THE COUPLED EINSTEIN-EULER SYSTEM WITH THE COSMOLOGICAL CONSTANT

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ABSTRACT
We prove the global existence of solutions to the coupled Einstein-Euler system with the cosmological constant Λ. We discuss this global existence using the sign of the derivatives of the initial data of potentials of gravitation a, b and the cosmological constant. To obtain the global existence, we suppose that $\Phi > 0,$ where Φ represents the scalar massive field for the Euler equations, because, physically; $\Phi \sim G^{-1}$, G standing for the variable gravitational ``constant". Thereafter, we tranform the Einstein-Euler system which is a second orther differential system into a first order differential system and we apply the standard theory.

Keywords: *Bianchi type; Differential system; potentials of gravitation, scalar massive field, problem of constraints; local existence; global existence.*

Mathematics Subject Classifications: 83C*xx.*

1. INTRODUCTION

In this paper, we study the Einstein-Euler system; the background space-time being the time-oriented Bianchi type 1 space-time, which is an immediate generalization of the flat Friedman-Lemaitre-Robertson-Walker space-time, also known to be the basic space-time of Cosmology. In Cosmology, homogeneous phenomena such as the one we consider here are relevant. The whole universe is modeled and particles in the kinetic theory may be particles of ionized gas as nebular galaxies or even cluster of galaxies, burning reactors, solar wind, for which only the evolution in time is really significant. In the case we consider, the evolution is governed by the coupled Einstein-Euler system, the Einstein equations for gravitational field inquiring about gravitational effects, whereas the Euler equations for the scalar massive field traducing the conservation of the energy momentum tensor which represents the energetic and material content of the space.

The Einstein theory stipulates that the gravitational field, which in the case we consider, depends on the two real valued functions a and b, called potentials of gravitation, is determined through the Einstein equations, by the material and energetic content of space-time.

The Euler equations are important in the sense that they have contributed to modify the General Theory of Relativity. In addition to the space-time, Brans and Dicke introduced a dynamical scalar field Φ in this new modified gravitational theory corresponding to the variable gravitational ``constant" : $\Phi \sim G^{-1}$. That's why, in this paper , we have assumed the homogeneous case $\Phi = \Phi(t)$. The Brans and Dicke Theory also contains a dimensionless free coupling parameter ω between the scalar field Φ and the tensor components of gravitation. In the limit $\omega \to \infty$, the Brans and Dicke Theory reduces to the General Theory of Relativity for a constant scalar field Φ. In our work we will suppose that Φ is not a constant. The system is coupled in the sense that the scalar field Φ which is subjected to the Euler equations is also present in the Einstein equations in a and b while the potentials of gravitation a and b which are subjected to the Einstein equations are also present in the Euler equations.

The coupled Einstein-Euler system turns out to be a non-linear differential system to determine a, b and Φ.

In this paper, we prove using a change of variables, that if the cosmological constant $\Lambda > 0$ and if the initial datum $\dot{b}_0$ of the derivative with respect to t of b is positive, then there exists a global solution to the coupled system. And we also prove that if this derivative is negative, or if the cosmological constant $\Lambda < 0$ even if $\dot{b}_0 > 0$, then there cannot exist global solutions. The fact that we prove global existence with cosmological

constant $\Lambda,$ is also with a great interest. In fact, some recent observations show that the whole universe is in an accelerated expansion, and it is the cosmological constant present in the Einstein equations which mathematically explains this phenomenon.
The paper organizes as follows:
In section $2,$ we present the system and give the equations.
In section $3,$ we study the local existence.
In section $4,$ we study the global existence.

2. EQUATIONS AND PRELIMINARIES

Greek indexes $\alpha,\beta,\gamma,...$ range from 0 to $3,$ and Latin indexes $i,j,k,...$ from 1 to $3.$ We adopt the Einstein summation convention :

$$A_\alpha B^\alpha = \sum_\alpha A_\alpha B^\alpha .$$

2.1 THE EINSTEIN-EULER SYSTEM

We consider the Bianchi type -I space-time $(\mathbb{R}^4, g)$ and denote by $x^\alpha = (x^0, x^i) = (t, x^i)$ the usual coordinates in $\mathbb{R}^4;$ where $x^0 = t$ represents the time and (x^i) the space. $g = (g_{\alpha\beta})$ stands for the metric tensor of hyperbolic signature $(-,+,+,+)$ that can be written :

$$g = -(dt)^2 + a^2(t)(dx^1)^2 + b^2(t)\left[(dx^2)^2 + (dx^3)^2\right] \tag{1}$$

in which $a > 0, b > 0$ are unknown functions of the single variable t.
We require the assumption that a and b be two continuously differentiable functions with respect to t.
Following $[11]$, the Einstein-Euler system with the cosmological constant Λ reads :

