# STABILITY OF AN SIR EPIDEMIC MODEL WITH INFORMATION VARIABLE AND LIMITED MEDICAL RESOURCES

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**ABSTRACT**

In this paper, an SIR epidemic model involving information-related vaccination, non-linear incidence rate and the information variable is studied. We get a critical value $\mathcal{R}_0$. When $(1-p_0)\mathcal{R}_0 < 1$, the disease-free Equilibrium is stable, on the contrary, if $(1-p_0)\mathcal{R}_0 > 1$, the disease-free equilibrium is unstable and an endemic equilibrium appears. We study the local stability and global stability by using the Routh-Hurwitz criterion and the geometric method due to Li and Muldowney, respectively. A numerical analysis is given to show the effectiveness of the main results.

**Keywords:** *SIR epidemic model; Vaccination; Information variable; equilibrium; Global stability.*

## 1. INTRODUCTION

Mathematical modeling has been playing a more and more important role in the study of epidemiology. Various epidemic models have been proposed and explored extensively. Great progress has been achieved in the study of disease control and prevention. Of course, the more reasonable, the better. In order to achieve this goal, some authors used a new variable named information variable, which summarizes information about the current state of the disease and also summarizes information about past values of state variables[5]. Alberto d`Onofrio, Piero Manfredi and Ernesto Salinelli provided many different possibilities, which was including (a): information variable only depended on current values of state variables and (b): it also summarized information about past values of state variables. Moreover, they explored a family of models for information-related vaccinating behaviour[3]. Later, Bruno Buonomo, Alberto d`Onofrio, and Deborah Lacitignola studied the following model, which adopt $M(t) = \int_{-\infty}^{t} g(S(\tau), I(\tau))K_a(t-\tau)\mathrm{d}\tau$ as information variable[4]:

$$\begin{cases} S'(t) = \mu(1-p(M)) - \mu S - \beta SI, \\ M(t) = \int_{-\infty}^{t} g(S(\tau), I(\tau))K_a(t-\tau)\mathrm{d}\tau, \\ I'(t) = \beta SI - (\mu+\nu)I. \end{cases} \qquad (1)$$

The non-decreasing positive function $p(M)$ modeled the proportion of vaccinated newborns, and it split as follows:

$$p(M) = p_0 + p_1(M).$$

As special choices, in [1], it was set $g(S, I) = kI$, where $k$ was a positive constant, and the kernel $K_a(t) = a\exp(-a(t))$, which was called the weak exponential delay kernel[8]. Some authors realized that every community or country had an appropriate or limited capacity for treatment other than sufficient. So, Wang and Ruan introduced a constant treatment function[10] in an SIR model and progressed later[9]. Several authors adopted the staged treatment function and explored the dynamics of some epidemic models with standard incidence rate[6] and general incidence rate[13]. Recently, Zhang and Liu[14] introduced a new continually differentiable treatment function $h(I) = \dfrac{rI}{1+\alpha I}$ to characterize the saturation phenomenon of the limited medical resources. Linhua Zhou and Meng Fan improved this function and studied the following model[11]:

$$\begin{cases} S'(t) = \Lambda - \dfrac{\beta SI}{1+kI} - dS, \\ I'(t) = \dfrac{\beta SI}{1+kI} - (d+\gamma+\varepsilon)I - \dfrac{\alpha I}{\omega+I}, \\ R'(t) = \gamma I - dR + \dfrac{\alpha I}{\omega+I}, \end{cases} \qquad (2)$$

where the incidence rate $\dfrac{\beta SI}{1+kI}$ was of saturated type and reflected the "psychological" effect or the inhibition effect [2].The treatment function was $h^*(I) = \dfrac{\alpha I}{\omega+I}$ , where $\alpha \geq 0$ represented the maximal supply of medical resources per unit time and $\omega > 0$ was half-saturation constant, which measured the efficiency of the medical resources supply, in the sense that if $\omega$ was smaller, then the efficiency was higher.

Besides, others introduced the nonlinear incidence rate[12]. T.K.Kar and Prasanta Kumar Mondal adopt it and studied an SIR model with information variable as follows[5]:

$$\begin{cases} S'(t) = rS(1-\dfrac{S}{k}) - \dfrac{\beta SI}{1+\alpha S}, \\ I'(t) = \dfrac{\beta IS}{1+\alpha S} - \mu_1 I - \gamma I, \\ R'(t) = \gamma I - \mu_2 R. \end{cases} \qquad (3)$$

In this paper,we further explore the dynamics of the SIR model containing the information variable, the nonlinear incidence rate and saturated type treatment function.

This paper is organized as follows. In Section 2, we present the model and explore the existence of its equilibria. In Section 3, we analyze the local asymptotical stability of the disease-free equilibrium and the endemic equilibrium. Bifurcation is also given in this section. Section 4 describes the sufficient conditions to global asymptotical stability of the endemic equilibrium. Finally, some numerical simulations are given in Section 5.

## 2. THE MODEL AND ITS EQUILIBRIA

We consider the following model:

$$\begin{cases} S'(t) = \mu(1-p_0) - \mu p_1(M) - \mu S - \dfrac{\beta SI}{1+\alpha S}, \\ M(t) = \int_{-\infty}^{t} kIa\exp(-a(t-\tau))\mathrm{d}\tau, \\ I'(t) = \dfrac{\beta SI}{1+\alpha s} - (\mu+\nu+\varepsilon)I - \dfrac{\gamma I}{\omega+I}, \\ R'(t) = \nu I - \mu R + \dfrac{\gamma I}{\omega+I}, \end{cases} \qquad (4)$$

where $S(0), I(0), R(0) \geq 0$, and $S(t), I(t), R(t)$ is the density of susceptible, infective and recovered individuals at time $t$ within the population, respectively. And the parameters are: $\mu$ is the natural death rate of the population, $\nu$ is the rate of recovery from infection, $\varepsilon$ is the disease-related mortality. The coverage function $p_1(M)$ to be a generic function of the information variable $M$ and assume the following properties to hold: $(i)\, 0 \leq p_1(M) \leq 1-p_0$ for all $M$, $(ii)\, p_1(0) = 0$, $(iii)\, p_1(M)$ is continuous and differentiable, except in some cases, at finite number points, and $(iv)\, 0 < p_1'(M) < \Pi$, for some constant $\Pi$ .Then the non-linear integro-differential system (4) can be transformed into the following set of non-linear ordinary differential equations:

$$\begin{cases} S'(t) = \mu(1-p_0) - \mu p_1(M) - \mu S - \dfrac{\beta SI}{1+\alpha S}, \\ M'(t) = akI - aM, \\ I'(t) = \dfrac{\beta SI}{1+\alpha s} - (\mu+\nu+\varepsilon)I - \dfrac{\gamma I}{\omega + I}, \\ R'(t) = \nu I - \mu R + \dfrac{\gamma I}{\omega + I}. \end{cases} \quad (5)$$

