

COUPLED FIXED POINT THEOREMS WITH CLRg PROPERTY IN FUZZY METRIC SPACES

Sumitra & Ibtisam Masmali

Department of Mathematics, Faculty of Sciences, Jazan University, Jazan, Saudi-Arabia

E-mail: math.research30_d@yahoo.com

ABSTRACT

In this paper, we introduce the notions of E.A. property and CLRg property for coupled mappings and prove common coupled fixed point theorems without exploiting the notion of continuity, completeness of the whole space or any of its range spaces. Our theorems generalize the result of [5] and [10-14]. We also find an affirmative answer in fuzzy metric space to the problem of Rhoades[2]. Illustrative examples supporting our results have also been cited.

Keywords: *Coupled fixed point, Weakly Compatible maps, E.A. property, CLRg property.*

1. INTRODUCTION

Recently, Bhaskar and Lakshmikantham [3] introduced the concepts of coupled fixed points and mixed monotone property and illustrated these results by proving the existence and uniqueness of the solution for a periodic boundary value problem. Later on these results were extended and generalized by Sedghi et al. [7], Fang [4] and Xin-Qi Hu [5] etc.

In the study of common fixed points of compatible mappings we often require assumption on completeness of the space or continuity of mappings involved besides some contractive condition but the study of fixed points of noncompatible mappings can be extended to the class of nonexpansive or Lipschitz type mapping pairs even without assuming the continuity of the mappings involved or completeness of the space. Aamri and El Moutawakil [1] generalized the concepts of noncompatibility by defining the notion of (E.A) property and proved common fixed point theorems under strict contractive condition. Although E.A property is a generalization of the concept of noncompatible maps yet it requires either completeness of the whole space or any of the range space or continuity of maps. But on the contrary, the new notion of CLR(g) property recently given by Sintunavarat and Kumam [8] does not impose such conditions. The importance of CLRg property ensures that one does not require the closeness of range subspaces.

The intent of this paper is to establish the concept of E.A. property and (CLRg) property for coupled mappings and an affirmative answer of question arising by Rhoades[2] whether, by using the concept of noncompatibility or its generalized notion, can we find equally interesting results in fuzzy metric space also?

So, our improvement in this paper is four fold as

- (i) Relaxed continuity of maps completely
- (ii) Completeness of the whole space or any of its range space removed.
- (iii) Minimal type contractive condition used.
- (iv) The condition $\lim_{t \rightarrow \infty} M(x, y, t) = 1$ is not used.
- (v) Weakened the concept of compatibility by a more general concept of weak compatible.

2. DEFINITIONS AND PRELIMINARIES

Definition 2.1[9]. A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

Definition 2.2 [9]. A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norm if $([0,1], *)$ is a topological abelian monoid with unit 1 s.t. $a * b \leq c * d$ whenever $a \leq c$ and

$b \leq d, \forall a, b, c, d \in [0,1]$. Some examples are below:

- (i) $*(a, b) = ab$,
- (ii) $*(a, b) = \min(a, b)$.

Definition 2.3[5]. Let $0 < t < 1$ ^{sup.} $\Delta(t, t) = 1$. A t-norm Δ is said to be of H-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

$\Delta^1(t) = t, \Delta^{m+1}(t) = t \Delta(\Delta^m(t)), m = 1, 2, \dots, t \in [0, 1]$ i.e. a t-norm Δ is a H-type t-norm iff for any $\lambda \in (0, 1)$, there exists $\delta(\lambda) \in (0, 1)$ such that $\Delta^m(t) > (1-\lambda)$ for all $m \in \mathbb{N}$, when $t > (1-\delta)$.

The t-norm $\Delta_M = \min$ is an example of t-norm of H-type.

Definition 2.4[9].The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(FM-1) $M(x, y, 0) > 0$,

(FM-2) $M(x, y, t) = 1$ iff $x=y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM-5) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous, for all $x, y, z \in X$ and $s, t > 0$.

We note that $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$.

Definition 2.5[6]. An element $(x, y) \in X \times X$ is called a

(i) coupled fixed point of the mapping $f: X \times X \rightarrow X$ if

$$f(x, y) = x, \quad f(y, x) = y.$$

(ii) coupled coincidence point of the mappings $f: X \times X \rightarrow X$ and $g: X \rightarrow X$ if

$$f(x, y) = g(x), \quad f(y, x) = g(y).$$

(iii) common coupled fixed point of the mappings $f: X \times X \rightarrow X$ and $g: X \rightarrow X$ if

$$x = f(x, y) = g(x), \quad y = f(y, x) = g(y).$$

Definition 2.6[5]. An element $x \in X$ is called a common fixed point of the mappings