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + g_{\alpha\beta}\Lambda = 8\pi T_{\alpha\beta} \tag{2}$$

$$\nabla_\alpha T^{\alpha\beta} = 0 \tag{3}$$

where :
• (2) are the Einstein equations for the metric tensor g that represents the gravitational field, and :
$R_{\alpha\beta}$ is the Ricci tensor, contracted of the curvature tensor;
$R = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature, contracted of the Ricci tensor;
$T_{\alpha\beta}$ is the energy-momentum tensor we specify below :
• (3) are the Euler equations which simply express the conservation of the energy-momentum tensor $T_{\alpha\beta}.$
∇_α is the usual covariant derivative in g and indexes are raised and lowered by $g,$ using the usual rules :

$$V^\alpha = g^{\alpha\beta} V_\beta = g_{\alpha\beta} V^\beta$$

where $(g^{\alpha\beta})$ stands for the inverse matrix of $(g_{\alpha\beta}).$
The general expression of the energy-momentum tensor is :

$$T_{\alpha\beta} = \nabla_\alpha \Phi \nabla_\beta \Phi - \frac{g_{\alpha\beta}}{2}\left[\nabla^\lambda \Phi \nabla_\lambda \Phi + m_0^2 \Phi^2\right] \tag{4}$$

in which Φ is a continuous unknown function, non constant, depending only on the time t and representing the scalar field whose mass is denoted by $m_0,$ with $m_0 > 0.$ Since $\Phi \sim G^{-1},$ where G stands for the variable gravitational ``constant", we can assume that $\Phi > 0.$
The Christoffel symbols of g are given by:

$$\Gamma_{\alpha\beta}^{\lambda} = \frac{1}{2} g^{\lambda\mu} \left[\partial_{\alpha} g_{\mu\beta} + \partial_{\beta} g_{\alpha\mu} - \partial_{\mu} g_{\alpha\beta} \right] \tag{5}$$

The expression (1) shows that the only no vanishing Christoffel symbols $\Gamma_{\alpha\beta}^{\lambda}$ are : Γ_{ii}^{o} , $\Gamma_{i0}^{i}, i = 1,2,3,$ and a direct calculation gives:

$$\begin{cases} \Gamma_{11}^{0} = a\dot{a}; \Gamma_{22}^{0} = \Gamma_{33}^{0} = b\dot{b}; \Gamma_{01}^{1} = \dfrac{\dot{a}}{a}; \Gamma_{02}^{2} = \Gamma_{03}^{3} = \dfrac{\dot{b}}{b} \\ \Gamma_{\alpha\beta}^{\lambda} = 0 \, \text{otherwise.} \end{cases} \tag{6}$$

In (6), the dots stand for derivation with respect to t. Recall that $\Gamma_{\alpha\beta}^{\lambda} = \Gamma_{\beta\alpha}^{\lambda}$.

2.2 Expression of the Euler equations in Φ

According to (3) we have :

$$\nabla_{\alpha} T^{\alpha\beta} = \nabla^{\beta} \Phi \left(\Box_g \Phi - m_0^2 \right) \tag{7}$$

where $\Box_g = \nabla_{\alpha} \nabla^{\alpha}$ is the D'Alembertian.

We deduce from (7) that the Euler equations (3) are satisfied if Φ verifies the second order differential equation:

$$\nabla^{\beta} \Phi \left(\Box_g \Phi - m_0^2 \right) = 0. \tag{8}$$

For $\beta = i,$ using (1), we see that (8) is automatically verified.

For $\beta = 0,$ (8) leads to a non linear differential equation of second order:

$$\dot{\Phi} \left(\ddot{\Phi} + H \dot{\Phi} + m_0^2 \Phi \right) = 0 \tag{9}$$

where H is defined by:

$$H = \frac{\dot{a}}{a} + 2 \frac{\dot{b}}{b} = \frac{1}{2} g^{kl} \partial_0 g_{kl}. \tag{10}$$

Setting in (9):

$$U = \frac{1}{2} \left(\dot{\Phi} \right)^2, \tag{11}$$

we obtain:

$$\begin{cases} U \geq 0 \\ \dot{\Phi} = \pm \sqrt{2U}. \end{cases} \tag{12}$$

We suppose in what follows that Φ *is continuously differentiable, is not a constant and is decreasing . This implies that:*

$$\begin{cases} \dot{\Phi} = -\sqrt{2U} \\ \Phi(t) \leq \Phi(0), t \in \mathbb{R}^{+}. \end{cases} \tag{13}$$

(9) is then equivalent to the non linear first order differential system given below:

$$\dot{\Phi} = -\sqrt{2U} \tag{14}$$

$$\dot{U} = -2HU + m_0^2\Phi\sqrt{2U}. \qquad (15)$$

2.3 Expression of the Einstein equations in a and b.

Proposition 1.

$$S_{00} = 2\frac{\dot{a}\dot{b}}{ab} + \left(\frac{\dot{b}}{b}\right)^2 \qquad (16)$$

$$S_{11} = -a^2\left[2\frac{\ddot{b}}{b} + \left(\frac{\dot{b}}{b}\right)^2\right], S_{22} = S_{33} = -b^2\left[\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab}\right] \qquad (17)$$

$$S_{\alpha\beta} = 0, if\ \alpha \neq \beta. \qquad (18)$$

Proof. See [19].