Since the dynamical behaviour of the last equation of the system (5) depends only the dynamics of $I$ , so we do not consider it in our discussion. Here we will study the following non-linear ordinary differential equations:

$$\begin{cases} S'(t) = \mu(1-p_0) - \mu p_1(M) - \mu S - \dfrac{\beta SI}{1+\alpha S}, \\ M'(t) = akI - aM, \\ I'(t) = \dfrac{\beta SI}{1+\alpha s} - (\mu+\nu+\varepsilon)I - \dfrac{\gamma I}{\omega + I}. \end{cases} \quad (6)$$

Introduce the parameter $\mathcal{R}_0 = \dfrac{\beta - \alpha(\mu+\nu+\varepsilon+\frac{\gamma}{\omega})}{\mu+\nu+\varepsilon+\frac{\gamma}{\omega}}$ , which is referred to as the basic reproductive number.

We have the following result:

**Theorem 2.1.** The system (6) has a disease-free equilibrium $E_0(1-p_0, 0, 0)$ . If $(1-p_0)\mathcal{R}_0 > 1$ , and

$$\frac{\gamma\beta S^2}{[(\mu+\nu+\varepsilon)(\omega+I)+\gamma]^2}(\mu + \frac{\beta I}{(1+\alpha S)^2}) < \frac{\beta S}{1+\alpha S},$$

then the system also has a unique endemic equilibrium $E^*(S^*, M^*, I^*)$ . The components of $E^*$ are given by:

$$S^* = \frac{(\mu+\nu+\varepsilon)(\omega+I^*)+\gamma}{\beta(\omega+I^*) - \alpha[(\mu+\nu+\varepsilon)(\omega+I^*)+\gamma]}, \quad M^* = kI^* \text{ and } I^*,$$

which is the unique positive solution of:

$$\mu(1-p_0) - \mu p_1(kI^*) - \mu S^* - \frac{\beta S^* I^*}{1+\alpha S^*} = 0. \quad (7)$$

**Proof.** The equilibria satisfy the following equations:

$$\begin{cases} \mu(1-p_0) - \mu p_1(M) - \mu S - \dfrac{\beta SI}{1+\alpha S} = 0, \\ akI - aM = 0, \\ \dfrac{\beta SI}{1+\alpha s} - (\mu+\nu+\varepsilon)I - \dfrac{\gamma I}{\omega + I} = 0. \end{cases} \quad (8)$$

Firstly, setting $I = 0$ , it is easy to get $M = 0$ , and $S = 1-p_0$ , so we can easily get the disease-free equilibrium $E_0(1-p_0, 0, 0)$ .

Then setting $I = I^* \neq 0$ , we consider the second and the third equation of (8). We can get $M^* = kI^*$ ,and

$S^* = \dfrac{(\mu+\nu+\varepsilon)(\omega+I^*)+\gamma}{\beta(\omega+I^*)-\alpha[(\mu+\nu+\varepsilon)(\omega+I^*)+\gamma]}$. In order to make $S^* > 0$, we must set

$$\beta-\alpha(\mu+\nu+\varepsilon)-\frac{\alpha\gamma}{\omega+I^*} > 0.$$

Note that if $(1-p_0)\mathcal{R}_0 > 1$, then $\mathcal{R}_0 > 0$, so

$$0 < \beta-\alpha(\mu+\nu+\varepsilon)-\frac{\alpha\gamma}{\omega} < \beta-\alpha(\mu+\nu+\varepsilon)-\frac{\alpha\gamma}{\omega+I^*}.$$

$I^*$ satisfy the first equation:

$$\mu(1-p_0)-\mu p_1(kI^*)-\mu S^*-\frac{\beta S^* I^*}{1+\alpha S^*}=0,$$

and we say $I^*$ is the unique positive solution of this equation. Next, we will prove it.

Defining $f(I)=\mu p_1(kI)$ and $g(I)=\mu(1-p_0)-\mu S(I)-\dfrac{\beta S(I)I}{1+\alpha S(I)}$. Then we want to say $f(I)$ and $g(I)$ have a unique point of intersection $I$, i.e. $I^*$.

Note that $f(0)=\mu p_1(0)=0,$ and $g(0)=\mu(1-p_0)-\mu S(0)=\mu[1-p_0-\dfrac{\mu+\nu+\varepsilon+\frac{\gamma}{\omega}}{\beta-\alpha(\mu+\nu+\varepsilon+\frac{\gamma}{\omega})}]$, as $(1-p_0)\mathcal{R}_0>1,$ then we have

$$1-p_0 > \frac{\mu+\nu+\varepsilon+\frac{\gamma}{\omega}}{\beta-\alpha(\mu+\nu+\varepsilon+\frac{\gamma}{\omega})}.$$

So $g(0)>0.$ Thus, $f(0)<g(0).$

Furthermore, $f'(I)=k\mu p_1'(kI)>0,$ hence $f(I)$ is strictly increasing.

$$\begin{aligned} g'(I) &= -\mu S'(I)-\frac{(\beta S'(I)I+\beta S)(1+\alpha S)-\beta SI\alpha S'(I)}{(1+\alpha S)^2} \\ &= -S'(I)(\mu+\frac{\beta I}{(1+\alpha S)^2})-\frac{\beta S}{1+\alpha S} \end{aligned}$$

Note that

$$S^* = \frac{(\mu+\nu+\varepsilon)(\omega+I^*)+\gamma}{\beta(\omega+I^*)-\alpha(\mu+\nu+\varepsilon)(\omega+I^*)-\alpha\gamma},$$

so

$$S'(I) = -\frac{\gamma\beta S^2}{[(\mu+\nu+\varepsilon)(\omega+I)+\gamma]^2}.$$

Thus

$$g'(I)=\frac{\gamma\beta S^2}{[(\mu+\nu+\varepsilon)(\omega+I)+\gamma]^2}(\mu+\frac{\beta I}{(1+\alpha S)^2})-\frac{\beta S}{1+\alpha S}.$$