$f: X \times X \rightarrow X$ and $g: X \rightarrow X$ if

$$x = f(x, x) = g(x).$$

Definition 2.7. Let $A, B : X \times X \rightarrow X$ and $S, T : X \rightarrow X$ be four mappings. Then, the pair of maps (B, S) and (A, T) are said to have Common Coupled Coincidence Point if there exist a, b in X such that

$$B(a, b) = S(a) = T(a) = A(a, b) \text{ and } B(b, a) = S(b) = T(b) = A(b, a).$$

Definition 2.8[3]. The mappings $f: X \times X \rightarrow X$ and $g: X \rightarrow X$ are said to be compatible if

$$\lim_{n \rightarrow \infty} M(gf(x_n, y_n), f(g(x_n), g(y_n)), t) = 1, \\ \lim_{n \rightarrow \infty} M(gf(y_n, x_n), f(g(y_n), g(x_n)), t) = 1,$$

for all $t > 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X , such that

$$\lim_{n \rightarrow \infty} f(x_n, y_n) = \lim_{n \rightarrow \infty} g(x_n) = x, \quad \lim_{n \rightarrow \infty} f(y_n, x_n) = \lim_{n \rightarrow \infty} g(y_n) = y, \text{ for all } x, y \in X.$$

Now we introduce the followings:

Definition 2.9. The mappings $f: X \times X \rightarrow X$ and $g: X \rightarrow X$ are said to be compatible of type (A) if

$$\lim_{n \rightarrow \infty} M(f(gx_n, gy_n), g^2 x_n, t) = 1, \quad \lim_{n \rightarrow \infty} M(f(gy_n, gx_n), g^2 y_n, t) = 1$$

and

$$\lim_{n \rightarrow \infty} M(gf(x_n, y_n), f(f(x_n, y_n), f(y_n, x_n)), t) = 1,$$

$$\lim_{n \rightarrow \infty} M(gf(y_n, x_n), f(f(y_n, x_n), f(x_n, y_n)), t) = 1,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n \rightarrow \infty} f(x_n, y_n) = \lim_{n \rightarrow \infty} g(x_n) = x, \lim_{n \rightarrow \infty} f(y_n, x_n) = \lim_{n \rightarrow \infty} g(y_n) = y$ for some $x, y \in X$ and $t > 0$.

Definition 2.10. The mappings $f: X \times X \rightarrow X$ and $g: X \rightarrow X$ are said to be compatible of type (B) if

$$\lim_{n \rightarrow \infty} M(f(gx_n, gy_n), g^2 x_n, t) \geq \frac{1}{2} \left\{ \begin{array}{l} \lim_{n \rightarrow \infty} M(f(gx_n, gy_n), f(x, y), t) \\ + \lim_{n \rightarrow \infty} M(f(x, y), f(f(x_n, y_n), f(y_n, x_n)), t) \end{array} \right\},$$

$$\lim_{n \rightarrow \infty} M(f(gy_n, gx_n), g^2 y_n, t) \geq \frac{1}{2} \left\{ \begin{array}{l} \lim_{n \rightarrow \infty} M(f(gy_n, gx_n), f(y, x), t) \\ + \lim_{n \rightarrow \infty} M(f(y, x), f(f(y_n, x_n), f(x_n, y_n)), t) \end{array} \right\}$$

and

$$\lim_{n \rightarrow \infty} M(gf(x_n, y_n), f(f(x_n, y_n), f(y_n, x_n)), t) \geq \frac{1}{2} \left\{ \begin{array}{l} \lim_{n \rightarrow \infty} M(gf(x_n, y_n), gx, t) \\ + \lim_{n \rightarrow \infty} M(gx, g^2 x_n, t) \end{array} \right\},$$

$$\lim_{n \rightarrow \infty} M(gf(y_n, x_n), f(f(y_n, x_n), f(x_n, y_n)), t) \geq \frac{1}{2} \left\{ \begin{array}{l} \lim_{n \rightarrow \infty} M(gf(y_n, x_n), gy, t) \\ + \lim_{n \rightarrow \infty} M(gy, g^2 y_n, t) \end{array} \right\}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n \rightarrow \infty} f(x_n, y_n) = \lim_{n \rightarrow \infty} g(x_n) = x, \lim_{n \rightarrow \infty} f(y_n, x_n) = \lim_{n \rightarrow \infty} g(y_n) = y$ for some $x, y \in X$ and $t > 0$.

Definition 2.11. The mappings $f: X \times X \rightarrow X$ and $g: X \rightarrow X$ are said to be compatible of type (P) if

$$\lim_{n \rightarrow \infty} M(f(f(x_n, y_n), f(y_n, x_n)), g^2 x_n, t) = 1,$$

$\lim_{n \rightarrow \infty} M(f(f(y_n, x_n), f(x_n, y_n)), g^2 y_n, t) = 1$
 whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n \rightarrow \infty} f(x_n, y_n) = \lim_{n \rightarrow \infty} g(x_n) = x$, $\lim_{n \rightarrow \infty} f(y_n, x_n) = \lim_{n \rightarrow \infty} g(y_n) = y$ for some $x, y \in X$ and $t > 0$.

