for modelling biological growth in 1968. L-system or Lindenmayer system is an algorithmic method for generating branched forms and structures such as plants. The components of an L-system are the following: a) an alphabet which is a finite set V of formal symbols containing elements that can be replaced (variables); b) the constants which is a set S of symbols containing elements that remain fixed; c) the axiom (also called the initiator) which is a string **ω** of symbols from **V** defining the initial state of the system; d) production (or rewriting rule) P that is a set of rules or productions defining the way variables can be replaced with combinations of constants and other variables. A production consists of two strings - the predecessor and the successor. The rules of the L-system grammar are applied iteratively starting from the initial state. L-systems are also commonly known as parametric L systems, and they are defined as a tuple **G** = {V, S, ω, P}. L-system can be also defined as a formal grammar (a set of rules and symbols) most famously used for modelling the growth processes of plant development, and it has been thought able for modelling the morphology of a variety of organisms. The differences between L-systems and Chomsky grammars are well described by Prusinkiewicz and Lindenmayer that affirmed: "The essential difference between Chomsky grammars and L-systems lies in the method of applying productions. In Chomsky grammars productions are applied sequentially, whereas in L-systems they are applied in parallel and simultaneously replace all letters in a given word. This difference highlights the biological motivation of L-systems. Productions are intended to capture cell divisions in multicellular organisms, where many divisions may occur at the same time. Parallel production application has an essential impact on the formal properties of rewriting systems" [23]. Strings generated by L-systems may be interpreted geometrically in different ways. For example, L-system strings serve a drawing commands for LOGO-style turtle. Prusinkiewicz and Lindenmayer defined a state of the turtle as a triplet (x, y, α), where the Cartesian coordinates (x, y) represent the turtle's position, and the angle α, called the heading, is interpreted as the direction in which the turtle is facing. Given the step size s and the angle increment δ, the turtle can respond to commands represented by the symbols in the table 1.

Symbols	Meaning
F	Move forward a step of length s. The state of the turtle changes, now it is (x', y', α), where $x' = x + s \cdot \cos \alpha$ and $y' = y + s \cdot \sin \alpha$. A segment between (x, y), starting point, and the point (x', y') is drawn.
f	Move forward a tep of length s without drawing a line.
+	Turn left by angle δ. The positive orientation of angles is counterclockwise, and the next state of the turtle is (x, y, α+δ).
-	Turn right by angle δ. The next state of the turtle is (x, y, α - δ).
[	Push the current state of the turtle onto a pushdown operations stack. The information saved on the stack contains the turtle's position and orientation, and possibly other attributes such as the color and width of lines being drawn.
]	Pop a state from the stack and make it the current state of the turtle. No line is drawn, although in general the position of the turtle changes.

Table 1 Commands for LOGO-style turtle derived by L-systems

Originally the L-systems were devised to provide a formal description of the development of such simple multicellular organisms, and to illustrate the neighbourhood relationships between plant cells. Later on, this system was extended to describe higher plants and complex branching structures. Smith (1984) was the first to prove that L-systems were useful in computer graphics for describing the structure of certain plants, in his paper: "Plants, Fractals, and Formal Languages" [24]. He described that these objects should not be labeled as "fractals" for their similarity to fractals, introducing a new class of objects which Smith called "graftals". This class had of great interest in the Computer Imagery [24, 25]. Figure 11a shows an example of plant-like structures generated after five iterations by bracketed L-systems with the initial string F (angle δ= 20°), and the replacement rule F $\rightarrow$F[+F] F[-F] [F]. In figure 11b there is a plant-like structures generated after four iterations by bracketed L-systems with the initial string F (angle δ= 22.5°), and the replacement rule F $\rightarrow$FF+[+F-F-F] -[-F+F+F].



a) b)

Figure 11. Plant-like structures generated by bracketed L-systems with five (a) and four iterations (b)

3.5 Fractal geometry and landscapes

Other interesting application of fractal geometry is to model landscapes which include terrain, mountains, trees. Fournier *et al.* [26] developed a mechanism for generating a kind of fractal mountains based on recursive subdivision algorithm for a triangle. Here, the midpoints of each side of the triangle are connected, creating four new subtriangles. Figure 12 shows the subdivision of the triangle into four smaller triangle, figure 12b illustrates how the midpoints of the original triangle are perturbed in the y direction [25]. To perturb these points, can be use the properties of the self-similarity, and the conditional expectation properties of fractional Brownian motion (abbreviated to fBm).