Therefore, if

$$\frac{\gamma\beta S^2}{[(\mu+\nu+\varepsilon)(\omega+I)+\gamma]^2}(\mu+\frac{\beta I}{(1+\alpha S)^2}) < \frac{\beta S}{1+\alpha S},$$

then $g'(I) < 0$. So $g(I)$ is strictly decreasing. Thus, $f(I)$ and $g(I)$ have a unique point of intersection $I = I^*$. Furthermore, $f(1) = \mu p_1(k) > 0,$ and $g(1) = \mu(1-p_0) - \mu S(1) - \frac{\beta S(1)}{1+\alpha S(1)},$

where

$$S(1) = \frac{\mu+\nu+\varepsilon+\frac{\gamma}{\omega+1}}{\beta-\alpha(\mu+\nu+\varepsilon+\frac{\gamma}{\omega+1})} > 0.$$

Note that

$$\frac{\beta S(1)}{1+\alpha S(1)} = \mu+\nu+\varepsilon+\frac{\gamma}{\omega+1},$$

we have

$$g(1) = -\mu p_0 - \mu S(1) - \mu - \nu - \varepsilon - \frac{\gamma}{\omega+1} < 0,$$

so, $g(1) < 0 < f(1)$. Thus, $0 < I^* < 1$, we notice that $I^*$ will always be epidemiologically meaningful. Our proof is completed.

## 3. THE LAS OF THE EQUILIBRIA AND HOPF BIFURCATION

**Theorem 3.1.** The equilibrium point $E_0$ is locally asymptotically stable, if $(1-p_0)\mathcal{R}_0 < 1$, and if $(1-p_0)\mathcal{R}_0 > 1$, then $E_0$ is unstable.

**Proof.** The Jacobian matrix at $E_0$ is

$$J(1-p_0,0,0) = \begin{pmatrix} -\mu & -\mu\frac{\partial p_1(0)}{\partial M} & -\frac{\beta(1-p_0)}{1+\alpha(1-p_0)} \\ 0 & -a & ak \\ 0 & 0 & \frac{\beta(1-p_0)}{1+\alpha(1-p_0)} - (\mu+\nu+\varepsilon+\frac{\gamma}{\omega}) \end{pmatrix}.$$

The eigenvalues are $-\mu$ , $-a$ , $\frac{\beta(1-p_0)}{1+\alpha(1-p_0)} - (\mu+\nu+\varepsilon+\frac{\gamma}{\omega})$ .

If $\frac{\beta(1-p_0)}{1+\alpha(1-p_0)} - (\mu+\nu+\varepsilon+\frac{\gamma}{\omega}) < 0$, i.e. if $(1-p_0)\mathcal{R}_0 < 1$, it follows that any eigenvalues has negative real part, so the equilibrium point $E_0$ is locally asymptotically stable. On the contrary, if $(1-p_0)\mathcal{R}_0 > 1$, then the equilibrium $E_0$ is unstable.

The local stability properties of the endemic equilibrium $E^*$ and the occurrence of Hopf bifurcations are described in the following theorem:

**Theorem 3.2.** Set $D = \mu + \frac{\beta I^*}{(1+\alpha S^*)^2} - \frac{\gamma}{\omega+I^*} + \frac{\gamma\omega}{(\omega+I^*)^2}$, then if $\frac{\beta I^*}{(1+\alpha S^*)^2} > \frac{\gamma}{\omega+I^*}$, and

$$D^2 - \frac{\beta I}{(1+\alpha S^*)^2} k\mu\frac{\partial p_1}{\partial M} + 2D\sqrt{\frac{\beta I^*}{(1+\alpha S^*)^2}(\mu+\nu+\varepsilon+\frac{\gamma\omega}{(\omega+I^*)^2}) + \mu(-\frac{\gamma}{\omega+I^*}+\frac{\gamma\omega}{(\omega+I^*)^2})} < 0, \quad (9)$$

there exist two values $a_1, a_2 > 0$, for the parameter $a$, such that $E^*$ is unstable for $a \in (a_1, a_2)$, otherwise it is

locally asymptotically stable. At the points $a_1$ and $a_2$, Hopf bifurcation occurs. If the reverse of (9) holds, then $E^*$ is always locally asymptotically stable.

**Proof.** The Jacobian matrix at $E^*$ is

$$J(S^*, M^*, I^*) = \begin{pmatrix} -\mu - \frac{\beta I^*}{(1+\alpha S^*)^2} & -\mu \frac{\partial p_1(M^*)}{\partial M} & -\frac{\beta S^*}{1+\alpha S^*} \\ 0 & -a & ak \\ \frac{\beta I^*}{(1+\alpha S^*)^2} & 0 & \frac{\beta S^*}{1+\alpha S^*} - (\mu+\nu+\varepsilon) - \frac{\gamma\omega}{(\omega+I^*)^2} \end{pmatrix}.$$

The characteristic equation is

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0, (10)$$

where

$$c_1 = \frac{\beta I^*}{(1+\alpha S^*)^2} + \mu + a - \frac{\gamma}{\omega + I^*} + \frac{\gamma\omega}{(\omega+I^*)^2},$$

$$c_2 = a\mu + \frac{\beta I^*}{(1+\alpha S^*)^2}[a + (\mu+\nu+\varepsilon) + \frac{\gamma\omega}{(\omega+I^*)^2}] + (a+\mu)[-\frac{\gamma}{\omega+I^*} + \frac{\gamma\omega}{(\omega+I^*)^2}],$$

$$c_3 = \frac{a\beta I^*}{(1+\alpha S^*)^2}[(\mu+\nu+\varepsilon) + \frac{\gamma\omega}{(\omega+I^*)^2} + k\mu\frac{\partial p_1(M^*)}{\partial M}] + a\mu[-\frac{\gamma}{\omega+I^*} + \frac{\gamma\omega}{(\omega+I^*)^2}].$$