Definition 2.12. The mappings $f: X \times X \rightarrow X$ and $g: X \rightarrow X$ are said to be compatible of type (C) if

$$\lim_{n \rightarrow \infty} M(f(gx_n, gy_n), g^2 x_n, t) \geq \frac{1}{3} \left\{ \begin{array}{l} \lim_{n \rightarrow \infty} M(f(gx_n, gy_n), f(x, y), t) \\ + \lim_{n \rightarrow \infty} M(f(x, y), g^2 x_n, t) \\ + \lim_{n \rightarrow \infty} M(f(x, y), f(f(x_n, y_n), f(y_n, x_n)), t) \end{array} \right\},$$

$$\lim_{n \rightarrow \infty} M(f(gy_n, gx_n), g^2 y_n, t) \geq \frac{1}{3} \left\{ \begin{array}{l} \lim_{n \rightarrow \infty} M(f(gy_n, gx_n), f(y, x), t) \\ + \lim_{n \rightarrow \infty} M(f(y, x), g^2 y_n, t) \\ + \lim_{n \rightarrow \infty} M(f(y, x), f(f(y_n, x_n), f(x_n, y_n)), t) \end{array} \right\}$$

and

$$\lim_{n \rightarrow \infty} M(gf(x_n, y_n), f(f(x_n, y_n), f(y_n, x_n)), t) \geq \frac{1}{3} \left\{ \begin{array}{l} \lim_{n \rightarrow \infty} M(gf(x_n, y_n), gx, t) \\ + \lim_{n \rightarrow \infty} M(gx, f(f(x_n, y_n), f(y_n, x_n)), t) \\ + \lim_{n \rightarrow \infty} M(gx, g^2 x_n, t) \end{array} \right\}$$

$$\lim_{n \rightarrow \infty} M(gf(y_n, x_n), f(f(y_n, x_n), f(x_n, y_n)), t) \geq \frac{1}{3} \left\{ \begin{array}{l} \lim_{n \rightarrow \infty} M(gf(y_n, x_n), gy, t) \\ + \lim_{n \rightarrow \infty} M(gy, f(f(y_n, x_n), f(x_n, y_n)), t) \\ + \lim_{n \rightarrow \infty} M(gy, g^2 y_n, t) \end{array} \right\}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n \rightarrow \infty} f(x_n, y_n) = \lim_{n \rightarrow \infty} g(x_n) = x$, $\lim_{n \rightarrow \infty} f(y_n, x_n) = \lim_{n \rightarrow \infty} g(y_n) = y$ for some $x, y \in X$ and $t > 0$.

Definition 2.13. The mappings $f: X \times X \rightarrow X$ and $g: X \rightarrow X$ are said to be compatible of type (A_f) if

$\lim_{n \rightarrow \infty} M(f(gx_n, gy_n), gg x_n, t) = 1$, $\lim_{n \rightarrow \infty} M(f(gy_n, gx_n), gg y_n, t) = 1$
 whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n \rightarrow \infty} f(x_n, y_n) = \lim_{n \rightarrow \infty} g(x_n) = x$, $\lim_{n \rightarrow \infty} f(y_n, x_n) = \lim_{n \rightarrow \infty} g(y_n) = y$ for some $x, y \in X$ and $t > 0$.

Definition 2.14. The mappings $f: X \times X \rightarrow X$ and $g: X \rightarrow X$ are said to be compatible of type (A_g) if

$\lim_{n \rightarrow \infty} M(gf(x_n, y_n), f(f(x_n, y_n), f(y_n, x_n)), t) = 1$,
 $\lim_{n \rightarrow \infty} M(gf(y_n, x_n), f(f(y_n, x_n), f(x_n, y_n)), t) = 1$
 whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n \rightarrow \infty} f(x_n, y_n) = \lim_{n \rightarrow \infty} g(x_n) = x$, $\lim_{n \rightarrow \infty} f(y_n, x_n) = \lim_{n \rightarrow \infty} g(y_n) = y$ for some $x, y \in X$ and $t > 0$.

Definition 2.15. The mappings $f: X \times X \rightarrow X$ and $g: X \rightarrow X$ are called weakly compatible maps if

$f(x, y) = g(x)$, $f(y, x) = g(y)$ implies $gf(x, y) = f(gx, gy)$, $gf(y, x) = f(gy, gx)$, for all x, y in X .