Proposition 2.

$$\begin{cases} T_{00} = U + \frac{1}{2}m_0^2\Phi^2 \\ T_{11} = a^2U - \frac{1}{2}a^2m_0^2\Phi^2 \\ T_{22} = T_{33} = b^2U - \frac{1}{2}b^2m_0^2\Phi^2 \\ T_{\alpha\beta} = 0 \text{ if } \alpha \neq \beta \end{cases}. \qquad (19)$$

Proof. We have:

$$T_{\alpha\beta} = \nabla_\alpha\Phi\nabla_\beta\Phi - \frac{g_{\alpha\beta}}{2}\nabla^\lambda\Phi\nabla_\lambda\Phi - \frac{g_{\alpha\beta}}{2}m_0^2\Phi^2$$
$$= \partial_\alpha\Phi\partial_\beta\Phi - \frac{1}{2}g_{\alpha\beta}g^{\lambda\mu}\partial_\mu\Phi\partial_\lambda\Phi - \frac{1}{2}g_{\alpha\beta}m_0^2\Phi^2.$$

Invoking the fact that Φ only depends on t, we obtain the result using (1) and (11).

Proposition 3.

The Einstein system in a and b can be written in the form:

$$2\frac{\dot{a}\dot{b}}{ab} + \left(\frac{\dot{b}}{b}\right)^2 - \Lambda = 4\pi\left(2U + m_0^2\Phi^2\right) \qquad (20)$$

$$2\frac{\ddot{b}}{b} + \left(\frac{\dot{b}}{b}\right)^2 - \Lambda = 4\pi\left(-2U + m_0^2\Phi^2\right) \qquad (21)$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} - \Lambda = 4\pi\left(-2U + m_0^2\Phi^2\right), \qquad (22)$$

Proof. The Einstein equations (2) with the cosmological constant Λ write for:

- $\alpha=\beta=0$: $S_{00}+g_{00}\Lambda=8\pi T_{00}$;
- $\alpha=\beta=i$: $S_{ii}+g_{ii}\Lambda=8\pi T_{ii}$.

It is commonly known that $R_{\alpha\beta}-\frac{1}{2}Rg_{\alpha\beta}+g_{\alpha\beta}\Lambda=0$ for $\alpha\neq\beta$. So using (19) which also shows that $T_{\alpha\beta}=0$ if $\alpha\neq\beta$, we conclude that proposition 3 holds.

2.4 The Cauchy problem

The system $(14)-(15)$ which is equivalent to the Euler equations (3) is a system of first order differential equations in U and Φ, if we suppose that in the r.h.s, a and b are given.
The system $(20)-(21)-(22)$ is equivalent to the Einstein equations (2), and shows that those equations are a system of second order non-linear differential equations in a and b, if we suppose that in the r.h.s, Φ and of course U are known. We suppose that $a_0>0$, $b_0>0$, $\Phi_0>0$ and $U_0>0$ are given and we look for solutions U,Φ,a and b of the system $(14)-(15)-(20)-(21)-(22)$ satisfying:

$$\begin{cases} U(0)=U_0 \\ \Phi(0)=\Phi_0 \\ a(0)=a_0, b(0)=b_0, \dot{a}(0)=\dot{a}_0, \dot{b}(0)=\dot{b}_0. \end{cases} \tag{23}$$

Remark 1.

1) Recall that the choice $U(0)=U_0>0$ holds. In fact since Φ is not a constant, $\dot{\Phi}\neq 0$ and $U=\frac{1}{2}\dot{\Phi}^2$ never vanishes, so $U>0$.

2) Since $\Phi : G^{-1}$, physically it is reasonable to suppose that $\Phi>0$.

3) The aim of the present work is to prove the global existence of solutions on $[0,+\infty[$ of the above Cauchy problem, i.e looking for global solutions satisfying conditions (23) called initial conditions. The values prescribed at $t=0$ will be called initial data, and it is obvious that the signs of $\dot{a}_0$ and $\dot{b}_0$ play an important role.

2.5 The problem of constraints

Proposition 4.

1● *The Einstein equation* (20), *called the Hamiltonian constraint, is satisfied all over the domain of the solutions* a *and* b *, if and only if, the initial data* $U_0,\Phi_0,a_0,b_0,\dot{a}_0,\dot{b}_0$ *satisfy the initial condition:*

$$2\frac{\dot{a}_0\dot{b}_0}{a_0b_0}+\left(\frac{\dot{b}_0}{b_0}\right)^2=\Lambda+4\pi\left[2U_0+m_0^2\Phi_0^2\right] \tag{24}$$

2● The remaining Einstein equations:

$$S_{0i}+g_{0i}\Lambda=8\pi T_{0i};S_{ij}+g_{ij}\Lambda=8\pi T_{ij} \tag{25}$$

are identically satisfied by any solutions a and b of $(21)-(22)$.

