ON ϕ- SYMMETRIC *K*- CONTACT MANIFOLDS

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ABSTRACT
In the present paper we study ϕ-symmetric and ϕ-Ricci symmetric *K*-Contact manifold. We also discussed ϕ-symmetric and ϕ-Ricci symmetric *K*-Contact manifold in three-dimensional cases. An example of three-dimensional ϕ-Ricci symmetric *K*-Contact manifold constructed for illustration.

Keywords: $K-$*Contact manifold,* $\phi-$ *symmetric,* $\phi-$ *Ricci symmetric, Einstein manifold, Scalar curvature.*
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1. INTRODUCTION

The notion of locally $\phi-$symmetric Sasakian manifold was introduced by Takahashi $[3]$ as a weaker version of local symmetric of such manifold. In respect of contact Geometry, Boeckx, Buecken and Vanhecke $[6]$ introduced the notion of $\phi-$symmetry. Recently, U.C.De, A.A Shaikh and Sudipta Biswas $[4]$ introduced the notion of $\phi-$recurrent Sasakian manifold which generlized the notion of $\phi-$symmetric Sasakian manifold, the $\phi-$ Ricci symmetric Sasakian manifold was studied by U.C.De and Avijit Sarkar $[8]$. Also A.A.Shaikh and K.K.Baishya $[7]$ have studied $\phi-$symmetric LP-Sasakian manifolds. It is known that if the charcteristic vector field of a contact metric manifold is Killing vector field then the manifold is called a $K-$contact manifold. A Sasakian manifold is always a $K-$contact manifold and a three -dimensional $K-$contact manifold is a Sasakian manifold. Also a locally symmetric $K-$contact manifold is of constant curvature one. If a $K-$contact manifold is isometrically immersed in a manifold of constant curvature one $[5]$ then it is a Sasakian manifold for all vector field on M.

The present paper deals with the study of $\phi-$symmetric and $\phi-$ Ricci symmetric $K-$contact manifolds. The paper is organaized as follows:

Section 1, is introductary. In **section** 2, we recall some preliminary results. $\phi-$symmetric $K-$contact manifolds have been studied in **section** 3. In **section** 4, three dimensional locally $\phi-$symmetric $K-$contact manifolds have been studied. Also in **section** 5, we study $\phi-$symmetric $K-$contact manifold. In$[2]$, D.E. Blair, T. Koufogiorgos and R.Sharma framed the form of the Ricci tensor in three dimensional contact manifolds. Hence in the last section, with the help of this Ricci tensor we like to study $\phi-$ Ricci symmetric $K-$contact manifold of dimension three.

2. PRELIMINARIES

Let M be a $(2n+1)-$dimensional contact metric manifold $[2]$ with contact metric structure (ϕ,ξ,η,g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a $1-$form and g is a Riemannian metric on M such that

$$\phi^2(X) = -X + \eta(X)\xi, \qquad \phi(\xi) = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \qquad g(X,\xi) = \eta(X), \qquad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X,Y on M.

If the charactaristic vector field ξ is a Killing vector field then the contact metric structure on M is called a $K-$contact metric structure and the manifold M is called a $K-$contact metric manifold or $K-$contact Riemannian manifold or simply a $K-$contact manifold. In a $K-$contact manifold the following results are known

$$R(\xi, X)Y = (\nabla_X \phi)(Y) = g(X,Y)\xi - \eta(Y)X, \tag{2.4}$$

$$\nabla_X \xi = -\phi X, \tag{2.5}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.6}$$

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.7}$$

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.8}$$

$$S(X,\xi) = 2n\eta(X), \qquad Q\xi = 2n\xi \tag{2.9}$$

$$S(\phi X,\phi Y) = S(X,Y) - 2n\eta(X)\eta(Y), \tag{2.10}$$

for all the vector fields X,Y,Z on M, where ∇ denotes the operator of covarient differention with respect to g, ϕ is a skew symmetric tensor field of type $(1,1)$, S is the Ricci tensor of type $(0,2)$ and R is the Riemannian curvature tensor of type (1,3) of the manifold.