Then

$$c_1c_2 - c_3$$
$$= [\mu + \frac{\beta I^*}{(1+\alpha S^*)^2} - \frac{\gamma}{\omega+I^*} + \frac{\gamma\omega}{(\omega+I^*)^2}]a^2 + [(\mu + \frac{\beta I^*}{(1+\alpha S^*)^2} - \frac{\gamma}{\omega+I^*} + \frac{\gamma\omega}{(\omega+I^*)^2})^2 - \frac{\beta I}{(1+\alpha S^*)^2}k\mu\frac{\partial p_1(M^*)}{\partial M}]a$$
$$+ (\mu + \frac{\beta I^*}{(1+\alpha S^*)^2} - \frac{\gamma}{\omega+I^*} + \frac{\gamma\omega}{(\omega+I^*)^2})[\frac{\beta I^*}{(1+\alpha S^*)^2}(\mu+\nu+\varepsilon + \frac{\gamma\omega}{(\omega+I^*)^2}) + \mu(-\frac{\gamma}{\omega+I^*} + \frac{\gamma\omega}{(\omega+I^*)^2})].$$

As

$$\frac{\beta I^*}{(1+\alpha S^*)^2} > \frac{\gamma}{\omega+I^*},$$

thus $c_1, c_3 > 0$. Let us assume

$$f(a) = c_1c_2 - c_3 = Aa^2 + Ba + C,$$

where

$$A = D, \quad B = D^2 - \frac{\beta I}{(1+\alpha S^*)^2}k\mu\frac{\partial p_1(M^*)}{\partial M},$$

$$C = D[\frac{\beta I^*}{(1+\alpha S^*)^2}(\mu+\nu+\varepsilon + \frac{\gamma\omega}{(\omega+I^*)^2}) + \mu(-\frac{\gamma}{\omega+I^*} + \frac{\gamma\omega}{(\omega+I^*)^2})].$$

the coefficients $A > 0, C > 0$, whereas $B$ has variable sign. Thus if $B \geq 0$, then $f(a) > 0$ for every values of $a$. From the well-known Routh-Hurwitz criterion, it follows that any roots of (10) has negative real part, so that $E^*$ is LAS independently on the delay. If $B < 0$, instability is possible. Note that

$$\Delta = B^2 - 4AC = B^2 - 4D^2[\frac{\beta I^*}{(1+\alpha S^*)^2}(\mu+\nu+\varepsilon + \frac{\gamma\omega}{(\omega+I^*)^2}) + \mu(-\frac{\gamma}{\omega+I^*} + \frac{\gamma\omega}{(\omega+I^*)^2})].$$

When $\Delta > 0$, there exist two values $a_1$ and $a_2$, $(a_1 < a_2)$, which satisfy $f(a_1) = f(a_2) = 0$. If $a \in (a_1, a_2)$

then $f(a) < 0$, therefore $E^*$ is unstable, otherwise it is locally asymptotically stable. At $a_1, a_2$, the characteristic equation becomes

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_1c_2 = 0,$$

we have the following eigenvalues $\lambda_1 = -c_1,\ \lambda_2 = \sqrt{c_2}\ i,\ \overline{\lambda}_2 = -\sqrt{c_2}\ i,$ then a pair of simple conjugate pure imaginary roots exists at $a = a_1$ and $a = a_2$, which crosses the imaginary axis from left to right at $a_1$ and crosses the imaginary axis from right to left at $a_2$. Thus Hopf bifurcations occur.

Note that

$$\begin{aligned}\Delta &= B^2 - 4D^2[\frac{\beta I^*}{(1+\alpha S^*)^2}(\mu+\nu+\varepsilon+\frac{\gamma\omega}{(\omega+I^*)^2}) + \mu(-\frac{\gamma}{\omega+I^*}+\frac{\gamma\omega}{(\omega+I^*)^2})] \\ &= (B - 2D\sqrt{\frac{\beta I^*}{(1+\alpha S^*)^2}(\mu+\nu+\varepsilon+\frac{\gamma\omega}{(\omega+I^*)^2}) + \mu(-\frac{\gamma}{\omega+I^*}+\frac{\gamma\omega}{(\omega+I^*)^2})}) \\ &\times (B + 2D\sqrt{\frac{\beta I^*}{(1+\alpha S^*)^2}(\mu+\nu+\varepsilon+\frac{\gamma\omega}{(\omega+I^*)^2}) + \mu(-\frac{\gamma}{\omega+I^*}+\frac{\gamma\omega}{(\omega+I^*)^2})}).\end{aligned}$$

Because of $B < 0$, we should set

$$B + 2D\sqrt{\frac{\beta I^*}{(1+\alpha S^*)^2}(\mu+\nu+\varepsilon+\frac{\gamma\omega}{(\omega+I^*)^2}) + \mu(-\frac{\gamma}{\omega+I^*}+\frac{\gamma\omega}{(\omega+I^*)^2})} < 0,$$

i.e. (9) hold, to ensure $\Delta > 0$. Hence the proof is completed.

## 4. THE GAS OF THE ENDEMIC EQUILIBRIUM

Firstly, we give the positively invariant sets to show that the orbits of system (6) are bounded.

**Proposition 4.1.** The set :

$$\Gamma = \{(S, M, I) \in R_+^3 \mid 0 \le M \le k,\ 0 \le S + I \le 1 - p_0\},$$

is positively invariant and absorbing and, as a consequence, the orbits of (6) are bounded, provided that $(S(0), M(0), I(0)) \ge (0, 0, 0)$.

**Proof.** From the following inequalities

$$-aM \le M'(t) \le a(k - M),$$

according to the comparison theorem, it follows that

$$\liminf_{t\to+\infty} M(t) \ge 0,\quad \limsup_{t\to+\infty} M(t) \le k.$$

Furthermore, defining $\delta = S + I$ , one has that

$$\delta' = \mu(1 - p_0 - p_1(M)) - \mu\delta - (\nu+\varepsilon)I - \frac{\gamma I}{\omega+I}.$$

Note that $h^*(I) = \dfrac{\gamma I}{\omega+I}$ is an increasing function, thus

$$\frac{\gamma I}{\omega+I} < \frac{\gamma\delta}{\omega+\delta} < \frac{\gamma\delta}{\omega}.$$

So,

$$\mu(1 - p_0 - p_1(k)) - (\mu+\nu+\varepsilon+\frac{\gamma}{\omega})\delta \le \delta' \le \mu(1 - p_0) - \mu\delta.$$

From the comparison theorem, we have

$$\liminf_{t\to+\infty}(S(t)+I(t)) \geq \frac{\mu}{\mu+\nu+\varepsilon+\frac{\gamma}{\omega}}(1-p_0-p_1(k)) > 0.$$

and

$$\limsup_{t\to+\infty}(S(t)+I(t)) \leq 1-p_0.$$

Thus our claim is demonstrated.