Example 2.1. Let $X = (0, \infty)$. Define $a * b = ab$ and $M(x, y, t) = \frac{t}{t + |x - y|}$, for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a FM-space. Define $f: X \times X \rightarrow X$ as $f(x, y) = x + y$ for all x, y in X and $g: X \rightarrow X$ as $g(x) = x$ for all x in X . Then, clearly for all x, y in X and $t > 0$, and consider sequences $\{x_n\} = \{\frac{1}{n}\}$ and $\{y_n\} = \{\frac{2}{n}\}$ in X with $\lim_{n \rightarrow \infty} f(x_n, y_n) = \lim_{n \rightarrow \infty} g(x_n) = x$, $\lim_{n \rightarrow \infty} f(y_n, x_n) = \lim_{n \rightarrow \infty} g(y_n) = y$ implies $x = 0$, $y = 0$. It can be easily verified that the maps f and g satisfies the following properties.

- (i) Compatible of type (A) property,
- (ii) Compatible of type (B) property,
- (iii) Compatible of type (P) property,
- (iv) Compatible of type (C) property,
- (v) Compatible of type (A_f) property,
- (vi) Compatible of type (A_g) property.

Now, we fuzzify the newly defined concepts of E.A Property introduced by Aamri and Moutawakil [1] and (CLR_g) property given by Sintunavarat and Kuman [8] for coupled maps as follows:

Definition 2.16. Let $(X, M, *)$ be a FM space. Two maps $f: X \times X \rightarrow X$ and $g: X \rightarrow X$ are said to satisfy E.A. property if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} (f(x_n, y_n)) = \lim_{n \rightarrow \infty} gx_n = x \text{ and } \lim_{n \rightarrow \infty} (f(y_n, x_n)) = \lim_{n \rightarrow \infty} gy_n = y, \text{ for some } x, y \text{ in } X..$$

Definition 2.17. Let $(X, M, *)$ be a FM space .Two maps $f: X \times X \rightarrow X$ and $g: X \rightarrow X$ are said to satisfy CLRg property if there exists sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} (f(x_n, y_n)) = \lim_{n \rightarrow \infty} gx_n = g(p) \quad \text{and} \quad \lim_{n \rightarrow \infty} (f(y_n, x_n)) = \lim_{n \rightarrow \infty} gy_n = g(q), \text{ for some } p, q \text{ in } X.$$

Example 2.2. Let $(X, M, *)$ be a fuzzy metric space, $*$ being a continuous norm with $X = [0, 1]$. Define $M(x, y, t) = \frac{t}{t+|x-y|}$ for all x, y in X and $t > 0$.

Also define the maps $f: X \times X \rightarrow X$ and $g: X \rightarrow X$ by $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$ and $g(x) = \frac{x}{2}$ respectively. Note that 0 is the points of coincidence of f and g . It is clear that the pair (f, g) is weakly compatible on X . We next show that the pair (f, g) is not compatible.

Consider the sequences $\{x_n\} = \{\frac{1}{2} + \frac{1}{n}\}$ and $\{y_n\} = \{\frac{1}{2} - \frac{1}{n}\}$, then

$$f(x_n, y_n) = \frac{1}{4} + \frac{1}{n^2} = f(y_n, x_n), g(x_n) = \frac{1}{4} + \frac{1}{2n}, g(y_n) = \frac{1}{4} - \frac{1}{2n}$$

$$\lim_{n \rightarrow \infty} M(f(x_n, y_n), g(x_n), t) = \left[\frac{t}{t + \left| \frac{1}{n^2} - \frac{1}{2n} \right|} \right] \rightarrow 1 \neq g(p), \text{ for any } p.$$

$$\lim_{n \rightarrow \infty} M(f(y_n, x_n), g(y_n), t) = \left[\frac{t}{t + \left| \frac{1}{n^2} - \frac{1}{2n} \right|} \right] \rightarrow 1 \neq g(q), \text{ for any } q.$$

$$M(fgx_n, gy_n), gf(x_n, y_n), t) = \frac{t}{t+|f(gx_n, gy_n) - gf(x_n, y_n)|} = \frac{t}{t + \frac{1}{4}(\frac{1}{2} + \frac{2}{n^2})} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Hence the pair (f, g) is not compatible satisfying E.A property but not CLRg.

Example 2.3. Let $(X, M, *)$ be a fuzzy metric space, $*$ being a continuous norm with $X = \mathbb{R}$. Define $M(x, y, t) = \frac{t}{t+|x-y|}$ for all x, y in X and $t > 0$. Define mappings

$f: X \times X \rightarrow X$ and $g: X \rightarrow X$ by $f(x, y) = x - y$ and $g(x) = 2x$ for all x, y in X and consider the sequences $x_n = \{\frac{1}{n}\}$

and $y_n = \{-\frac{1}{n}\}$, then $f(x_n, y_n) = \frac{2}{n}, g(x_n) = \frac{2}{n} \Rightarrow \lim_{n \rightarrow \infty} M(f(x_n, y_n), g(x_n), t) \rightarrow 1 = g(\frac{1}{2})$

$f(y_n, x_n) = -\frac{2}{n}, g(y_n) = -\frac{2}{n} \Rightarrow \lim_{n \rightarrow \infty} M(f(y_n, x_n), g(y_n), t) \rightarrow 1 = g(\frac{1}{2})$

therefore, f and g satisfy both the. properties E.A and CLRg

Remark 2.1. From above examples we can say that

- (1) Weak compatibility does not imply compatibility
- (2) E.A does not imply (CLRg).
- (3) E.A and (CLRg) does not imply compatibility.
- (4) In the next example, we show that the maps satisfying (CLRg) property need not be continuous, i.e. continuity is not the necessary condition for maps to satisfy (CLRg) property.

