FIXED POINT AND HYERS-ULAM-RASSIAS STABILITY OF A QUADRATIC FUNCTIONAL EQUATION IN BANACH SPACES

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ABSTRACT

In this paper, using the fixed point alternative approach, we prove the Hyers Ulam-Rassias stability of the following quadratic functional equation

$$f(2x+y)+f(x+2y)=4f(x+y)+f(y)+f(x)$$

in various spaces.

Keywords: *Fixed point, Hyers-Ulam-Rassias stability,Quadratic functional equation*.

1. INTRODUCTION

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation D must be close to an exact solution of D?.

If the problem accepts a solution, we say that the equation D is stable. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940.

In the next year D.H. Hyres [2], gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces.

In 1978, Th. M. Rassias [3] proved a generalization of Hyres's theorem for additive mappings . The result of Th. M. Rassias has influenced the development of what is now called the Hyers-Ulam-Rassias stability theory for functional equations. In 1994, a generalization of Rassia's theorem was obtained by Gӑvruta [4] by replacing the bound $\varepsilon(\|x\|^p+\|y\|^p)$ by a general control function $\varphi(x,y)$.

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [5] for mappings $f:\mathcal{X}\to\mathcal{Y}$, where $\mathcal{X}$ is a normed space and $\mathcal{Y}$ is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain $\mathcal{X}$ is replaced by an Abelian group. Czerwik [7] proved the generalized Hyers-Ulam stability of the quadratic functional equation.

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem ([8],[9][12][13],[14]).

Definition 1.1 *Let $\mathcal{X}$ be a set. A function $d:\mathcal{X}\times\mathcal{X}\to[0,\infty]$ is called a generalized metric on $\mathcal{X}$ if d satisfies the following conditions:*

(a) $d(x,y)=0$ if and only if $x=y$ for all $x,y\in\mathcal{X}$;

(b) $d(x,y)=d(y,x)$ for all $x,y\in\mathcal{X}$;

(c) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in\mathcal{X}$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

Theorem 1.1 *[15] Let ($\mathcal{X}$,d) be a complete generalized metric space and $\mathcal{J}:\mathcal{X}\to\mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then, for all $x\in\mathcal{X}$, either*

$$d(\mathcal{J}^n x,\mathcal{J}^{n+1}x)=\infty \tag{1}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(a) $d(\mathcal{J}^n x,\mathcal{J}^{n+1}x)<\infty$ for all $n\ge n_0$;

(b) the sequence $\{\mathcal{J}^n x\}$ converges to a fixed point y^* of $\mathcal{J}$;

(c) y^* is the unique fixed point of $\mathcal{J}$ in the set $\mathcal{Y} = \{y \in \mathcal{X} : d(\mathcal{J}^{n_0} x, y) < \infty\}$;

(d) $d(y, y^*) \leq \dfrac{1}{1-L} d(y, \mathcal{J}y)$ for all $y \in \mathcal{Y}$.

2. RESULTS AND DISCUSSION

Theorem 2.1 Assume that $\varphi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function such that, for which $0 < L < 1$

$$\varphi(x, y) \leq 3^2 L \varphi\left(\frac{x}{3}, \frac{y}{3}\right) \tag{2}$$

and $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$ satisfying

$$\| f(2x+y) + f(x+2y) - 4f(x+y) - f(x) - f(y) \| \leq \varphi(x, y) \tag{3}$$

for all $x, y \in \mathcal{X}$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\| f(x) - Q(x) \| \leq \frac{1}{9-9L}\left[\frac{\varphi(x, x)}{2} + 2\varphi(x, 0)\right] \tag{4}$$

for all $x, y \in \mathcal{X}$.