Global stability analysis for the endemic equilibrium $E^*$ will be performed through the approach due to Li and Muldowney[7]. Firstly, we will describe the general method.

$D \subset R^n$ is an open set, $f: D \mapsto R^n$, and $f \in C^1(D)$. Consider the differential equation:

$$x' = f(x). \quad (11)$$

Assume that the following hypotheses hold:

$(H_1)$ there exists a compact absorbing set $K \subset D$;

$(H_2)$ the equation (11) has a unique equilibrium $x^* \in D$.

Li and Muldowney showed that if $(H_1)$ and $(H_2)$ hold, and (11) satisfies a Bendixson criterion that is robust under $C^1$ local $\varepsilon$-perturbation of $f$ at all non-equilibrium non-wandering points for (11), then $x^*$ is globally stable in $D$ provided it is stable. Then a new Bendixson criterion robust under $C^1$ local $\varepsilon$-perturbation and based on the use of the Lozinski $\breve{l}$ measure is introduced as follows:

Let $x \mapsto P(x)$ be a $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function that is $C^1$ on $D$. Assume that $P^{-1}(x)$ exists and continues on $x \in K$, where $K$ is a compact absorbing set. Defining

$$q = \limsup_{t\to\infty} \sup_{x_0\in K} \frac{1}{t}\int_0^t \mu(B(x(s,x_0)))ds, \quad (12)$$

where $B = P_f P^{-1} + P\frac{\partial f^{[2]}}{\partial x}P^{-1}$, $P_f$ is $(P_{ij}(x))_f = \frac{\partial P_{ij}(x)}{\partial x}\cdot f(x)$, $\mu(B)$ is the Lozinski $\breve{l}$ measure

$$\mu(B) = \lim_{h\to 0^+}\frac{\|I+hB\|-1}{h},$$

and $\frac{\partial f^{[2]}}{\partial x}$ is the second additive compound matrix of the Jacobian matrix $\frac{\partial f}{\partial x}$.

**Theorem 4.1.**(Li and Muldowney)[7]Assume that conditions $(H_1)$ and $(H_2)$ hold. Then $x^*$ is globally asymptotically stable in $D$ provided that a function $P(x)$ and a Lozinski $\breve{l}$ measure $\mu$ exist such that $q < 0$ is satisfied.

Now, we will find the conditions for which $q < 0$ is verified. Consider the following assumptions:

$$\begin{aligned} &\nu+\varepsilon+\frac{\gamma}{\omega+c}+ak < a, \quad \frac{\gamma}{\omega+c} < \frac{\gamma\omega}{(\omega+1-p_0)^2}+\mu, \\ &\frac{\gamma}{\omega+c}+\frac{\beta(1-p_0)}{(1+\alpha(1-p_0))^2}+\mu\Pi < \frac{\gamma\omega}{(\omega+1-p_0)^2}+a. \end{aligned} \quad .(13)$$

**Theorem 4.2.** Under the assumptions $(1-p_0)\mathcal{R}_0 > 1$ and (13), the endemic equilibrium $E^*$ of system (6) exists and is globally asymptotically stable with respect to solutions of (6) initiating in the interior of $\Gamma$.

**Proof.** In fact, the instability of $E_0$ implies the uniform persistence, i.e. there exists a constant $c > 0$ such that

any solution $(S(t), M(t), I(t))$ with $(S(0), M(0), I(0))$ in the interior of $\Gamma$, satisfies:

$$\min\{\liminf_{t\to\infty} S(t), \liminf_{t\to\infty} M(t), \liminf_{t\to\infty} I(t)\} > c. \quad (14)$$

The uniform persistence together with boundedness of $\Gamma$, is equivalent to the existence of a compact set in the interior of $\Gamma$ which is absorbing for (6). Thus, $(\mathrm{H}_1)$ is verified. Moreover, $E^*$ is the only positive equilibrium in the interior of $\Gamma$, so that $(\mathrm{H}_2)$ is also verified.

The Jacobian matrix $J(S, M, I)$ corresponding to (6):

$$J(S,M,I) = \begin{pmatrix} -\mu - \dfrac{\beta I}{(1+\alpha S)^2} & -\mu \dfrac{\partial p_1(M)}{\partial M} & -\dfrac{\beta S}{1+\alpha S} \\ 0 & -a & ak \\ \dfrac{\beta I}{(1+\alpha S)^2} & 0 & \dfrac{\beta S}{1+\alpha S} - (\mu+\nu+\varepsilon) - \dfrac{\gamma\omega}{(\omega+I)^2} \end{pmatrix}.$$

The second additive compound matrix is

$$J^{[2]}(S,M,I) = \begin{pmatrix} -\mu - \dfrac{\beta I}{(1+\alpha S)^2} - a & ak & \dfrac{\beta S}{1+\alpha S} \\ 0 & -\mu - \dfrac{\beta I}{(1+\alpha S)^2} + \dfrac{\beta S}{1+\alpha S} & -\mu \dfrac{\partial p_1(M)}{\partial M} \\ & -(\mu+\nu+\varepsilon) - \dfrac{\gamma\omega}{(\omega+I)^2} & \\ -\dfrac{\beta I}{(1+\alpha S)^2} & 0 & \dfrac{\beta S}{1+\alpha S} - (\mu+\nu+\varepsilon) - \dfrac{\gamma\omega}{(\omega+I)^2} - a \end{pmatrix}.$$

Now, we take the function

$$P = P(S,M,I) = \mathrm{diag}\{\frac{S}{I}, \frac{S}{I}, \frac{S}{I}\},$$

it follows,

$$P_f P^{-1} = \mathrm{diag}\{\frac{\dot S}{S} - \frac{\dot I}{I}, \frac{\dot S}{S} - \frac{\dot I}{I}, \frac{\dot S}{S} - \frac{\dot I}{I}\},$$

and $PJ^{[2]}P^{-1} = J^{[2]}$, so that,

$$B = P_f P^{-1} + PJ^{[2]}P^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where $B_{11} = \dfrac{\dot S}{S} - \dfrac{\dot I}{I} - \mu - \dfrac{\beta I}{(1+\alpha S)^2} - a$, $B_{12} = [ak \;\; \dfrac{\beta S}{1+\alpha S}]$, $B_{21} = [0 \;\; -\dfrac{\beta I}{(1+\alpha S)^2}]^{\mathrm{T}}$,

$$B_{22} = \begin{pmatrix} \dfrac{\dot S}{S} - \dfrac{\dot I}{I} - \mu - \dfrac{\beta I}{(1+\alpha S)^2} + \dfrac{\beta S}{1+\alpha S} & -\mu \dfrac{\partial p_1(M)}{\partial M} \\ -\mu - \nu - \varepsilon - \dfrac{\gamma\omega}{(\omega+I)^2} & \\ 0 & \dfrac{\dot S}{S} - \dfrac{\dot I}{I} + \dfrac{\beta S}{1+\alpha S} - \mu - \nu - \varepsilon \\ & -\dfrac{\gamma\omega}{(\omega+I)^2} - a \end{pmatrix}.$$