The fractional Brownian motion was originally introduced by Mandelbrot and Van Ness in 1968 as a generalization of the Brownian motion (Bm). FBm basically consists of steps in a random direction and with a step-length that has some characteristic value. Hence the random walk process. An important feature of fBm is the self-similarity (if we zoom in on any part of the function we will produce a similar random walk in the zoomed in part). Other polygons can be used to generate the grid (e.g., triangles and hexagons).

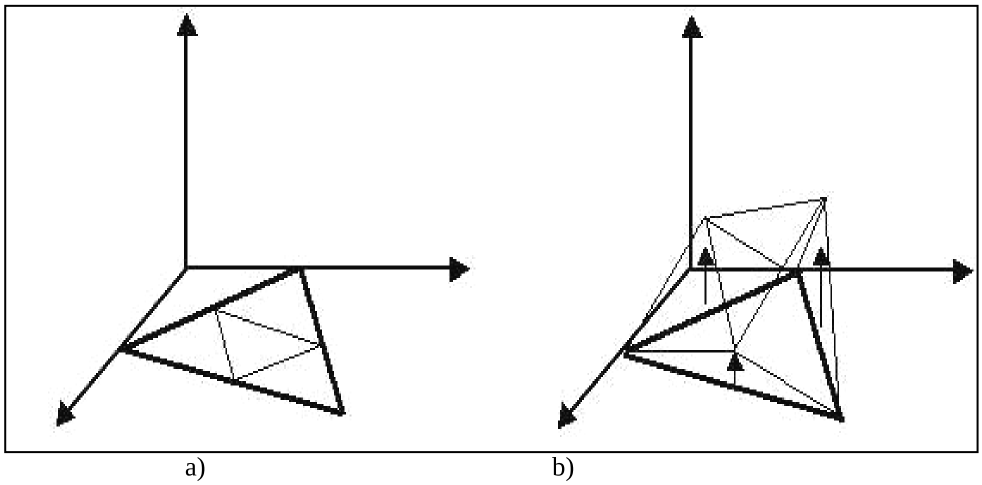


Figure 12. The subdivision of a triangle into four smaller triangle a). Perturbation in the y direction of the midpoints of the original triangle b)

This method evidences two problems, which are classified as internal and external consistency problems [26]. Internal consistency is the reproducibility of the primitive at any position in an appropriate coordinate space and at any level of detail, so the final shape is independent of the orientation of the subdivided triangle. This is satisfied by a Gaussian random number generator which depends on the point's position, thus it generates the same numbers in the same order at a given subdivision level. The external consistency concerns the midpoint displacement at shared edges and their direction of displacement.

This process, when iterated, produces a deformed grid which represents a surface, an example is shown in figure 13. After the rendering phase (that includes: hidden line, coloured, and shaded) can appear a realistic fractal mountain, as shown in figure 14.