Proof. It follows from (2)

$$\lim_{n\to\infty} \frac{\varphi(3^n x, 3^n y)}{3^{2n}} = 0 \tag{5}$$

for all $x, y \in \mathcal{X}$. Let consider the set $\Delta := \{h : \mathcal{X} \to \mathcal{Y} \mid h(0) = 0\}$ and the mapping d define on $\Delta \times \Delta$ by

$$d(g, h) := \inf\left\{\varepsilon \in (0, \infty) : \| g(x) - h(x) \| \leq \varepsilon\left(\frac{\varphi(x, x)}{2} + 2\varphi(x, 0)\right), \text{forall}, x \in \mathcal{X}\right\}$$

where $inf\varnothing = +\infty$. It is easy to show that (Δ, d) is a complete metric space (15). Let consider the mapping

$$\mathcal{J} : \Delta \to \Delta, \quad \mathcal{J}g(x) = 3^{-2}\, g(3x) \quad \text{forall } x \in \mathcal{X}.$$

Fix a $\varepsilon \in (0, \infty)$ and take $g, h \in \Delta$ such that $d(g, h) < \varepsilon$. By the definitions of d and $\mathcal{J}$, we have

$$\left\| 3^{-2} g(3x) - 3^{-2} h(3x) \right\| \leq \frac{\varepsilon}{3^2}\left(\frac{\varphi(3x, 3x)}{2} + 2\varphi(3x, 0)\right) \quad \text{for all } x \in \mathcal{X}$$

so by (2), we have

$$\left\| 3^{-2} g(3x) - 3^{-2} h(3x) \right\| \leq \varepsilon L\left(\frac{\varphi(x, x)}{2} + 2\varphi(x, 0)\right) \quad \text{for all } x \in \mathcal{X}.$$

This implies that

$$d(g, h) < \varepsilon \to d(\mathcal{J}g, \mathcal{J}h) \leq L\varepsilon\, d(g, h)$$

for all $g, h \in \Delta$. On the other hand, replacing y by x in (3), we obtain

$$\| f(3x) - 2f(2x) - f(x) \| \leq \frac{\varphi(x, x)}{2}. \tag{6}$$

for all $x \in \mathcal{X}$. Also, replacing y by 0 in ((3)), we have

$$\| f(2x) - 4f(x) \| \leq \varphi(x, 0). \tag{7}$$

for all $x \in \mathcal{X}$. Combining ((6)), ((7)) and using triangular inequality, we get

$$\begin{aligned} \| f(3x) - 3^2 f(x) \| &= \| f(3x) - 2f(2x) - f(x) + 2(f(2x) - 4f(x)) \| \\ &\leq \| f(3x) - 2f(2x) - f(x) \| + 2\| f(2x) - 4f(x) \| \end{aligned} \tag{8}$$

$$\leq \frac{\varphi(x,x)}{2}+2\varphi(x,0)$$

for all $x\in\mathcal{X}$. Therefore

$$\left\|3^{-2} f(3x)-f(x)\right\|\leq\frac{1}{9}\left(\frac{\varphi(x,x)}{2}+2\varphi(x,0)\right), \quad (9)$$

for all $x\in\mathcal{X}.$ This means that

$$d(\mathcal{J}f,f)\leq\frac{1}{9}<\infty.$$

By Theorem 1.1, there exists a mapping $Q:\mathcal{X}\to\mathcal{Y}$ satisfying the following:

(1) Q is a fixed point of $\mathcal{J}$, i.e.,

$$Q(3x)=3^2Q(x) \quad (10)$$

for all $x\in\mathcal{X}$. The mapping Q is a unique fixed point of $\mathcal{J}$ in the set $M=\{g\in S:d(h,g)<\infty\}.$ This implies that Q is a unique mapping satisfying (10) such that there exists a $\mu\in(0,\infty)$ satisfying

$$\| f(x)-Q(x)\|_Y\leq\mu\left(\frac{\varphi(x,x)}{2}+2\varphi(x,0)\right)$$

for all $x\in\mathcal{X}$;

(2) $d(\mathcal{J}^n f,Q)\to 0$ as $n\to\infty$. This implies the equality

$$\lim_{n\to\infty}\frac{1}{3^{2n}} f\left(3^n x\right)=Q(x) \quad (11)$$

for all $x\in\mathcal{X}$;