A K-contact manifold is said to be Einstein if the Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y),$$

where a is constant.

3. $\phi-$symmetric K-contact manifolds

At first, we recall:

Definition 3.1. *A K-contact manifold M is said to be locally $\phi-$symmetric if*

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{3.1}$$

for all vector fields X,Y,Z,W orthogonal to ξ. This notion was introduced by Takahashi for a Sasakian manifold .

Definition 3.2. *A K-contact manifold M is said to be $\phi-$symmetric if*

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{3.2}$$

for orbitary vector fields X,Y,Z,W on M.

Theorem 3.1. *$\phi-$ symmetric $K-$contact manifold is Einstein manifold.*

Proof. Let us assume that the manifold is $\phi-$ symmetric.Then in view of (3.2) and (2.1), we have

$$-(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi = 0, \tag{3.3}$$

taking inner product of (3.3) with U, we get

$$-g((\nabla_W R)(X,Y)Z,U) + \eta((\nabla_W R)(X,Y)Z)g(\xi,U) = 0. \tag{3.4}$$

Let $\{e_i\}, i = 1,2,...n$, be an orthonormal basis of the tangent space at any point P of the manifold. Then by putting $X = U = e_i$ in equation (3.4) and taking summation over $i, 1 \le i \le n,$ we get

$$-(\nabla_W S)(Y,Z) +_{i=1}^{n} \eta((\nabla_W R)(e_i,Y)Z)g(\xi,e_i) = 0. \tag{3.5}$$

Replacing Z by ξ in equation (3.5), we have

$$-(\nabla_W S)(Y,\xi) + \sum_{i=1}^{n}\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) = 0. \tag{3.6}$$

The second term of (3.6), takes the form

$$\eta((\nabla_W R)(e_i,Y)\xi) = g((\nabla_W R)(e_i,Y)\xi,\xi) \tag{3.7}$$
$$= g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(e_i,Y)\nabla_X \xi,\xi)$$
$$- g(R(\nabla_W e_i,Y)\xi,\xi) - g(R(e_i,\nabla_W Y)\xi,\xi).$$

Since $\{e_i\}$ is an orthonormal basis, $\nabla_W e_i = 0$ at P. Also using equation (2.6), we have

$$g(R(e_i,\nabla_W Y)\xi,\xi) = 0. \tag{3.8}$$

Using (3.8) in (3.7), we have

$$\eta((\nabla_W R)(e_i,Y)\xi = g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi). \tag{3.9}$$

Now, since

$$g(R(e_i,Y)\xi,\xi) = -g(R(\xi,\xi)Y,e_i) = 0.$$

We have

$$g((\nabla_W R)(e_i,Y)\xi,\xi) + g(R(e_i,Y)\xi,\nabla_W \xi) = 0. \tag{3.10}$$

Using equation (3.10) in (3.9), we obtain

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = -g(R(e_i,Y)\xi,\nabla_W \xi) - g(R(e_i,Y)\nabla_W \xi,\xi). \tag{3.11}$$

Again using equation (2.5) in above, we get

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = -g(R(e_i,Y)\xi,-\phi W) - g(R(e_i,Y)-\phi W,\xi) = 0. \tag{3.12}$$

The equation (3.12) and (3.6), implies that

$$(\nabla_W S)(Y,\xi) = 0.$$

Which gives

$$\nabla_W (S(Y,\xi)) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi) = 0.$$

Using equation (2.9) and (2.5), we get

$$2n(\nabla_W \eta(Y)) - 2n\eta(\nabla_W Y) + S(Y,\phi W) = 0.$$

Replacing Y by ϕY in above, we get

$$S(\phi Y,\phi W) = 2ng((\nabla_W \phi)Y,\xi).$$

Using equation (2.4) and (2.10), we get

$$S(Y,W) = 2ng(Y,W).$$

Which prove the theorem.

4. Three-dimensional locally $\phi-$symmetric $K-$contact manifolds

We know that three dimensional K-contact manifold is a Sasakian manifold. So three dimensional locally ϕ symmetric $K-$contact manifold is also three dimensional locally ϕ-symmetric Sasakian manifold.