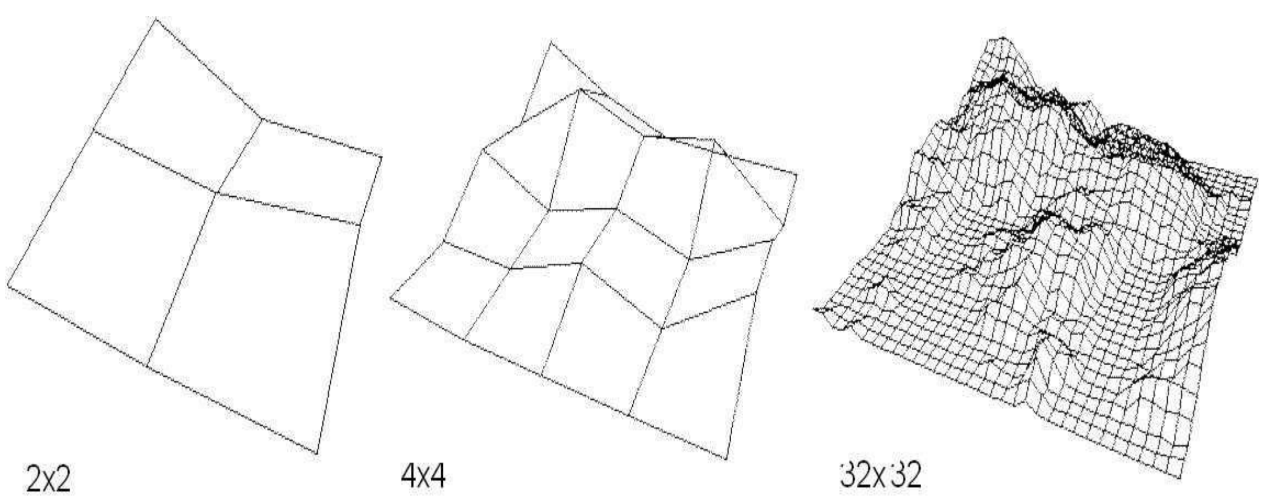


Figure 13. Grid of squares generated by a recursive subdivision and applying the fractional Brownian motion

Figure 14. Fractal mountains

These examples describe how to realize fractal mountains but not their erosion. Musgrave *et al.* [27] introduced techniques which are independent of the terrain creation. The algorithm can be applied to already generated data represented as regular height fields require separate processes to define the mountain and the river system. Prusinkiewicz and Hammel [28] combined the midpoint-displacement method for mountain generation with the squig-curve model of a non-branching river originated by Mandelbrot [29]. Their method created one non-branching river as result of context sensitive L-system operating on geometric objects (a set of triangles). Three key problems remained open (i) the river flowed at a constant altitude, (ii) the river flowed in an asymmetric valley, and (iii) the river had no tributaries. Figure 15 shows an example of a squig-curve construction (recursion levels 0–7) [28]

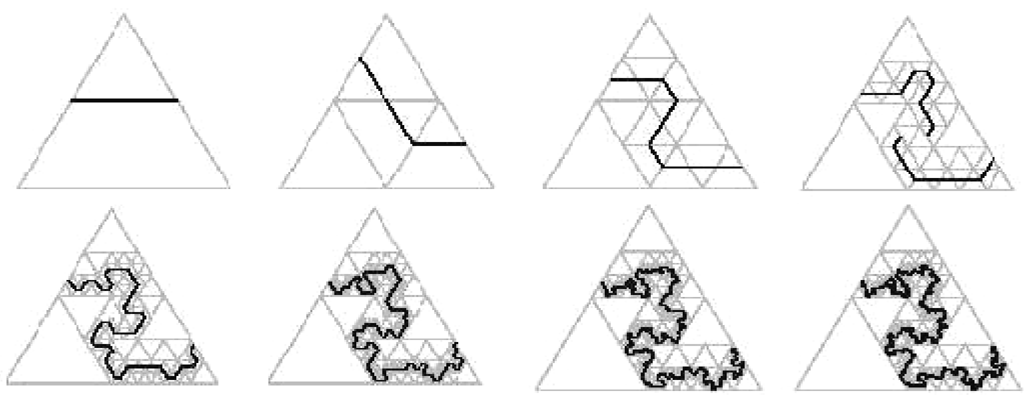


Figure 15. Squid-curve construction (recursion level 0-7)

Maràk *et al* [30] reported a method for synthetical terrain erosion, that is based on rewriting process of matrices representing terrain parts, which were rewritten using certain user-defined set of rules which represented an erosion process. The method consisted in three kinds of rewriting process [30].