(3) $d(f,Q)\leq\dfrac{1}{1-L}d(f,\mathcal{J}f)$, which implies the inequality

$$d(f,Q)\leq\frac{d(f,\mathcal{J}f)}{1-L}\leq\frac{1}{9-9L}.$$

This implies that the inequalities ((30)) holds. Replacing x,y by $3^n x,3^n y$ in (3) and applying ((2)) and ((11)), we have

$$\begin{aligned}&\| Q(2x+y)+Q(x+2y)-4Q(x+y)-Q(x)-Q(y)\| \quad (12)\\ &=\lim_{n\to\infty}\frac{\| f(2(3^n x)+3^n y)+f(3^n x+2(3^n y))-4f(3^n x+3^n y)-f(3^n x)-f(3^n y)\|}{3^{2n}}\\ &\leq\lim_{n\to\infty}\frac{\varphi(3^n x,3^n y)}{3^{2n}}=0.\end{aligned}$$

So $Q(2x+y)+Q(x+2y)=4Q(x+y)+Q(x)+Q(y).$ Therefore Q is a quadratic mapping, this completes the proof.

Corollary 2.1 *Let* r,θ *be non-negative real numbers such that* $r<2$ *and let* $f:\mathcal{X}\to\mathcal{Y}$ *be a mapping with* $f(0)=0$ *satisfying*

$$\| f(2x+y)+f(x+2y)-4f(x+y)-f(x)-f(y)\|\leq\theta(\| x\|^r+\| y\|^r) \quad (13)$$

for all $x,y\in\mathcal{X}.$ Then there exists a unique quadratic mapping $Q:\mathcal{X}\to\mathcal{Y}$ such that

$$\| f(x)-Q(x)\|\leq\frac{3\theta\| x\|^r}{3^2-3^r},$$

for all $x\in\mathcal{X}$.

Proof. The result follow from Theorem 2.1, when $\varphi(x,y)=\theta(\|x\|^r+\|y\|^r)$ for all $x,y\in\mathcal{X}$ and $L=3^{r-2}$.

Theorem 2.2 *Assume that* $\varphi:\mathcal{X}\times\mathcal{X}\to[0,\infty)$ *be a function such that, for which* $0<L<1$

$$\varphi\left(\frac{x}{3},\frac{y}{3}\right)\le\frac{L\varphi(x,y)}{3^2} \tag{14}$$

and $f:\mathcal{X}\to\mathcal{Y}$ be a mapping with $f(0)=0$ satisfying ((3)). Then there exists a unique quadratic mapping $Q:\mathcal{X}\to\mathcal{Y}$ such that

$$\|f(x)-Q(x)\|\le\frac{L}{9-9L}\left[\frac{\varphi(x,x)}{2}+2\varphi(x,0)\right] \tag{15}$$

for all $x\in\mathcal{X}$.

Proof. Replacing x by $\frac{x}{3}$ in (8), we obtain

$$\left\|f(x)-3^2f\left(\frac{x}{3}\right)\right\|\le\frac{1}{2}\varphi\left(\frac{x}{3},\frac{x}{3}\right)+2\varphi\left(\frac{x}{3},0\right) \tag{16}$$
$$\le\frac{L}{3^2}\left[\frac{\varphi(x,x)}{2}+2\varphi(x,0)\right]$$

for all $x\in\mathcal{X}$. Let (Δ,d) be the generalized metric space defined in the proof of Theorem 2.1. Consider a linear mapping $\mathcal{J}:\Delta\to\Delta$ such that

$$\mathcal{J}h(x):=3^2h\left(\frac{x}{3}\right) \tag{17}$$

for all $x\in\mathcal{X}$. Let $g,h\in\Delta$ be such that $d(g,h)=\varepsilon$. Then

$$\|g(x)-h(x)\|\le\varepsilon\left[\frac{\varphi(x,x)}{2}+2\varphi(x,0)\right]$$

for all $x\in\mathcal{X}$ and so

$$\|\mathcal{J}g(x)-\mathcal{J}h(x)\|=\left\|3^2g\left(\frac{x}{3}\right)-3^2h\left(\frac{x}{3}\right)\right\|\le3^2\varepsilon\left[\frac{1}{2}\varphi\left(\frac{x}{3},\frac{x}{3}\right)+2\varphi\left(\frac{x}{3},0\right)\right]$$
$$\le3^2\varepsilon.\frac{L}{3^2}\left[\frac{\varphi(x,x)}{2}+2\varphi(x,0)\right]$$