Proof. ● For the first assertion, see [19].

● The second assertion is obvious since (25) is naturally equivalent to $(21)-(22)$.

In what follows, we suppose that the initial data $U_0, \Phi_0, a_0, b_0, \dot{a}_0, \dot{b}_0$ satisfy the constraint (24). One must also remark that if the cosmological constant Λ is non negative and if

U_0, Φ_0, a_0, b_0 are given, (24) requires that $2\frac{\dot{a}_0\dot{b}_0}{a_0 b_0} + \left(\frac{\dot{b}_0}{b_0}\right)^2 > 0,$ or equivalently:

$$\frac{\dot{a}_0}{a_0}\left(\frac{\dot{b}_0}{b_0} + 2\frac{\dot{a}_0}{a_0}\right) > 0. \tag{26}$$

The above inequality will particularly come true if $\dot{a}_0 > 0$ and $\frac{\dot{b}_0}{b_0} + 2\frac{\dot{a}_0}{a_0} > 0.$ It suffices then that:

$$\dot{a}_0 > 0, \dot{b}_0 > 0. \tag{27}$$

In the next sections we look for global existence of solutions a and b of equations (21) and (22), called evolution equations. The Hamiltonian constraint (20) will be used as a property of the solutions.

3. LOCAL EXISTENCE OF SOLUTIONS

The Einstein-Euler system $(2)-(3)$ is , using $(14),(15),(20),(21),(22),(24)$ equivalent to :

$$\begin{cases} \dot{\Phi} = -\sqrt{2U} \\ \dot{U} = -2HU + m_0^2\Phi\sqrt{2U} \\ 2\frac{\ddot{b}}{b} + \left(\frac{\dot{b}}{b}\right)^2 - \Lambda = 4\pi\left(m_0^2\Phi^2 - 2U\right) \\ \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} - \Lambda = 4\pi\left(m_0^2\Phi^2 - 2U\right). \end{cases} \tag{28}$$

Equations (28) easily show that we must have :

$$\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} - \left(\frac{\dot{b}}{b}\right)^2 = 0. \tag{29}$$

If we set :

$$u = \frac{\dot{a}}{a}, v = \frac{\dot{b}}{b}, \tag{30}$$

then :

$$\frac{\ddot{a}}{a} = \dot{u} + u^2, \frac{\ddot{b}}{b} = \dot{v} + v^2. \tag{31}$$

So (28) becomes :

$$\begin{cases} \dot{\Phi} = -\sqrt{2U} \\ \dot{U} = -2(u+2v)U + m_0^2\Phi\sqrt{2U} \\ \dot{u} = -u^2 + \frac{1}{2}v^2 - uv + \frac{\Lambda}{2} + 2\pi m_0^2\Phi^2 - 4\pi U \\ \dot{v} = -\frac{3}{2}v^2 + \frac{\Lambda}{2} + 2\pi m_0^2\Phi^2 - 4\pi U, \end{cases} \tag{32}$$

where using (29), u and v must be linked by the relation :

$$\dot{u} - \dot{v} + uv + u^2 - 2v^2 = 0. \tag{33}$$

But invoking (29), we clearly obtain that (33) is satisfied by any solutions u and v of (32).
We have now to solve the system (32) which is equivalent, owing to (20) and (24), to the Einstein-Euler system $(2)-(3)$ with the initial conditions:

$$u(0) = u_0 = \frac{\dot{a}_0}{a_0}, v(0) = v_0 = \frac{\dot{b}_0}{b_0}, U(0) = U_0, \Phi(0) = \Phi_0. \tag{34}$$

Setting:

$$X = (U, \Phi, u, v)^T, \tag{35}$$

the system of equations (32) can be written in the form :

$$\dot{X} = f(X), \tag{36}$$

where the function f is continuously differentiable with respect to the time t and all its arguments.
As a result, f is locally Lipschitzian. Then by the standard theory on first order differential systems, the system $(32)-(34)$ admits a *unique local solution*.
It remains to show that this solution is **global** on $[0,+\infty[$.

4. GLOBAL EXISTENCE OF SOLUTIONS

We are searching the global existence of solutions a, b and Φ to the system $(2)-(3)$ using the cosmological constant Λ. It is sufficient to search the global existence of solutions to the equivalent system $(9)-(21)-(22)$ subjected to the Hamiltonian constraint (24).