4. The "Superformula"

Belgian biologist Johan Gielies has introduced, in his report *A generic geometric transformation that unifies a wide range of natural and abstract shapes* (2003), a geometrical approach for modeling and understanding various natural shapes [31]. He started from the concept of the circle, showing that a large variety of shapes can be described by a single and simple geometrical equation, that he has called the Superformula:

$$r = f(\phi)\frac{1}{\sqrt[n_1]{\left(\left|\frac{1}{a}\cos\left(\frac{m}{4}\phi\right)\right|\right)^{n_2} + \left(\left|\frac{1}{b}sin\left(\frac{m}{4}\phi\right)\right|^{n_3}\right)}}$$

Modification of the parameters permits the generation of various natural polygons. For example, for a = b = 1, n_1 = 2, $n_2 = n_3$ = 13, m = 5 e f (ϕ) = 1, we obtain a marine diatom like *Triceratium*.
A shape like a fern is obtained replacing r (the ray) with y (vertical axis) and a = b = 1, $n_1 = n_2 = n_3$ = 1, m = 4 e f (ϕ) = cos ϕ. A shell similar to a *Pleuroploca trapezium* is obtained: a = b = 1, $n_1 = n_2 = n_3$ = 5, m = 10 e f (ϕ) = $e^{0,2\,\phi}$.
Natural shapes generated by Superformula are in figure 16.

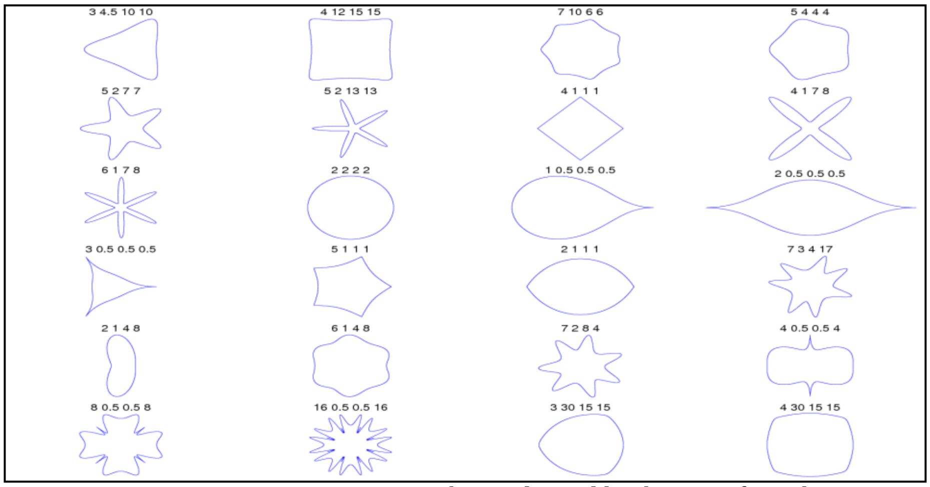


Figure 16. Some shapes derived by the Superformula

5. CONCLUSIONS

Scholars and scientists have recognized that many natural shapes are better characterized using fractal geometry [16, 18, 32, 33, 34, 35, 36, 37, 38]. Losa (2012) wrote: "The application of the fractal principle is very valuable for measuring dimensional properties and spatial parameters of irregular biological structures, for understanding the architectural/ morphological organization of living tissues and organs, and for achieving an objective comparison among complex morphogenetic changes occurring through the development of physiological, pathologic, and neoplastic processes. Emphasis will be laid on the fractal contribution to the knowledge of cell membranes, hematological tumors, cell tissue cancers, and brain tissues in healthy and diseased states**.**" [8].
Superformula represents an attempt to provide the precise mathematical relation between Euclidean measurements and the internal non-Euclidean metrics of shapes, looking beyond Euclidean circles and Pythagorean measures. About his Superformula Gielies wrote: "Considering that the mathematics behind the Superformula are easily understood and given the wide range of applications, both in technology and science, I believe that the Superformula has the potential to transform the way we look at symmetry and shape in a profound manner." [31]

More critic Mandelbrot on his fractal geometry, he affirmed [39]: "fractals are not a panacea; they are not everywhere…", but it is recognised that fractal geometry is a key to understand the Nature, its complexity, its biological patterns and its phenomena [8, 33, 34, 35, 36, 37, 38, 40].

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