A NINTH-ORDER NEWTON-TYPE METHOD TO SOLVE SYSTEMS OF NONLINEAR EQUATIONS

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ABSTRACT
In this paper, modification of Newton's method with ninth-order convergence is presented. The modification of Newton's method is based on Darvishi and Barati's third-order method. The new method requires three-step per iteration. Analysis of convergence demonstrates that the order of convergence is 9. Some numerical examples illustrate that the algorithm is more efficient and performs better than classical Newton's method and other methods.

Keywords: System of non-linear equations, Iterative methods, Newton-like's method, Order of convergence.

1. INTRODUCTION

Consider the system of nonlinear equations

$$\begin{aligned} &f_1(x_1,x_2,...,x_n)=0,\\ &f_2(x_1,x_2,...,x_n)=0,\\ &\vdots\\ &f_n(x_1,x_2,...,x_n)=0, \end{aligned} \tag{1}$$

where each function f_i maps a vector $x=(x_1,x_2,...,x_n)$ of the $n-$dimensional space R^n into the real line R. The system (1.1) of n nonlinear equations in n unknowns can also be represented by defining a function F mapping R^n into R^n as

$$F(\mathrm{x})=(f_1(x),f_2(x),...,f_n(x))^t. \tag{2}$$

Thus, the system (1.1) can be written in the form $F(x)=0$, where the functions $f_1(x)$, $f_2(x)$, ..., $f_n(x)$ are the coordinate functions of F. In recent years, several iterative methods have been developed to solve the nonlinear system of equations $F(x)=0$ including essentially Taylor's polynomial, decomposition, homotopy perturbation method, quadrature formulas methods. For more details, see[1-26] and the references therein. M.T. Darvishi and A. Barati [5] developed third-order method, which is written as:

$$x_{k+1}=x_k-F'(x_k)^{-1}(F(x_k)+F(x_{k+1}^*)), \tag{3}$$

where $x_{k+1}^*=x_k-F'(x_k)^{-1}F(x_k)$. Motivated and inspired by the on-going activities in this direction, we construct a modification (based on the above M.T. Darvishi and A. Barati's method) of Newton's method with higher-order convergence for solving the nonlinear system of equations. It has been shown that this three-step iterative method is ninth-order convergence. Several numerical examples are given to illustrate the efficiency and the performance of the new iterative methods. Our results can be viewed as an improvement and refinement of the previously known results.

2. THE METHOD AND ANALYSIS OF CONVERGENCE

From (3), we construct a three-step iterative method

$$\begin{cases} y_k = x_k - F'(x_k)^{-1}F(x_k), \\ z_k = x_k - F'(x_k)^{-1}F(x_k) - F'(x_k)^{-1}F(y_k), \\ x_{k+1} = z_k - F'(z_k)^{-1}F(z_k) - F'(z_k)^{-1}F(z_k - F'(z_k)^{-1}F(z_k)), \end{cases} \tag{4}$$

where $\mathrm{x}_k = (x_{1k}, x_{2k}, ..., x_{nk})^t$, $\mathrm{y}_k = (y_{1k}, y_{2k}, ..., y_{nk})^t$, $\mathrm{z}_k = (z_{1k}, z_{2k}, ..., z_{nk})^t$ and $\mathrm{F}'(x_k)$ is Jacobian matrix. We let $\mathrm{F}'(\mathrm{x})^{-1} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, ... \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1}, \frac{\partial f_2}{\partial x_2}, ... \frac{\partial f_2}{\partial x_n} \\ \dots \\ \frac{\partial f_n}{\partial x_1}, \frac{\partial f_n}{\partial x_2}, ... \frac{\partial f_n}{\partial x_n} \end{pmatrix}^{-1} = G(\mathrm{x}).$ So the (4) can be expressed by

$$\begin{cases} y_k = x_k - G(x_k)F(x_k), \\ z_k = x_k - G(x_k)F(x_k) - G(x_k)F(y_k), \\ x_{k+1} = z_k - F'(z_k)^{-1}F(z_k) - F'(z_k)^{-1}F(z_k - F'(z_k)^{-1}F(z_k)). \end{cases} \tag{5}$$

We can easily prove that scheme (5) is ninth-order convergent. In order to avoid the computation of the first derivative and inverse function $F'(z_k)^{-1}$, we use $G(z_k)$ instead of $F'(z_k)^{-1}$. Therefore, a new scheme is as follows:

$$\begin{cases} y_k = x_k - G(x_k)F(x_k), \\ z_k = x_k - G(x_k)F(x_k) - G(x_k)F(y_k), \\ x_{k+1} = z_k - G(z_k)F(z_k) - G(z_k)F(z_k - G(z_k)F(z_k)). \end{cases} \tag{6}$$

This proves that the method defined by (6) has ninth-order convergence.