Remark **2.** *The Hamiltonian constraint* (24) *can be written:*

$$\left(\frac{\dot{a}_0}{a_0} + \frac{\dot{b}_0}{b_0}\right)^2 = \Lambda + \left(\frac{\dot{a}_0}{a_0}\right)^2 + 4\pi\left(2U_0 + m_0^2\Phi_0^2\right) \tag{37}$$

and shows that we must have :

$$\Lambda \in \left[-\left(\frac{\dot{a}_0}{a_0}\right)^2 - 4\pi\left(2U_0 + m_0^2\Phi_0^2\right), +\infty\right[. \tag{38}$$

Lemma 1. *Let* x *and* y *be two real valued and differentiable functions of the variable* t *such that :*

$$\begin{cases} \dot{x} \le c_0^2 - \gamma^2 x^2 \left(\text{resp } \dot{x} \ge c_0^2 - \gamma^2 x^2\right) \\ \dot{y} = c_0^2 - \gamma^2 y^2 \\ x_0(t_0) = y_0(t_0), c_0, t_0, \gamma \in \mathbb{R}, \end{cases}$$

then :

$$x(t) \le y(t) \left(\text{resp } x(t) \ge y(t)\right), \forall t \ge t_0.$$

Proof. The system $\begin{cases} \dot{x} \le c_0^2 - \gamma^2 \\ \dot{y} = c_0^2 - \gamma^2 y^2 \end{cases}$ implies that $\dot{x} - \dot{y} \le \gamma^2 \left(y^2 - x^2\right)$. If we set $Y = x - y$ and $A = \gamma^2 (x + y)$, we obtain $\dot{Y} + AY \le 0$.

Multiplying the above inequality by $\exp\left(\int_{t_0}^t A(s) ds\right)$, we get :

$$\frac{d}{dt}\left(Y \exp\left(\int_{t_0}^t A(s) ds\right)\right) \le 0.$$

Integrating then over $[t_0, t[$ leads to the inequality:

$$Y(t) \exp\left(\int_{t_0}^t A(s) ds\right) \le Y(0),$$

which writes :

$$x(t) \le y(t) \forall t \ge t_0.$$

Lemma 2. $H = \frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}$ *satisfies the relations:*

$$\dot{H} + \left(\frac{\dot{a}}{a}\right)^2 + 2\left(\frac{\dot{b}}{b}\right)^2 = \Lambda - 16\pi U + 4\pi m_0^2 \Phi^2 \tag{39}$$

$$\dot{H} + H^2 = 3\Lambda + 12\pi m_0^2 \Phi^2. \tag{40}$$

Proof. If we compute $-\frac{1}{2}(20) + \frac{1}{2}(21) + (22)$, we obtain :

$$\frac{\ddot{a}}{a} + 2\frac{\ddot{b}}{b} = \Lambda + 4\pi m_0^2 \Phi^2 - 16\pi U. \tag{41}$$

But we have :

$$\dot{H} = \frac{\ddot{a}}{a} + 2\frac{\ddot{b}}{b} - \left(\frac{\dot{a}}{a}\right)^2 - 2\left(\frac{\dot{b}}{b}\right)^2. \tag{42}$$

Substituting (41) in (42) gives the relation (39).

It follows clearly using $(20),(42)$ and $H=\frac{\dot a}{a}+2\frac{\dot b}{b}$ that:

$$\dot H+H^2=3\Lambda+12\pi m_0^2\Phi^2.$$

Lemma 3. *The Cauchy problem in* W :

$$\begin{cases}\dot W=c_0^2-W^2\\ W(0)=W_0=H(0),\end{cases}\tag{43}$$

where $H(0)=H_0\neq 0$ and $c_0\neq 0$ are given, has a unique solution defined by:

$$W(t)=c_0\left[\frac{(H_0+c_0)e^{2c_0t}+(H_0-c_0)}{(H_0+c_0)e^{2c_0t}-(H_0-c_0)}\right].\tag{44}$$

Proof. c_0 is a trivial solution of (43). Setting :

$$W=c_0+x,\tag{45}$$

x verifies the equation :

$$\dot x=-2c_0x-x^2.\tag{46}$$

Since (46) is a Bernoulli equation, we can set $f=\frac{1}{x}$ and we obtain :

$$-\dot f+2c_0f+1=0.\tag{47}$$

Solving (47), we find:

$$f(t)=-\frac{1}{2c_0}+\frac{H_0+c_0}{2c_0(H_0-c_0)}e^{2c_0t}=\frac{1}{x}.\tag{48}$$

Invoking (45), we obtain the result.

4.1 Study of the case $\Lambda<0$

Proposition 5. *If the cosmological constant* $\Lambda<0$, *then the Einstein-Euler system has no global solution on* $[0,+\infty[$.

Proof. See $[20]$.

4.2 Study of the case $\Lambda\geq 0$

Observe first of all that when $\Lambda\geq 0$, we have $\dot b_0\neq 0$. This comes from equation (24) which implies that :

$$\frac{\dot b_0}{b_0}\left(\frac{\dot a_0}{a_0}+2\frac{\dot b_0}{b_0}\right)>0,$$

giving then the result.