Example 2.4. Let $(X, M, *)$ be a fuzzy metric space, $*$ being a continuous norm with $X = [0, \infty)$.

Define $M(x, y, t) = \frac{t}{t+|x-y|}$ for all x, y in X and $t > 0$. Define mappings

$f: X \times X \rightarrow X$ and $g: X \rightarrow X$ as follows

$$f(x, y) = \begin{cases} x + y, x \in [0, 1), y \in X \\ \frac{x + y}{2}, x \in [1, \infty), y \in X \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 + x, x \in [0, 1) \\ \frac{x}{2}, x \in [1, \infty) \end{cases}$$

We consider the sequences $x_n = \left\langle \frac{1}{n} \right\rangle$ and $y_n = \left\langle 1 + \frac{1}{n} \right\rangle$ Then,

$$f(x_n, y_n) = f\left(\frac{1}{n}, 1 + \frac{1}{n}\right) = 1 + \frac{2}{n}, f(y_n, x_n) = f\left(1 + \frac{1}{n}, \frac{1}{n}\right) = \frac{1}{2} + \frac{1}{n}$$

$$g(x_n) = g\left(\frac{1}{n}\right) = 1 + \frac{1}{n}, g(y_n) = g\left(1 + \frac{1}{n}\right) = \frac{1}{2} + \frac{1}{2n}$$

$$\lim_{n \rightarrow \infty} M\left(f(x_n, y_n), g(x_n), t\right) \rightarrow 1 = g(0) \quad \text{and} \quad \lim_{n \rightarrow \infty} M\left(f(y_n, x_n), g(y_n), t\right) \rightarrow 1 = g(0)$$

therefore, the maps f and g satisfy (CLRg) property but the maps are not continuous.

3. MAIN RESULTS

Proposition 3.1. Let $(X, M, *)$ be a Fuzzy Metric Space, $*$ being continuous t-norm of H-type with $a * b \geq ab, \forall a, b \in [0, 1]$. Let $A : X \times X \rightarrow X$ and $S : X \rightarrow X$ be compatible mappings such that $A(x, y) = Sx$ and $A(y, x) = Sy$ for some x, y in X , then

$$AA(x, y) = A(Sx, Sy) = SSx = SA(x, y) \quad \text{and} \quad AA(y, x) = A(Sy, Sx) = SSy = SA(y, x)$$

Proof : Let $\{x_n\}$ and $\{y_n\}$ be the sequences in X such that $\{x_n\} = x$ and $\{y_n\} = y, n = 1, 2, 3, \dots$ and $A(x, y) = Sx, A(y, x) = Sy$ so $A(x_n, y_n), Sx_n \rightarrow Sx$ and $A(y_n, x_n), Sy_n \rightarrow Sy$. Also the maps A, S are compatible, so

$$\lim_{n \rightarrow \infty} M(A(Sx_n, Sy_n), SA(x_n, y_n), t) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} M(A(Sy_n, Sx_n), SA(y_n, x_n), t) = 1$$

$$M(A(Sx, Sy), SSx, t) = M(A(Sx, Sy), SA(x, y), t) \\ = \lim_{n \rightarrow \infty} M(A(Sx_n, Sy_n), SA(x_n, y_n), t) \rightarrow 1$$

Thus, $A(Sx, Sy) = SSx$, similarly $SA(x, y) = AA(x, y)$. But $A(x, y) = Sx$, hence

$$AA(x, y) = A(Sx, Sy) = SSx = SA(x, y),$$

Similarly we can have $AA(y, x) = A(Sy, Sx) = SSy = SA(y, x)$

Remark 3.1. Proposition 3.1 is true if compatibility condition is replaced by any of its variants. Now, we prove the following result for quadruple maps.

Theorem 3.2. Let $(X, M, *)$ be a Fuzzy Metric Space, $*$ being continuous t-norm of H-type with $a * b \geq ab, \forall a, b \in [0, 1]$. Let $A, B : X \times X \rightarrow X$ and $S, T : X \rightarrow X$ be four mappings satisfying following conditions:

(3.1) The pairs (A, S) and (B, T) satisfy CLR(g) property

$$(3.2) \quad M(A(x, y), B(u, v), kt) \geq \text{Min}\{M(Sx, Tu, t), M(A(x, y), Sx, t), M(B(u, v), Tu, t)\}$$

$\forall x, y, u, v \in X, t > 0, 0 < k < 1$. Then

- (i) A and S have point of coincidence.
- (ii) B and T have point of coincidence.