Consider now the norm in $R^3$ as: $|(u,v,w)| = \max\{|u|, |v|+|w|\}$, where $(u,v,w)$ denotes the vector in $R^3$

. Denote by $\mu$ as the Lozinski $\breve{l}$ measure with respect to this norm. It follows:

$$\mu(B) \le \sup\{g_1, g_2\} \equiv \sup\{\mu_1(B_{11}) + |B_{12}|, \mu_1(B_{22}) + |B_{21}|\},$$

where $|B_{12}|$ and $|B_{21}|$ are matrix norms with respect to the $L^1$ vector norm and $\mu_1$ denotes the Lozinski $\breve{l}$ measure with respect to the $L^1$ norm. Thus,

$$\mu_1(B_{11}) = \frac{\dot{S}}{S} - \frac{\dot{I}}{I} - \mu - \frac{\beta I}{(1+\alpha S)^2} - a, \ |B_{12}| = \max\{ak, \frac{\beta S}{1+\alpha S}\}, \ |B_{21}| = \frac{\beta I}{(1+\alpha S)^2}.$$

Next, we will compute $\mu_1(B_{22})$. For each column of matrix $B_{22}$, adding the absolute value of the non-diagonal element to the diagonal element of the corresponding column, we get

$$B'_{22} = \begin{pmatrix} \frac{\dot{S}}{S} - \frac{\dot{I}}{I} - \mu - \frac{\beta I}{(1+\alpha S)^2} + \frac{\beta S}{1+\alpha S} & -\mu \frac{\partial p_1(M)}{\partial M} \\ -\mu - \nu - \varepsilon - \frac{\gamma\omega}{(\omega+I)^2} & \\ 0 & \frac{\dot{S}}{S} - \frac{\dot{I}}{I} + \frac{\beta S}{1+\alpha S} - \mu - \nu - \varepsilon \\ & -\frac{\gamma\omega}{(\omega+I)^2} - a + \mu \frac{\partial p_1(M)}{\partial M} \end{pmatrix},$$

then, $\mu_1(B_{22})$ is the maximum value of the two diagonal elements, i.e.

$$\mu_1(B_{22}) = \frac{\dot{S}}{S} - \frac{\dot{I}}{I} + \frac{\beta S}{1+\alpha S} - (\mu+\nu+\varepsilon) - \frac{\gamma\omega}{(\omega+I)^2} + \max\{-\mu - \frac{\beta I}{(1+\alpha S)^2}, -a + \mu \frac{\partial p_1(M)}{\partial M}\}.$$

Observe that system (6) provides the following equality:

$$\frac{\dot{I}}{I} = \frac{\beta S}{1+\alpha S} - (\mu+\nu+\varepsilon) - \frac{\gamma}{\omega+I},$$

So

$$g_1 = \frac{\dot{S}}{S} + \max\{-\frac{\beta S}{1+\alpha S} - \frac{\beta I}{(1+\alpha S)^2} + \nu + \varepsilon - a + ak + \frac{\gamma}{\omega+I}, \nu + \varepsilon + \frac{\gamma}{\omega+I} - \frac{\beta I}{(1+\alpha S)^2} - a\},$$

$$g_2 = \frac{\dot{S}}{S} + \max\{\frac{\gamma}{\omega+I} - \frac{\gamma\omega}{(\omega+I)^2} - \mu, \frac{\gamma}{\omega+I} - \frac{\gamma\omega}{(\omega+I)^2} + \frac{\beta I}{(1+\alpha S)^2} - a + \mu \frac{\partial p_1(M)}{\partial M}\}.$$

Thus

$$\mu(B) \le \sup\{g_1, g_2\}$$

$$\le \frac{\dot{S}}{S} + \max\{-\frac{\beta c}{1+\alpha c} - \frac{\beta c}{(1+\alpha c)^2} + \nu + \varepsilon - a + ak + \frac{\gamma}{\omega+c}, \nu + \varepsilon + \frac{\gamma}{\omega+c} - \frac{\beta c}{(1+\alpha c)^2} - a,$$
$$\frac{\gamma}{\omega+c} - \frac{\gamma\omega}{(\omega+1-p_0)^2} - \mu, \frac{\gamma}{\omega+c} - \frac{\gamma\omega}{(\omega+1-p_0)^2} + \frac{\beta(1-p_0)}{(1+\alpha(1-p_0))^2} - a + \mu\Pi\}$$

where c is the constant of uniform persistence.
Now, impose that:

$$\nu + \varepsilon + ak + \frac{\gamma}{\omega+c} < a + 2\frac{\beta c}{1+\alpha c}, \quad \nu + \varepsilon + \frac{\gamma}{\omega+c} < a + \frac{\beta c}{1+\alpha c},$$
$$\frac{\gamma}{\omega+c} < \frac{\gamma\omega}{(\omega+1-p_0)^2} + \mu, \qquad \frac{\gamma}{\omega+c} + \frac{\beta(1-p_0)}{(1+\alpha(1-p_0))^2} + \mu\Pi < \frac{\gamma\omega}{(\omega+1-p_0)^2} + a. \tag{15}$$

This allows to conclude that:

$$\mu(B) \le \frac{\dot{S}}{S} - \eta,$$

where

$$\eta = \min\{a+2\frac{\beta c}{1+\alpha c}-\nu-\varepsilon-ak-\frac{\gamma}{\omega+c}, a+\frac{\beta c}{1+\alpha c}-\nu-\varepsilon-\frac{\gamma}{\omega+c}, \frac{\gamma\omega}{(\omega+1-p_0)^2}+\mu-\frac{\gamma}{\omega+c}, \frac{\gamma\omega}{(\omega+1-p_0)^2}+a-\frac{\gamma}{\omega+c}-\frac{\beta(1-p_0)}{(1+\alpha(1-p_0))^2}-\mu\Pi\},$$

and $\eta > 0$. Hence,

$$\frac{1}{t}\int_0^t \mu(B)\mathrm{d}s \le \frac{1}{t}\int_0^t (\frac{\dot{S}}{S}-\eta)\mathrm{d}s = \frac{1}{t}\ln\frac{S(t)}{S(0)} - \eta,$$

so

$$q = \limsup_{t\to\infty}\sup_{x_0\in K}\frac{1}{t}\int_0^t \mu(B)\mathrm{d}s \le -\eta < 0,$$

and the Bendixson criterion is thus verified. Finally, because of $c > 0$, conditions (15) are fulfilled if the inequalities (13) hold true.