for all $x\in\mathcal{X}$. Thus $d(g,h)=\varepsilon$ implies that $d(\mathcal{J}g,\mathcal{J}h)\le L\varepsilon$. This means that $d(\mathcal{J}g,\mathcal{J}h)\le Ld(g,h)$ for all $g,h\in\Delta$. It follows from ((16)) that

$$d(f,\mathcal{J}f)\le\frac{L}{9}<+\infty. \tag{18}$$

By Theorem 1.1, there exists a mapping $Q:\mathcal{X}\to\mathcal{Y}$ satisfying the following:
(a) Q is a fixed point of $\mathcal{J}$, that is,

$$Q\left(\frac{x}{3}\right)=\frac{1}{9}Q(x) \tag{19}$$

for all $x\in\mathcal{X}$. The mapping Q is a unique fixed point of $\mathcal{J}$ in the set $\Omega=\{h\in S:d(g,h)<\infty\}$. This implies that Q is a unique mapping satisfying ((19)) such that there exists $\mu\in(0,\infty)$ satisfying

$$\| f(x)-Q(x) \| \leq \mu \left[\frac{\varphi(x,x)}{2} + 2\varphi(x,0) \right]$$

for all $x \in \mathcal{X}$.

(b) $d(\mathcal{J}^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 3^{2n} f(\frac{x}{3^n}) = Q(x) \tag{20}$$

for all $x \in \mathcal{X}$.

(c) $d(f,Q) \leq \frac{d(f,\mathcal{J}f)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f,Q) \leq \frac{L}{9-9L}. \tag{21}$$

The rest of the proof is similar to the proof of Theorem 2.1. This completes the proof.

Corollary 2.2 *Let r, θ be non-negative real numbers such that $r > 2$ and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$ satisfying ((13)) . Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that*

$$\| f(x)-Q(x) \| \leq \frac{3\theta \| x \|^r}{3^r - 3^2},$$

for all $x \in \mathcal{X}$.

Proof. The result follow from Theorem 2.2, when $\varphi(x,y) = \theta(\| x \|^r + \| y \|^r)$ for all $x, y \in \mathcal{X}$ and $L = 3^{2-r}$.

Theorem 2.3 *Assume that $\varphi : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ be a function such that, for which $0 < L < 1$*

$$\varphi(x,y) \leq 4L\varphi\left(\frac{x}{2},\frac{y}{2}\right) \tag{22}$$

and $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$ satisfying ((3)). Then there exists a unique quadratic mapping $Q^* : \mathcal{X} \to \mathcal{Y}$ such that

$$\| f(x)-Q^*(x) \| \leq \frac{\varphi(x,0)}{4-4L} \tag{23}$$

for all $x, y \in \mathcal{X}$.

Proof. By the same reasoning as in the proof of Theorem 2.1, we obtain that

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{4^n} = 0. \quad \forall x, y \in \mathcal{X} \tag{24}$$

Let consider the set $S := \{h : \mathcal{X} \to \mathcal{Y} \mid h(0) = 0\}$ and the mapping d define on $S \times S$ by

$$d(g,h) := \inf\{\varepsilon \in (0,\infty) : \| g(x)-h(x) \| \leq \varepsilon\varphi(x,0) \text{ for all } x \in \mathcal{X}\} \tag{25}$$

where $inf \varnothing = +\infty$. It is easy to show that (S,d) is a complete metric space. Replacing y by 0 in ((3)), we have

$$\left\| \frac{f(2x)}{4} - f(x) \right\| \leq \frac{\varphi(x,0)}{4}, \tag{26}$$