Moreover, if the pairs (A, S) and (B, T) are weakly compatible, then there exists unique x in X such that $A(x, x) = T(x) = B(x, x) = S(x) = x$.

Proof. Since the pairs (A, S) and (B, T) satisfy CLRg property, there exist sequences $\{x_n\}, \{y_n\}, \{x_n'\}$ and $\{y_n'\}$ in X such that $\lim_{n \rightarrow \infty} A(x_n, y_n) = \lim_{n \rightarrow \infty} S(x_n) = Sa,$

$\lim_{n \rightarrow \infty} A(y_n, x_n) = \lim_{n \rightarrow \infty} S(y_n) = Sb$ and $\lim_{n \rightarrow \infty} B(x'_n, y'_n) = \lim_{n \rightarrow \infty} T(x'_n) = Ta'$, $\lim_{n \rightarrow \infty} B(y'_n, x'_n) = \lim_{n \rightarrow \infty} T(y'_n) = Tb'$, for some a, b, a', b' in X

Step 1: We now show that the pairs (A, S) and (B, T) have common coupled coincidence point. We first show that $Sa = Ta'$. Using (3.2), we have ,

$$M(A(x_n, y_n), B(x_n, y_n), kt) \geq \text{Min}\{M(Sx_n, Tx_n, t), M(A(x_n, y_n), Sx_n, t), M(B(x_n, y_n), Tx_n, t)\}$$

Taking $n \rightarrow \infty$, we get $M(Sa, Ta', kt) \geq \text{Min}\{M(Sa, Ta', t), 1, 1\}$

i.e $M(Sa, Ta', kt) \geq M(Sa, Ta', t) \Rightarrow Sa = Ta'$, similarly we can have $Sb = Tb'$.

Also ,

$$M(A(y_n, x_n), B(x'_n, y'_n), kt) \geq \text{Min}\{M(Sy_n, Tx'_n, t), M(A(y_n, x_n), Sy_n, t), M(B(x'_n, y'_n), Tx'_n, t)\} \text{ i.e}$$

$$M(Sb, Ta', kt) \geq M(Sb, Ta', t) \Rightarrow Sb = Ta'$$

Hence $Sb = Ta' = Sa = Tb'$. Now, for all $t > 0$, using condition (3.2), we have

$$M(A(x_n, y_n), B(a', b'), kt) \geq \text{Min}\{M(Sx_n, Ta', t), M(A(x_n, y_n), Sx_n, t), M(B(a', b'), Ta', t)\}$$

Taking $n \rightarrow \infty$, we get , $M(Sa, B(a', b'), kt) \geq M(Sa, B(a', b'), t)$,

which implies that $Sa = B(a', b')$. Similarly, we can get that $Sb = B(b', a')$.

In a similar fashion , we can have $Ta' = A(a, b)$ and $Tb' = A(b, a)$.

Thus, $B(a', b') = Sa = Ta' = A(a, b)$ and $B(b', a') = Sb = Tb' = A(b, a)$. Thus the pairs (A, S) and (B, T) have common coincidence points.

Let $Sa = A(a, b) = B(a', b') = Ta' = x$ and $Sb = A(b, a) = B(b', a') = Tb' = y$. Since (A, S) and (B, T) are weakly compatible , so

$$Sx = SA(a, b) = A(Sa, Sb) = A(x, y) \text{ and } Sy = SA(b, a) = A(Sb, Sa) = A(y, x).$$

$$Tx = TB(a', b') = B(Ta', Tb') = B(x, y) \text{ and } Ty = TB(b', a') = B(Tb', Ta') = B(y, x).$$

Step 2: We next show that $x = y$. From (3.2) ,

$$\begin{aligned} M(x, y, kt) &= M(A(a, b), B(a', b'), kt) \\ &= \text{Min}\{M(Sa, Ta', t), M(A(a, b), Ta, t), M(B(a', b'), Ta', t)\} = 1 \end{aligned}$$

Thus , $x = y$.

Step 3: Now , we prove that $Sx = Tx$, using (3.2) again

$$\begin{aligned} M(Sx, Tx, kt) &= M(Sx, Ty, kt) = M(A(x, y), B(y, x), kt) \\ &\geq \text{Min}\{M(Sx, Ty, t), M(A(x, y), Sx, t), M(B(y, x)Ty, t)\} \\ &= \text{Min}\{M(Sx, Tx, t), M(A(x, y), Sx, t), M(B(y, x)Ty, t)\} \end{aligned}$$

i.e $M(Sx, Tx, kt) \geq M(Sx, Ty, t) \Rightarrow Sx = Tx = Ty$.