## 5. NUMERICAL SIMULATIONS

To demonstrate the theoretical results obtained in this paper, we will give some numerical simulations.

We set the function $p_1(M) = (1-p_0-\varepsilon_1)\dfrac{DM}{DM+1}$, see [1]. Firstly, we consider the parameter values as: $\beta = 2, \alpha = 0.01, \gamma = 6, \omega = 7, \mu = 0.0001, \nu = 0.12, \varepsilon = 0.02, a = 0.88, \varepsilon_1 = 0.01, k = 0.1, D = 500, p_0 = 0.8$. It is easy to get $(1-p_0)\mathcal{R}_0 = 0.399106 < 1$, so according to theorem 3.1, we will know that the disease-free equilibrium $E_0(1-p_0, 0, 0) = (0.2, 0, 0)$ is locally asymptotically stable. Now, from the figure 1, we can notice that it is also globally asymptotically stable.

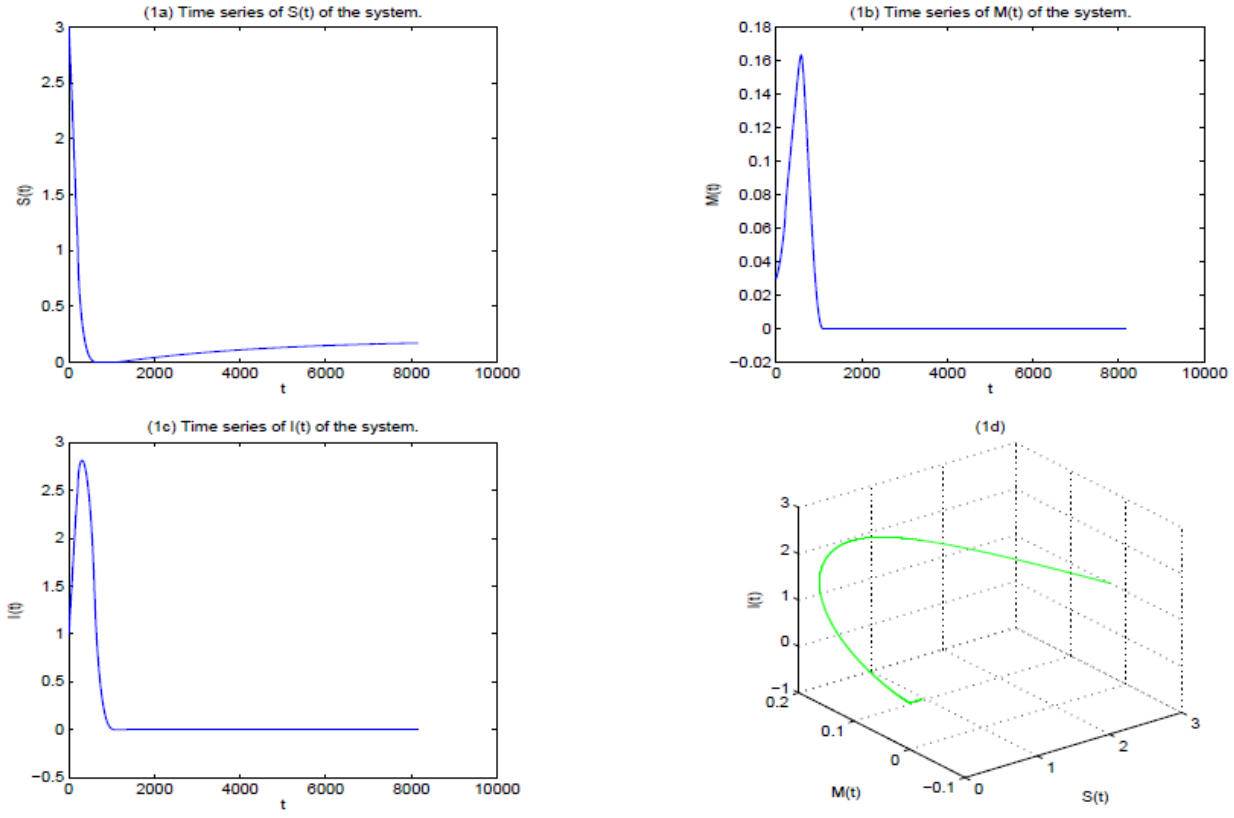


Fig 1. (1a)-(1d) are showed that the disease-free equilibrium $E_0$ of system (6) with $(S(0), M(0), I(0)) = (3, 0.03, 1)$ is globally asymptotically stable.

Next, we chose the the parameter values as:
$\beta = 4, \alpha = 0.1, \gamma = 6, \omega = 35, \mu = 0.002, \nu = 0.12, \varepsilon = 0.0036, a = 0.88, \varepsilon_1 = 0.1, k = 0.1, D = 500, p_0 = 0.8$
. It is also easy to get $(1-p_0)\mathcal{R}_0 = 2.6736 > 1$, and condition (13) are all satisfied, so according to theorem 4.2, we will know that the positive equilibrium $E^*(S^*, M^*, I^*) = (0.07481191, 0.000063556, 0.00063556)$ is globally asymptotically stable with respect to solutions of (6) initiating in the interior of $\Gamma$. And the value of $I^*$ is the unique positive solution of the equation (7) which is got by the MATLAB software.