In a three-dimensional Riemannian manifold, we have

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \tag{4.1}$$

where R denote the curvature tensor of type $(1,3)$, S is the Ricci tensor of type $(0,2)$, r is the scalar curvature and Q is the Ricci operator defined by

$$S(X,Y) = g(QX,Y).$$

Putting $Z=\xi$ in equation (4.1) and using (2.6) and (2.7) ($for \ \ n=3$), we get

$$\eta(Y)QX-\eta(X)QY=\frac{1}{2}(r-2)[\eta(Y)X-\eta(X)Y]. \tag{4.2}$$

Again putting $Y=\xi$ in equation (4.2) and using (2.9) $(\,for\ \ n=3),$ we get

$$QX=\frac{1}{2}[(r-2)X-(r-6)\eta(X)\xi], \tag{4.3}$$

and

$$S(X,Y)=\frac{1}{2}[(r-2)g(X,Y)-(r-6)\eta(X)\eta(Y)]. \tag{4.4}$$

Using equation (4.3) and (4.4) in $(4.1),$ we get the curvature tensor of three-dimensional $K-$contact manifold

$$\begin{aligned} R(X,Y)Z=&\frac{r-4}{2}[g(Y,Z)X-g(X,Z)Y] \\ &-\frac{r-6}{2}[g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi \\ &+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y]. \end{aligned} \tag{4.5}$$

Theorem 4.1. *A three-dimensional $K-$contact manifold is locally $\phi-$symmetric if and only if the scalar curvature r is constant.*

Proof. In a three-dimensional $K-$contact manifold the curvature tensor is given by

$$\begin{aligned} R(X,Y)Z=&\frac{r-4}{2}[g(Y,Z)X-g(X,Z)Y]-\frac{r-6}{2}[g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi \\ &+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y]. \end{aligned}$$

Taking covaiant differention of (4.5) with respect to $W,$ we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z=&\frac{dr(W)}{2}[g(Y,Z)X-g(X,Z)Y]-\frac{dr(W)}{2}[g(Y,Z)\eta(X)\xi \\ &-g(X,Z)\eta(Y)\xi+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y] \\ &-\frac{r-6}{2}[g(Y,Z)(\nabla_W\eta)(X)\xi+g(Y,Z)\eta(X)\nabla_W\xi \\ &-g(X,Z)(\nabla_W\eta)(Y)\xi-g(X,Z)\eta(Y)\nabla_W\xi+(\nabla_W\eta)(Y)\eta(Z)X \\ &+\eta(Y)(\nabla_W\eta)(Z)X-(\nabla_W\eta)(X)\eta(Z)Y-\eta(X)(\nabla_W\eta)(Z)Y]. \end{aligned}$$

On applying ϕ^2 on both sides of above equation, we get

$$\begin{aligned} \phi^2(\nabla_W R)(X,Y)Z=&-\frac{dr(W)}{2}[g(Y,Z)X-g(X,Z)Y-g(Y,Z)\eta(X)\xi. \\ &+g(X,Z)\eta(Y)\xi-\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y] \\ &-\frac{r-6}{2}[g(X,Z)\eta(Y)\nabla_W\xi-g(Y,Z)\eta(X)\nabla_W\xi \\ &-(\nabla_W\eta)(Y)\eta(Z)X+(\nabla_W\eta)(Y)\eta(Z)\eta(X)\xi \\ &-\eta(Y)(\nabla_W\eta)(Z)X+\eta(Y)(\nabla_W\eta)(Z)\eta(X)\xi \\ &+(\nabla_W\eta)(X)\eta(Z)Y-(\nabla_W\eta)(X)\eta(Z)\eta(Y)\xi \\ &+\eta(X)(\nabla_W\eta)(Z)Y-\eta(X)(\nabla_W\eta)(Z)\eta(Y)\xi \end{aligned}$$

Now taking X, Y, Z orthogonal to ξ, then the above equation gives

$$\phi^2(\nabla_W R)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{4.6}$$

Hence from equation (4.6), the theorem is proved.