Theorem 1. The iterative method (6) has local order of convergence at least nine with the following error equation

$$z_k - G(z_k)F(z_k) - G(z_k)F(z_k - G(z_k)F(z_k)) = \alpha + 16c_2^8 e_k^9 + o(\|e_k^{10}\|). \tag{7}$$

Proof: Let α be a simple zero of F. As F is a sufficiently differentiable function, by expanding $F(x_k)$ and $F'(x_k)$ about α, we get

$$F(x_k) = F'(\alpha)[e_k + c_2e_k^2 + c_3e_k^3 + c_4e_k^4 + c_5e_k^5 + c_6e_k^6 + c_7e_k^7 + c_8e_k^8] + o(\|e_k^9\|), \tag{8}$$

and

$$F'(x_k) = F'(\alpha)[1 + 2c_2e_k + 3c_3e_k^2 + 4c_4e_k^3 + 5c_5e_k^4 + 6c_6e_k^5 + 7c_7e_k^6 + 8c_8e_k^7] + o(\|e_k^8\|), \tag{9}$$

where $c_k = (1/k!)F'(\alpha)^{-1}F^{(k)}(\alpha)$, $k = 2,3,...$ and $e_k = x_k - \alpha$. As the terms in the square brackets are polynomials in terms of e_k, direct division gives us

$$\begin{aligned} & F'(x_k)^{-1}F(x_k) = G(x_k)F(x_k) \\ & = e_k - c_2e_k^2 + 2(c_2^2 - c_3)e_k^3 + (7c_2c_3 - 3c_4 - 4c_2^3)e_k^4 + (10c_2c_4 - 4c_5 + 6c_3^2 - 20c_3c_2^2 + 8c_2^4)e_k^5 \\ & + (17c_3c_4 - 33c_2c_3^2 + 52c_3c_2^3 - 28c_4c_2^2 + 13c_2c_5 - 5c_6 - 16c_2^5)e_k^6 + o(\|e_k^7\|). \end{aligned} \tag{10}$$

From (10), we have

$$\mathrm{y}_k = \alpha + c_2 e_k^2 - 2(c_2^2 - c_3)e_k^3 - (7c_2c_3 - 3c_4 - 4c_2^3)e_k^4 - (10c_2c_4 - 4c_5 + 6c_3^2 - 20c_3c_2^2 + 8c_2^4)e_k^5$$
$$-(17c_3c_4 - 33c_2c_3^2 + 52c_3c_2^3 - 28c_4c_2^2 + 13c_2c_5 - 5c_6 - 16c_2^5)e_k^6 + o(\|e_k^7\|). \quad (11)$$

By (9) and (11), we have

$$G(x_k)F(y_k) = c_2e_k^2 + (2c_3 - 4c_2^2)e_k^3 + (3c_4 - 14c_2c_3 + 13c_2^3)e_k^4 + (-12c_3^2 + 64c_3c_2^2$$
$$-20c_2c_4 - 38c_2^4)e_k^5 + (-34c_3c_4 + 90c_4c_2^2 - 240c_3c_2^3 + 104c_2^5$$
$$+103c_2c_3^2)e_k^6 + (288c_2c_3c_4 - 24c_4^2 - 336c_4c_2^3 - 558c_3^2c_2^2 + 800c_3c_2^4 \quad (12)$$
$$+54c_3^3 - 272c_2^6)e_k^7 + o(\|e_k^8\|).$$

Combining (10) with (12), we get

$$z_k = \alpha + 2c_2^2e_k^3 + (7c_2c_3 - 9c_2^3)e_k^4 + (6c_3^2 - 44c_3c_2^2 + 10c_2c_4 + 30c_2^4)e_k^5$$
$$+(17c_3c_4 - 62c_4c_2^2 + 188c_3c_2^3 - 88c_2^5 - 70c_2c_3^2)e_k^6 + o(\|e_k^7\|). \quad (13)$$

With (13), we obtain

$$F(z_k) = F'(\alpha)(2c_2^2e_k^3 + (7c_2c_3 - 9c_2^2)e_k^4 + (6c_3^2 - 44c_3c_2^2 + 10c_2c_4 + 30c_2^4)e_k^5$$
$$-(84c_2^5 - 17c_3c_4 + 62c_4c_2^2 - 188c_3c_2^3 + 70c_2c_3^2)e_k^6) + o(\|e_k^7\|), \quad (14)$$

$$G(z_k)F(z_k) = 2c_2^2e_k^3 + (7c_2c_3 - 9c_2^3)e_k^4 + (6c_3^2 - 44c_3c_2^2 + 10c_2c_4 + 30c_2^4)e_k^5$$
$$+(17c_3c_4 - 92c_2^5 - 62c_4c_2^2 + 188c_3c_2^3 - 70c_2c_3^2)e_k^6 + o(\|e_k^7\|), \quad (15)$$

and

$$z_k - G(z_k)F(z_k) = \alpha + 4c_2^5e_k^6 + (28c_3c_2^4 - 36c_2^6)e_k^7 + o(\|e_k^8\|). \quad (16)$$

With (16), we get

$$F(z_k - G(z_k)F(z_k)) = 4c_2^5e_k^6 + (28c_3c_2^4 - 36c_2^6)e_k^7 + o(\|e_k^8\|). \quad (17)$$

From (13),(15),(17), we have

$$z_k - G(z_k)F(z_k) - G(z_k)F(z_k - G(z_k)F(z_k)) = \alpha + 16c_2^8e_k^9 + o(\|e_k^{10}\|). \quad (18)$$

This shows the ninth-order convergence of the method. Hence, the proof is completed.