Proposition 6. *If* $\dot b_0<0$ *and* $\Lambda\geq 0$, *the Einstein-Euler system has no global solution on* $[0,+\infty[$.

Proof. Let us assume that $\Lambda \geq 0$ and $\dot{b}_0 < 0,$ and that the Einstein-Euler system has a global solution on $[0,+\infty[.$

By $(20),$ we have:

$$v(v+2u) \geq 8\pi U = c_0^2 U > 0 \tag{49}$$

where $c_0^2 = 8\pi$ and since by hypothesis $U > 0.$

We deduce from (49) that $v \neq 0$ and $v + 2u \neq 0.$

Since $\dot{b}_0 < 0,$ then $v(0) = \dfrac{\dot{b}_0}{b_0} < 0.$

So by the Weierstrass intermediate value theorem:

$$v < 0. \qquad (a)$$

Thus

$$v + 2u < 0. \quad (b)$$

Now:

$$v(v+2u) = v^2 + 2uv = (u+v)^2 - u^2 \geq c_0^2 U,$$

and

$$(u+v)^2 \geq c_0^2 U > 0.$$

Consequently

$$|u+v| \geq c_0 \sqrt{U}. \quad (c)$$

But by (a) and $(b),$ we have:

$$v + u = \frac{1}{2}(2v+2u) = \frac{1}{2}[v + (v+2u)] < 0.$$

So also invoking $(c),$ we obtain:

$$|u+v| = -u - v \geq c_0 \sqrt{U},$$

which leads to:

$$u + v \leq -c_0 \sqrt{U}.$$

It comes from (32) where $\dot{U} = -2(u+2v)U + m_0^2 \Phi \sqrt{2U}$ that:

$$\frac{d}{dt}\left[\frac{1}{\sqrt{U}}\right] = \frac{-\dot{U}}{2U\sqrt{U}} = \frac{2(u+2v)U - m_0^2 \Phi \sqrt{2U}}{2U\sqrt{U}}$$

$$\leq \frac{2(u+2v)U}{2U\sqrt{U}} = \frac{(u+2v)}{\sqrt{U}} = \frac{(u+v)+v}{\sqrt{U}}$$

$$\leq \frac{(u+v)}{\sqrt{U}} \leq -c_0.$$

Thus:

$$\frac{d}{dt}\left[\frac{1}{\sqrt{U}}\right] \le -c_0.$$

Integrating over $[0,t]$, we get:

$$\frac{1}{\sqrt{U}} \le \frac{1}{\sqrt{U_0}} - c_0 t,$$

where $t > 0$ is an arbitrary real number , since the solution is global.
The r.h.s of the above inequality vanishes after a finite time

$$t^* = \frac{1}{c_0\sqrt{U_0}}$$

and this implies that $\frac{1}{\sqrt{U}}$ also vanishes. This is a contradiction because $U > 0$ never vanishes.

Theorem 1. 1) *If* $\Lambda \ge 0$ *and* $\dot{b}_0 > 0$, *then the Cauchy problem* $(32)-(34)$ *in* (Φ, U, u, v) *has a global solution on* $[0,+\infty[$.
2) The Einstein-Euler system $(2)-(3)$ in (a,b,Φ) with initial data $a(0) = a_0, b(0) = b_0, \dot{a}(0) = \dot{a}_0, \dot{b}(0) = \dot{b}_0, \Phi(0) = \Phi_0$ has a global solution on $[0,+\infty[$.

Proof. 1) We proved in section 3 that the Cauchy problem $(32)-(34)$ has a solution (Φ, U, u, v) on every interval $[0,T], T > 0$.
By the standard theory on first order differential system, to show that this solution (Φ, U, u, v) is global, it will be sufficient to prove that for every $T \in [0,+\infty[$, (Φ, U, u, v) is *uniformly bounded* on $[0,T[$.

- $t \mapsto \Phi(t)$ is uniformly bounded on $[0,T[$.

In fact, we have using (13) that :

$$\Phi(t) \le \Phi_0, \forall t \ge 0.$$

This implies that :

$$\Phi^2(t) \le \max\{\Phi_0^2, \Phi^2(T)\}, \forall t \in [0,T[. \tag{50}$$

- $t \mapsto u(t)$ and $t \mapsto v(t)$ are uniformly bounded on $[0,T[$.

a) First of all by equations (19) and (20) we have :

$$v(v+2u) = \Lambda + 4\pi\left(2U + m_0^2\Phi^2\right).$$

This shows that $v(v+2u) \ge 0$.