Step 4: Lastly , we prove that $Sx = x$

$$\begin{aligned} M(Sx, x, kt) &= M(Sx, y, kt) = M(A(x, y), B(x, y), kt) \\ &\geq \text{Min}\{M(Sx, Tx, t), M(A(x, y), Sx, t), M(B(x, y), Tx, t)\} \end{aligned}$$

Hence $x = Sx = Tx = A(x, x) = B(x, x)$. This shows that A, B, S, T have a common fixed point and uniqueness of x follows easily from (3.2).

Theorem 3.3. Theorem 3.2 remains true if the ‘compatible property’ is replaced by any one (retaining the rest of the hypothesis) of the following:

- (i) compatible maps.
- (ii) compatible of type (A) property,
- (iii) compatible of type (B) property,
- (iv) compatible of type (P) property,
- (v) compatible of type (C) property,
- (vi) compatible of type (A_f) property,
- (vii) compatible of type (A_g) property,

Proof. Theorem follows from proposition 3.1 and theorem 3.2.

Corollary 3.1 . Let $(X, M, *)$ be a Fuzzy Metric Space, $*$ being continuous t-norm of H-type with $a * b \geq ab, \forall a, b \in [0, 1]$. Let $A, B : X \times X \rightarrow X$ and $S, T : X \rightarrow X$ be four mappings satisfying following conditions:

(3.3) The pairs (A, S) and (B, T) satisfy CLR(g) property

$$(3.4) \quad M(A(x, y), B(u, v), kt) \geq \text{Min}\{M(Sy, Tv, t), M(A(y, x), Sy, t), M(B(v, u), Tv, t)\}$$

$\forall x, y, u, v \in X, t > 0, 0 < k < 1$. Then

(iii) A and S have point of coincidence.

(iv) B and T have point of coincidence.

Moreover if the pairs (A, S) and (B, T) are compatible, then there exists unique x in X such that $A(x, x) = T(x) = B(x, x) = S(x) = x$.

Corollary 3.2. Let $(X, M, *)$ be a Fuzzy Metric Space, $*$ being continuous t-norm of H-type with $a * b \geq ab, \forall a, b \in [0, 1]$. Let $A, B : X \times X \rightarrow X$ and $S, T : X \rightarrow X$ be four mappings satisfying following conditions:

(3.5) The pairs (A, S) and (B, T) satisfy CLR(g) property

$$(3.6) \quad M(A(x, y), B(u, v), kt) \geq \{M(Sx, Tu, t) * M(A(x, y), Sx, t) * M(B(u, v), Tu, t)\}$$

$\forall x, y, u, v \in X, t > 0, 0 < k < 1$. Then

(v) A and S have point of coincidence.

(vi) B and T have point of coincidence.

Moreover if the pairs (A, S) and (B, T) are compatible, then there exists unique x in X such that $A(x, x) = T(x) = B(x, x) = S(x) = x$.

4. AN APPLICATION

Corollary 3.3. Let $(X, M, *)$ be a Fuzzy Metric Space, $*$ being continuous t-norm of H-type with $a * b \geq ab, \forall a, b \in [0, 1]$. Let $A, B : X \times X \rightarrow X$ and $S, T : X \rightarrow X$ be four mappings satisfying following conditions:

(3.7) The pairs (A, S) and (B, T) satisfy (E.A) property

$$(3.8) \quad A(X \times X) \subseteq T(X), B(X \times X) \subseteq S(X),$$

(3.9) $S(X)$ and $T(X)$ are closed subsets of X .

$$(3.10) \quad M(A(x, y), B(u, v), kt) \geq \{M(Sx, Tu, t) * M(A(x, y), Sx, t) * M(B(u, v), Tu, t)\}$$

$\forall x, y, u, v \in X, t > 0, 0 < k < 1$. Then

(vii) A and S have point of coincidence.

(viii) B and T have point of coincidence.

Moreover if the pairs (A, S) and (B, T) are compatible, then there exists unique x in X such that $A(x, x) = T(x) = B(x, x) = S(x) = x$.