3. NUMERICAL RESULTS

We now present some examples to illustrate the efficiency and the comparison of the newly developed method, see Table 1. We compare Newton's method (NM), the Darvishi and Barati method [5], the methods NAd1 and NAd2 [26] and our Newton-type method. All computations were done using Visual C++ 6.0. We used $\varepsilon = 10^{-12}$. The following stopping criteria is used for computer programs[26]:

(i) $\|\mathrm{x}_{k+1} - \mathrm{x}_k\|_\infty + \|F(\mathrm{x}_k)\|_\infty < \varepsilon$

(1) $F(x_1, x_2) = (\exp(x_1^2) - \exp(\sqrt{2}x_1), x_1 - x_2), \bar{x}_1 = (\sqrt{2}, \sqrt{2})^T, \bar{x}_2 = (0.0, 0.0)^T.$

(2) $F(x_1, x_2) = (x_1 + \exp(x_2) - \cos(x_2), 3x_1 - x_2 - \sin(x_2)), \bar{x} = (0.0, 0.0)^T.$

(3) $F(x_1, x_2) = (x_1^2 - 2x_1 - x_2 + 0.5, x_1^2 + 4x_2^2 - 4), \bar{x} = (1.9007, 0.3112)^T.$

(4) $F(x_1, x_2) = (x_1^2 + x_2^2 - 1, x_1^2 - x_2^2 + 0.5), \bar{x}_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})^T, \bar{x}_2 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})^T.$

(5) $F(x_1, x_2) = (\sin(x_1) + x_2\cos(x_1), x_1 - x_2), \bar{x} = (0.0, 0.0)^T$.

(6) $F(x) = (f_1(x), f_2(x), ..., f_n(x))$,

where $\mathrm{x} = (x_1, x_2, x_3, ..., x_n)^T$ and $f_i : R^n \to R, i = 1,2,...,n,$ such that

$$f_i(x) = x_i x_{i+1} - 1, i = 1,2,...,n-1$$

$$f_n(x) = x_n x_1 - 1.$$

When n is odd, the exact zeros of $\mathrm{F(x)}$ are $\bar{x}_1 = (1,1,...,1)^T$ and $\bar{x}_2 = (-1,-1,...,-1)^T$. Results appearing in Table1 are obtained for $n = 9$.

(7) $F(x) = (f_1(x), f_2(x), f_3(x), f_4(x)),$

where $\mathrm{x} = (x_1, x_2, x_3, x_4)^T$ and $f_i : \mathrm{R}^4 \to \mathrm{R}, i = 1,2,3,4$, such that

$$f_1(x) = x_2 x_3 + x_4(x_2 + x_3)$$

$$f_2(x) = x_1 x_3 + x_4(x_1 + x_3)$$

$$f_3(x) = x_1 x_2 + x_4(x_1 + x_2)$$

$$f_4(x) = x_1 x_2 + x_1 x_3 + x_2 x_3 - 1.$$

The zeros of $F(x)$ are $\bar{x}_1 = (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{2\sqrt{3}})^T$ and

$$\bar{x}_2 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{2\sqrt{3}})^T.$$

Table 1: Numerical results for nonlinear systems.

F(x)	$\mathrm{x}^{(0)}$	Iterations				
		NM	Darvishi and Barati	NAd1	NAd2	Newton-type
(1)	$(2.3,2.3)^T$	10	8	7	6	3
	$(1.8,1.8)^T$	7	5	5	4	2
	$(0.8,0.8)^T$					
(2)	$(1.5,2.0)^T$	5	4	3	3	2
	$(0.3,0.5)^T$	7	6	5	4	2
(3)	$(3.0,2)^T$	5	4	4	3	2
	$(1.6,0.0)^T$	7	5	5	4	2
(4)	$(0.7,1.2)^T$	5	4	4	4	2
	$(-1.0,-2.0)^T$	5	4	3	3	2
(5)	$(1.2,-1.5)^T$	6	4	4	4	2
	$(-0.6,0.6)^T$	6	4	4	3	2
(6)	$(2.0,...,2.0)^T$	5	3	3	3	2
	$(-4,...,-4)^T$	6	5	4	4	3
		7	6	5	4	2
(7)	$(-1,-1,-1,-1)^T$	6	5	4	4	3
	$(2.0,2.0,2.0,0.0)^T$	7	5	5	4	2

4. CONCLUSION

We have obtained a Newton-type iterative method for nonlinear systems and proved that its convergence order is nine. This method is efficient and effective for solving systems of nonlinear equations, with moderate computation and good accuracy.

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