On the other hand $v(0) = \frac{\dot{b}_0}{b_0} > 0$ and v is continuous, so by the intermediate value theorem, $v > 0$. Thus $v + 2u > 0$.

b) To show that u and v are uniformly bounded on $[0,T[$, it suffices to show that H is uniformly bounded on the same interval.
We have :

$$H = u + 2v = \frac{1}{2}(2u+v) + \frac{3}{2}v \geq \frac{3}{2}v$$

since $2u + v \geq 0.$
So :

$$H \geq 0.$$

But by lemma $2,$ equation $(40),$ we have :

$$\dot{H} + H^2 = 3\Lambda + 12m_0^2\Phi^2.$$

So using $(50),$ we get:

$$\dot{H} + H^2 \leq c_0^2, \tag{51}$$

where

$$c_0^2 = 3\Lambda + 12m_0^2 \max\{\Phi_0^2, \Phi^2(T)\}.$$

Consider the following system:

$$\begin{cases} \dot{H} \leq c_0^2 - H^2 \\ \dot{W} = c_0^2 - W^2 \\ W(0) = H(0) = H_0. \end{cases} \tag{52}$$

According to lemma $1,$ we have $H(t) \leq W(t), \forall t \geq 0.$
If we now consider the problem :

$$\begin{cases} \dot{W} = c_0^2 - W^2 \\ W(0) = H(0) = H_0, \end{cases}$$

by lemma 3, we will have :

$$W(t) = c_0\left[\frac{(H_0 + c_0)e^{2c_0 t} + (H_0 - c_0)}{(H_0 + c_0)e^{2c_0 t} - (H_0 - c_0)}\right]. \tag{53}$$

Setting then:

$$h_1(t) = (H_0 + c_0)e^{2c_0 t} + (H_0 - c_0)$$

and

$$h_2(t) = (H_0 + c_0)e^{2c_0 t} - (H_0 - c_0),$$

we find:

$$\dot{h}_1(t) = \dot{h}_2(t) = 2c_0(H_0 + c_0)e^{2c_0 t} > 0.$$

So

$$\begin{cases} h_1(t) > 2H_0 \\ h_2(t) > 2c_0. \end{cases}$$

This shows that

$$W(t) > 0, \forall t \geq 0.$$

Since $W > 0$ is continuous on $[0,+\infty[$ and since $\lim_{t\to+\infty} W(t) = c_0$, we conclude that W is uniformly bounded. Consequently, by $H(t) \le W(t)$, H is also uniformly bounded.

c) Now the inequality $\frac{3}{2}v \le H$ implies since $v \ge 0$, that v is uniformly bounded.

Since $2v + u = H$ and since $v \ge 0$, we have $u \le H$.

The inequality $2u + v \ge 0$ implies that $u \ge -\frac{v}{2}$. Consequently:

$$-\frac{v}{2} \le u \le H$$

and u is uniformly bounded.

● $t \mapsto U(t)$ is uniformly bounded on $[0,T[$.

By integrating the second equation in (32) over $[0,t], t \in [0,T]$, we find:

$$U(t) = U(0) - 2\int_0^t (u+2v)U ds + m_0^2 \int_0^t \Phi\sqrt{2U}\, ds.$$

So

$$|U(t)| \le U_0 + 2\int_0^t |u+2v| U ds + m_0^2 \int_0^t \left|\Phi\sqrt{2U}\right| ds \tag{54}$$

Since H is uniformly bounded, there exists $p_0 > 0$ such that $|u+2v| \le p_0$. If we also invok (14) and (50), the inequality (54) yields :

$$|U(t)| \le U_0 + \frac{1}{2}m_0^2 \max\{\Phi_0^2, \Phi^2(T)\}T + 2p_0\int_0^t U(s) ds. \tag{55}$$

Setting $\alpha_0 = 2p_0$ and $\beta_0 = U_0 + \frac{1}{2}m_0^2 \max\{\Phi_0^2, \Phi^2(T)\}T$ and using the Gromwall inequality on $[0,t], t \in [0,T]$, we obtain :

$$|U(t)| \le \beta_0 e^{\alpha_0 T}. \tag{56}$$

Consequently, U is uniformly bounded.

We deduce from all the preceding steps that any solution (Φ, U, u, v) of the Cauchy problem $(32)-(34)$ on $[0,T[, T > 0$ is global on $[0,+\infty[$.

$2)$ Since the Cauchy problem $(32)-(34)$ and the system $(2)-(3)$ in (a,b,Φ) with the initial conditions $a(0) = a_0, b(0) = b_0, \dot{a}(0) = \dot{a}_0, \dot{b}(0) = \dot{b}_0, \Phi(0) = \Phi_0$ are equivalent, we conclude owing to the step $1)$, that the Einstein-Euler system $(2)-(3)$ with initial data $a(0) = a_0, b(0) = b_0, \dot{a}(0) = \dot{a}_0, \dot{b}(0) = \dot{b}_0, \Phi(0) = \Phi_0$ has a global solution on $[0,+\infty[$.