for all $x \in \mathcal{X}$. Consider a linear mapping $\mathcal{J} : S \to S$ such that $\mathcal{J}h(x) := \frac{1}{4}h(2x)$ for all $x \in \mathcal{X}$. Let $g, h \in S$ be such that $d(g,h) = \varepsilon$. Then $\| g(x)-h(x) \| \leq \varepsilon\varphi(x,0)$ for all $x \in \mathcal{X}$ and so

$$\| \mathcal{J}g(x) - \mathcal{J}h(x) \| = \left\| \frac{g(2x)}{4} - \frac{h(2x)}{4} \right\| \leq \frac{\varepsilon\varphi(2x,0)}{4} \leq \frac{4L\varepsilon\varphi(x,0)}{4} = L\varphi(x,0)$$

for all $x \in \mathcal{X}$. Thus $d(g,h) = \varepsilon$ implies that $d(\mathcal{J}g, \mathcal{J}h) \le L\varepsilon$. This means that $d(\mathcal{J}g, \mathcal{J}h) \le Ld(g,h)$ for all $g,h \in S$. It follows from ((26)) that

$$d(f, \mathcal{J}f) \le \frac{1}{4} < +\infty. \tag{27}$$

By Theorem 1.1, there exists a mapping $Q^*: \mathcal{X} \to \mathcal{Y}$ satisfying the following:

(a) Q^* is a fixed point of $\mathcal{J}$, that is,

$$Q^*(2x) = 4Q^*(x) \tag{28}$$

for all $x \in \mathcal{X}$. The mapping Q^* is a unique fixed point of $\mathcal{J}$ in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that Q^* is a unique mapping satisfying ((28)) such that there exists $\mu \in (0,\infty)$ satisfying $\| f(x) - Q(x) \| \le \mu\varphi(x,0)$ for all $x \in \mathcal{X}$.

(b) $d(\mathcal{J}^n f, Q^*) \to 0$ as $n \to \infty$. This implies the equality $\lim_{n\to\infty} \frac{f(2^n x)}{4^n} = Q^*(x)$ for all $x \in \mathcal{X}$.

(c) $d(f, Q^*) \le \frac{d(f, \mathcal{J}f)}{1-L}$ with $f \in \Omega$, which implies the inequality $d(f, Q^*) \le \frac{1}{4-4L}$. The rest of the proof is similar to the proof of Theorem 2.1. This completes the proof.

Corollary 2.3 *Let* r, θ *be non-negative real numbers such that* $r < 1$ *and let* $f: \mathcal{X} \to \mathcal{Y}$ *be a mapping with* $f(0) = 0$ *satisfying ((13)). Then there exists a unique quadratic mapping* $Q: \mathcal{X} \to \mathcal{Y}$ *such that*

$$\| f(x) - Q^*(x) \| \le \frac{\theta \| x \|^r}{4 - 4^r},$$

for all $x \in \mathcal{X}$.

Proof. The result follow from Theorem 2.3, when $\varphi(x,y) = \theta(\| x \|^r + \| y \|^r)$ for all $x, y \in \mathcal{X}$ and $L = 4^{r-1}$.

Similarly we have the following and we will omit the proofs.

Theorem 2.4 *Assume that* $\varphi: \mathcal{X} \times \mathcal{X} \to [0,\infty)$ *be a function such that, for which* $0 < L < 1$

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L\varphi(x,y)}{4} \tag{29}$$

and $f: \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$ satisfying ((3)). Then there exists a unique quadratic mapping $Q^*: \mathcal{X} \to \mathcal{Y}$ such that

$$\| f(x) - Q^*(x) \| \le \frac{L\varphi(x,0)}{4 - 4L} \tag{30}$$

for all $x \in \mathcal{X}$.

Corollary 2.4 *Let* r, θ *be non-negative real numbers such that* $r > 1$ *and let* $f: \mathcal{X} \to \mathcal{Y}$ *be a mapping with* $f(0) = 0$ *satisfying ((13)) . Then there exists a unique quadratic mapping* $Q^*: \mathcal{X} \to \mathcal{Y}$ *such that*

$$\| f(x) - Q^*(x) \| \le \frac{\theta \| x \|^r}{4^r - 4},$$

for all $x \in \mathcal{X}$.

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