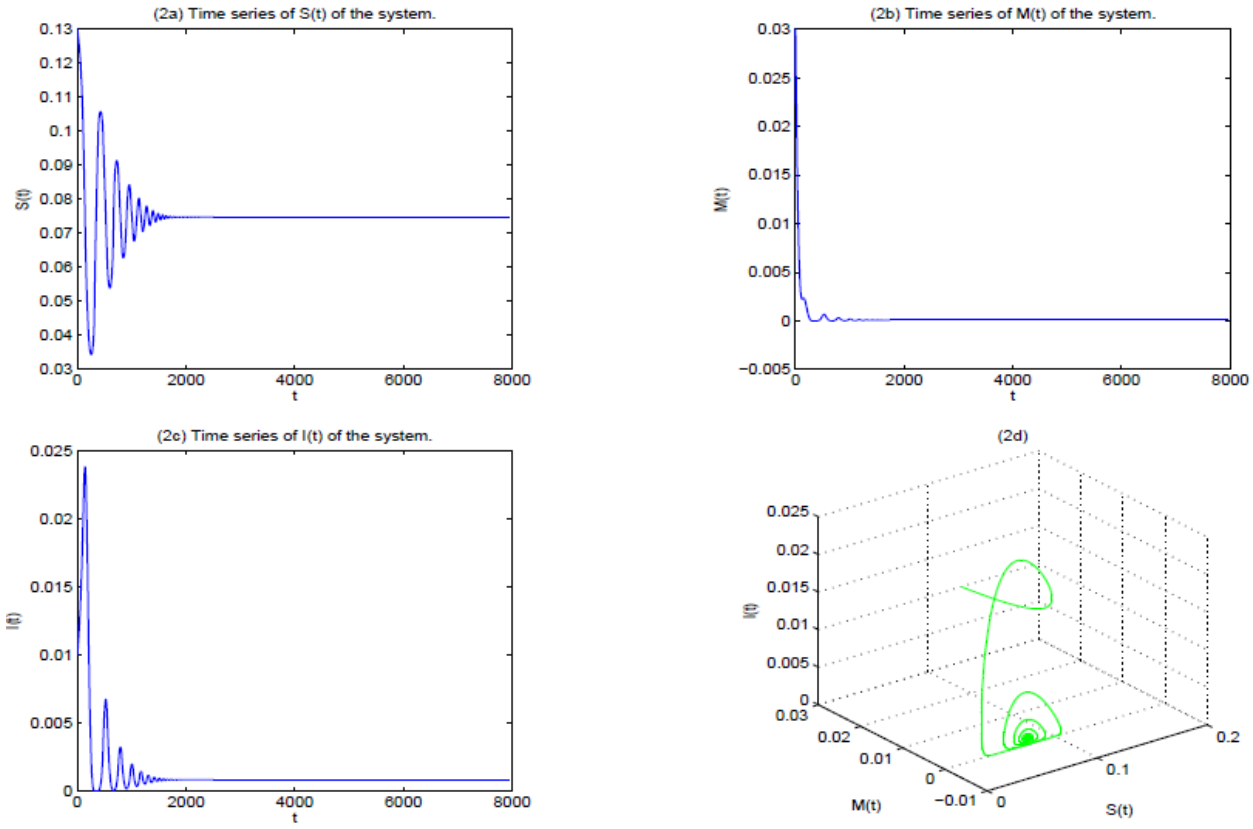


Fig 2. (2a)-(2d) are showed that the positive equilibrium $E^*$ of system (6) with $(S(0), M(0), I(0)) = (0.13, 0.03, 0.01)$ is globally asymptotically stable.

**6 . CONCLUSIONS**

In this paper, we consider an SIR epidemic model with information dependent vaccination, information variable and limited medical resources, i.e. model (6). We explore the conditions for the existence of the disease-free equilibrium and the endemic equilibrium. We also analyse their stability properties. We study the local asymptotical stability by applying Routh-Hurwitz criterion and global asymptotical stability by using the geometric method due to Li and Muldowney. We find when $(1-p_0)\mathcal{R}_0 < 1$, the disease-free equilibrium is globally asymptotically stable, and when $(1-p_0)\mathcal{R}_0 > 1$, the disease becomes prevailing.

## 11. REFERENCES

[1]. Bruno Buonomo, Alberto d`Onofrio, Deborah Lacitignola, Global stability of an SIR epidemic model with information dependent vaccination, Mathematical Bioscience 216(2008)9-16.

[2]. V. Capasso, G. Serio, A generalization of the Kermack-Mckendrick deterministic epidemic model, Math. Biosci. 42(1978)43-61.

[3]. Alberto d`Onofrio, Piero Manfredi, Ernesto Salinelli, Vaccinaying behaviour, information, and the dynamics of SIR vaccine preventable disease, Theoretical Population Biology 71(2007)301-317.

[4]. Alberto d`Onofrio, Piero Manfredi, Ernesto Salinelli, Bifurcation thresholds in an SIR model with information-dependent vaccination, Mathematical Modeling of Natural Phenomena, Epidemiology 2(1)(2007)23.

[5]. T.K. Kar, Prasanta Kumar Mondal, Global dynamics and bifurcation in delayed SIR model, Nonlinear Analysis: Real World Applications 12(2011)2058-2068.

[6]. Z.X. Liu, S. Liu, H. Wang, Backward bifurcation of an epidemic model with standard incidence rate and treatment rate, Nonlinear Analysis: Real World Applications 348(2008)433-443.

[7]. M.Y. Li, J.S. Muldowney, A geometric approach to global-stability problems, SIAMJ. Math. Anal. 27(4)(1996)1070.

[8]. N. MacDonald, Biological Delay System: Linear Stability Theory, Cambridge University, Cambridge, 1989.

[9]. W. Wang, Backward bifurcation of an epidemic model with treatment, Math. Biosci. 201(2006)58-71.

[10]. W. Wang, S. Ruan, Bifurcation of an epidemic model with constant removal rate of the infectives, J. Math. Anal. Appl. 291(2004)775-793.

[11]. Linhua Zhou, Meng Fan, Dynamics of an SIR epidemic model with limited medical resources revisited, Nonlinear Analysis: Real World Applications 13(2012)312-324.

[12]. J.Z. Zhang, Z. Jin, Q.X. Liu, Z.Y. Zhang, Analysis of a delayed SIR model with non-linear incidence rate, Discrete Dyn. Nat. Soc. (2008)16p. Article ID 636153.

[13]. X. Zhang, X.N. Liu, Backward bifurcation and global dynamics of an SIS epidemic model with general incidence rate and treatment, Nonlinear Analysis: Real World Applications 10(2009)565-575.

[14]. X. Zhang, X.N. Liu, Backward bifurcation of an epidemic model with saturated treatment function, J. Math. Anal. Appl. 348(2008)433-443.