Next, we give an example in support of our theorem 3.2

Example 3.1. Let $X = [-2, 2]$, $a * b = ab$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \begin{cases} \frac{t}{t + |x - y|}, t \neq 0 \\ 0, t = 0 \end{cases}$. Then $(X, M,$

$*$) is a Fuzzy Metric space. Define the mappings

$A, B : X \times X \rightarrow X$ and $S, T : X \rightarrow X$ as follows

$$A(x, y) = \begin{cases} x + y, x \in [0, 2], y \in X \\ 1, otherwise \end{cases} \quad \text{and} \quad B(x, y) = \begin{cases} x + y, x \in [0, 2], y \in X \\ 2, otherwise \end{cases}$$

$$S(x) = \begin{cases} x, x \in [0, 2], y \in X \\ 1, \text{otherwise} \end{cases} \quad \text{and} \quad T(x) = \begin{cases} 2x, x \in [0, 2], y \in X \\ 2, \text{otherwise} \end{cases}$$
 Consider the sequences

$$x_n = \left\{ \frac{1}{n} \right\}, y_n = \left\{ -\frac{1}{n} \right\}, x'_n = \left\{ 1 + \frac{1}{n} \right\}, y'_n = \left\{ 1 - \frac{1}{n} \right\}, n \in \mathbb{N}$$
 , then the pairs (A,S) and (B,T) are weakly compatible and we see

$$\lim_{n \rightarrow \infty} A(x_n, y_n) = \lim_{n \rightarrow \infty} A\left(\frac{1}{n}, -\frac{1}{n}\right) = \lim_{n \rightarrow \infty} S\left(\frac{1}{n}\right) = 0$$

$$\lim_{n \rightarrow \infty} A(y_n, x_n) = \lim_{n \rightarrow \infty} A\left(-\frac{1}{n}, \frac{1}{n}\right) = \lim_{n \rightarrow \infty} S\left(-\frac{1}{n}\right) = 0$$

$$\lim_{n \rightarrow \infty} B(x'_n, y'_n) = \lim_{n \rightarrow \infty} B\left(1 + \frac{1}{n}, 1 - \frac{1}{n}\right) = \lim_{n \rightarrow \infty} T\left(1 + \frac{1}{n}\right) = 2$$

$$\lim_{n \rightarrow \infty} M(B(y'_n, x'_n)) = \lim_{n \rightarrow \infty} B\left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right) = \lim_{n \rightarrow \infty} T\left(1 - \frac{1}{n}\right) = 2$$

So, all the conditions of our theorem are satisfied. Thus A,B,S and T have a unique common coupled fixed point in X. Indeed, x = 0 is the unique common fixed point which is also a point of discontinuity.

REFERENCES

- [1]. M. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, *Journal of Mathematical Analysis and Applications*, 27(2002), 181 – 188.
- [2]. B.E. Rhoades, Contractive definitions continuity, *Contemp. Math.*,72, (1988), 233-245.
- [3]. T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Analysis: Theory, Methods and Applications*, Vol.65, no.7, pp. 1379-1393, 2006.
- [4]. J.X.Fang, Common fixed point theorems of compatible and weakly compatible maps in Menger Spaces, *Nonlinear Analysis: Theory, Methods and Applications*, Vol.71, no. 5-6, pp.1833-1843, 2009.
- [5]. Xin-Qi Hu, Common Coupled Fixed Point Theorems for Contractive Mappings in Fuzzy Metric Spaces, *Fixed Point Theory and Applications*, Vol. 2011, article id 363716, 14 pages.
- [6]. V. Lakshmikantham and L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Analysis: Theory, Methods and Applications*, Vol.70, no.12, pp. 4341-4349, 2009.
- [7]. S.Sedghi, I. Altun and N.Shobe, Coupled fixed point theorems for contractions in fuzzy metric spaces, *Nonlinear Analysis: Theory, Methods and Applications*, Vol. 72, no. 3-4, pp. 1298-1304, 2010.
- [8]. W. Sintunavarat and P. Kuman, Common fixed point theorems for a pair of weakly compatible mappings in Fuzzy Metric Spaces, *Journal of Applied Mathematics*, Vol. 2011, Article ID 637958, 14 pages, doi: 10.1155/2011/637958.
- [9]. L.A.Zadeh, Fuzzy Sets, *Information and Control*, Vol. 89, pp. 338-353, 1965.
- [10]. M.K.Mishra et.al, Fixed point theorems in Fuzzy Metric Space for weakly compatible maps satisfying integral type inequality, *Int J.Applied Engineering Research*, Dindigul, vol1, No. 3,2010
- [11]. M.Imdad and Javid Ali, Some common fixed point theorems in Fuzzy Metric Space, *Mathematical Communications* 11(2006), 153-163.
- [12]. S.Sedhi and N.Shobe, Fixed point theorems in M- Fuzzy Metric Spaces with property (E), *Advances in Fuzzy Mathematics*, ISSN 0973-533X, Vol 1, No.1(2006), pp.55-65
- [13]. V.Pant and R.P.Pant, Fixed point theorems in Fuzzy Metric for non compatible maps, *Soocho J.Mathematics*, Vol. 33, No. 4, pp.647-665, October 2007.
- [14]. V.S.Chauhan et al, Fixed point theorems in Fuzzy Metric for weakly compatible maps, *Int.J.Contemp.Math Sci*. Vol. 5,2010,no. 3 145-151