5. $\phi-$Ricci symmetric $K-$contact manifold

Definition 5.1. *A $K-$contact manifold M is said to be locally $\phi-$ Ricci symmetric if the Ricci operator satisfies*

$$\phi^2((\nabla_X Q)Y)) = 0,$$

for all vector fields X and Y on M. And

$$S(X,Y) = g(QX,Y).$$

If X, Y are orthogonal to ξ, then the manifold is said to be locally $\phi-$ Ricci symmetric.

Theorem 5.1.. *An $(2n+1)$-dimensional $K-$contact manifold is $\phi-$Ricci symmetric if and only if the manifold is an Einstein manifold.*

Proof. *Let us suppose that the manifold is $\phi-$Ricci symmetric then, we have*

$$\phi^2((\nabla_X Q)Y)) = 0. \tag{5.1}$$

Using equation (2.1) in (5.1), we get

$$-(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi = 0. \tag{5.2}$$

Taking inner product of (5.2) with Z, we have

$$-g((\nabla_X Q)(Y), Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0, \tag{5.3}$$

which on simplifying gives

$$-g(\nabla_X Q(Y) - Q(\nabla_X Y), Z) + \eta((\nabla_W Q)(Y))\eta(Z) = 0,$$

$$-g(\nabla_X Q(Y), Z) + S(\nabla_X Y, Z) + \eta((\nabla_W Q)(Y))\eta(Z) = 0. \tag{5.4}$$

Putting $Y = \xi$, we get

$$-g(\nabla_X Q(\xi), Z) + S(\nabla_X \xi, Z) + \eta((\nabla_W Q)(\xi))\eta(Z) = 0. \tag{5.5}$$

Using equation (2.9) and (2.5) in (5.5), we get

$$g(2n\phi X, Z) - S(\phi X, Z) + \eta((\nabla_W Q)(\xi))\eta(Z) = 0. \tag{5.6}$$

Replacing Z by ϕZ in equation (5.4), we get

$$2ng(\phi X, \phi Z) = S(\phi X, \phi Z). \tag{5.7}$$

Using equation (2.3) and (2.10) and (5.7), we get

$$S(X,Z) = 2ng(X,Z).$$

Hence the manifold is an Einstein manifold.

Next, suppose that the manifold is an Einstein manifold then

$$S(X,Y) = ag(X,Y),$$

where $S(X,Y) = g(QX,Y)$ and a is a constant. Hence $QX = aX$ therefore, we have

$$\phi^2((\nabla_Y Q)(X)) = 0.$$

This complete the proof.

6. Three-dimensional locally $\phi-$Ricci symmetric $K-$contact manifold

Theorem 6.1. *If the scalar curvature r of a three-dimensional $K-$contact manifold is equal to 6, then the manifold is $\phi-$Ricci symmetric.*

Proof. The Ricci tensor of a three-dimensional $K-$contact manifold is

$$S(X,Y)=\frac{1}{2}[(r-2)g(X,Y)+(6-r)\eta(X)\eta(Y)]. \tag{6.1}$$

And

$$QX=\frac{1}{2}[(r-2)X+(6-r)\eta(X)\xi]. \tag{6.2}$$

Taking covariant differentation of above with respect to W, we get

$$(\nabla_W Q)(X)=\frac{1}{2}[dr(W)X+(6-r)(\nabla_W\eta)(X)\xi+(6-r)\eta(X)(\nabla_W\xi)-dr(W)\eta(X)\xi]. \tag{6.3}$$

Now applying ϕ^2 on both sides of (6.3) and using (2.2), we have

$$\phi^2((\nabla_W Q)(X))=\frac{1}{2}[dr(W)(-X+\eta(X)\xi)+(6-r)\eta(X)\phi^2(\nabla_X\xi)]. \tag{6.4}$$

Hence from equation (6.4), the proof is complete.