One should also remark that since u and v are uniformly bounded, there exist a constant $C > 0$ such that $\forall t \in [0,T], T > 0$:

$$a(t) \le a_0 e^{CT}; b(t) \le b_0 e^{CT}; \frac{1}{a}(t) \le \frac{1}{a_0} e^{CT}; \frac{1}{b}(t) \le \frac{1}{b_0} e^{CT}.$$

5. CONCLUSION

The physical significance of the work we did in the present paper, is the study of the global dynamics of a kind of fast moving, massive particles. We have coupled the Einstein equations whose unknowns are a and b representing the potentials of gravitation, to the Euler equations for the scalar massive field, represented by the unknown Φ. We have made some changes of variables to transform the initial system in (a, b, Φ) into a first order differential system with a view to apply the standard theory and to obtain regular solutions, for some given initial data.

In our future investigations, we intend to study the asymptotic behaviour and the geodesic completeness of our regular global solutions. We also intend to couple the Einstein-Euler system to the Boltzmann equation or to the Maxwell equations. Those investigations seem to have a great interest in the sence that, the systems obtained lead to a heavy problem of physical constraints which we find in several natural phenomena.

6. REFERENCES

[1]. Bancel, D. (1973) Probleme de Cauchy pour l'équation de Boltzmann en relativité Générale. Ann. Henri Poincaré. 18, 3-263.

[2]. Bancel, D, & Choquet-Bruhat, Y.(1973). Uniqueness and local stability for the Einstein-Maxwell-Boltzmann system. Comm.Math. Phys. 33, 83.

[3]. Barrow, D; Maartens, R, & Tsagas, C.G.(2007). Cosmology with inhomogeneous magnetic fields. Rep. 449, 131.

[4]. Chae, D. (2003). Global existence of solutions to the coupled Einstein and Maxwell-Higgs system in the spherical symmetry. Annales Henri Poincaré 4(1), 35.

[5]. Ehlers, J. A survey of General Relativity. Astrophysics and Cosmology. E.N.I.

[6]. El-Nabusi, R.A. (2010). Living with phantoms in a sheet spacetime. Gen.Rel.Grav. 42(6), 1381.

[7]. Georgiou,A. (2012). A Global solution of the Einstein-Maxwell field Equations for Rotating Charged Matter. J.Mod.Phys. 3,1301.

[8]. Glassey, R.T, & Strauss,W. (1992). Asymptotic stability of the relativistic Maxwellian. Pub.Math. RIMS Kyoto. 29, 301.

[9]. Horvat, D, & Ilijic, S. (2007). Regular and singular solutions for charged dust distributions in the Einstein-Maxwell theory. Can.J.P.Phys. 85957.

[10]. Horvat, D; Ilijic, S, & Marunovic, A. (2009). Electrically charged gravastar congurtions, Class.Quant.Grav.26, 025003.

[11]. Lichnerowich,A. (1955). Théories Relativistes de la gravitation et de l'électromagnétisme. Masson et Cie Editeurs.

[12]. Mucha, P.B. (1998). The Cauchy problem for the Einstein-Vlasov system. Journal of Applied Analysis. 4(1), 111-127.

[13]. Mucha, P.B. (1998). The Cauchy problem for the Einstein-Boltzmann system. Journal of Applied Analysis. 4(1), 129-141.

[14]. Mucha, P.B. (2000). Global existence of solutions of the Einstein-Boltzmann equation in the spatially homogeneous case. Evolution Equation, Existence, Regularity and Singularities. Banach Center Publications. institute of Mathematics. Polish Academy of Science, Warzawa. 52.

[15]. Noutchegueme,N; Dongho, D, & Takou, E. (2005). Global existence of solutions for the relativistic Boltzmann equation with arbitrarily large initial data on a Bianchi type 1 space-time. Gen.Relat.Grav. 37(12), 2047-2062.

[16]. Noutchegueme, N, & Dongho, D. (2006). Global existence of solutions for the Einstein-Boltzmann system in a Bianchi type 1 space-time for arbitrarily large initial data. Class Quantum Grav. 23, 2979-3003.

[17]. Noutchegueme, N, & Tetsadjio. T, M.E. (2009). (CQG).

[18]. Noutchegueme, N, & Ayissi, R.D.(2010). Global existence of solutions to the Maxwell-Boltzmann system in a Bianchi type 1 space-time. Adv. Studies Theor.phys., 4(18), 855-878.

[19]. Noutchegueme,N & Nangne,C. Global existence of solution to the coupled EINSTEIN-MAXWELL system for a perfect charged relativistic fluid in a ROBERTSON-WALKER space-time, Advanced Studies in Theoretical Physiscs, (2008).

[20]. Ayissi,R.D. (2013). The EINSTEIN-MAXWELL system for a perfect charged relativistic fluid in a BIANCHI type I space-time. Applied Physics Research (accepted paper to appear).