Taking X orthogonal to ξ in equation (6.4), we obtain

$$\phi^2((\nabla_W Q)(X))=\frac{1}{2}dr(W)X. \tag{6.5}$$

In view of (6.5), we have the following corollary :

Corollary 6.2. *A three-dimensional $K-$contact manifold is locally $\phi-$Ricci symmetric if and only if scalar curvature r is constant.*

7. Example

Here we constract an example of a three-dimensional $\phi-$Ricci symmetric $K-$contact manifold.

Let us consider three-dimensional manifold $M=\{(x,y,z\in R^3),(x,y,z)\neq(0,0,0)\}$, where (x,y,z) are the standared coordinates in R^3. The vector fields

$$e_1=\frac{\partial}{\partial y},\ e_2=\frac{\partial}{\partial z},\ e_3=2\frac{\partial}{\partial x}-y\frac{\partial}{\partial z}+z\frac{\partial}{\partial y},$$

are linearly independent at each point of M. Let g be the Riemannian metric defiend by

$$g(e_1,e_3)=g(e_2,e_3)=g(e_1,e_2)=0,\ \ g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1,$$

Let η be the $1-$form defined by $\eta(Z)=g(Z,e_3)$ for any Z belong to $X(M)$. Let ϕ be the $(1,1)$ tensor field defiend by $\phi e_1=-e_2$, $\phi e_2=e_1,\phi e_3=0,$ then using the linearty of ϕ and g, we have

$$\eta(e_3)=1,\ \ \phi^2(Z)=-Z+\eta(Z)\xi\ ,\ g(\phi Z,\phi W)=g(Z,W)-\eta(Z)\eta(W),$$

for any $Z,W\in X(M)$. Thus for $e_3=\xi,\ M$ defines an almost contact metric manifold.

Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g and R be the curvature tensor of g. Then, we have

$$[e_1,e_2]=0,\qquad [e_1,e_3]=-e_2\ ,\quad [e_2,e_3]=e_{1.}$$

The Riemannian connection ∇ of the metric g is given by

$$2g(\nabla_X Y,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)+g([X,Y],Z)-g([Y,Z],X)$$
$$+g([Z,X],Y),$$

which is Koszul's formula. Hence from above we have easily calculated that

$$\nabla_{e_1}e_3 = e_2, \quad \nabla_{e_1}e_2 = -e_3, \quad \nabla_{e_1}e_1 = -e_3,$$
$$\nabla_{e_2}e_3 = -e_1, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_1 = -e_3,$$
$$\nabla_{e_3}e_3 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_1 = o,$$

which satisfies the formula $\nabla_X\xi = -\phi X$. Hence M is a three-dimensional $K-$contact manifold. Using above result, we have

$$R(e_1,e_2)e_2 = e_1, \; R(e_1,e_3)e_3 = e_1, \; R(e_2,e_1)e_1 = -e_1 + e_2,$$
$$R(e_2,e_3)e_3 = e_2, \; R(e_3,e_1)e_1 = e_3, \; R(e_3,e_2)e_2 = e_3,$$
$$R(e_1,e_2)e_3 = e_3, \; R(e_2,e_3)e_1 = -e_3, \; R(e_3,e_1)e_3 = 0.$$

Since the Ricci tensor in three-dimensional manifold, we get

$$S(X,Y) = \sum_{i=1}^{3} g(R(e_i, X)Y, e_i).$$

From above equation and the component of the curvature tensor, we obtain that

$$S(e_1,e_1) = 2, \; S(e_2,e_2) = 2, \; S(e_3,e_3) = 2,$$
$$S(e_1,e_2) = 0, \; S(e_1,e_3) = 0, \; S(e_2,e_3) = 0.$$

From the above results it follows that the scalar curvature of the manifold is equal to 6 and $S(X,Y) = 2g(X,Y)$. Therefore $QX = 2X$, which implies that $\phi^2((\nabla_W Q)(X)) = 0$. Hence we conclude that the sclar curvature of the manifold under consideration is equal to 6 and it is ϕ-Ricci symmetric. Therefore this example supports theorem (6.1). We also note that the manifold is ϕ-Ricci symmetric and Einstein. So in case of three-dimensional this example also agrees with the theorem (5.1).

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