ON ALMOST SIMILARITY OPERATOR EQUIVALENCE RELATION

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ABSTRACT
We consider the almost similarity property which is a new class in operator theory and was first introduced by A. A. S. Jibril. We establish that almost similarity is an equivalence relation. Some results on almost similarity and isometries, compact operators, hermitian, normal and projection operator are also shown. By characterization of unitary equivalence operators in terms of almost similarity we prove that operators that are similar are almost similar. We also claim that quasi-similarity implies almost similarity under certain conditions (i.e. if the quasi-affinities are assumed to be unitary operators).
Furthermore, a condition under which almost similarity of operators implies similarity is investigated. Lastly, we show that two bounded linear operators A, B of a Banach algebra on a Hilbert space H are both completely non-unitary if they are contractions which are almost similar to each other.

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1. INTRODUCTION

Two bounded linear operators A, B of a Banach algebra on a Hilbert space H $(i.e.\ A, B \in B(H))$ are said to be almost similar (a.s) (denoted by $A \overset{a.s}{\sim} B$) if there exists an invertible operator N such that the following two conditions are satisfied:

$$A^*A = N^{-1}(B^*B)N$$
$$A^* + A = N^{-1}(B^* + B)N .$$

Recently, the class of almost similarity of operators has arisen keen interest to specialists in this area. Almost similarity was first introduced by A. A. S. Jibril (1996). He proved various results that relate almost similarity and other classes of operators, including isometries, normal operators, unitary operators, compact operators and characterization of θ −operators. θ −Operators were extensively studied by Campbell in [1]. Unitary equivalence of almost similarity of operators was also shown. In 2008, Nzimbi et al [6] results are also handy in enriching almost similarity where he attempts to classify those operators where almost similarity implies similarity.

If two operators are almost similar and one of them is isometric, then so is the other. Similar results hold true for hermitian, compact, partially isometric and θ −operators. We also note that if $A, B \in B(H)$ are such that $A \text{ and } B$ are unitarily equivalent, then they are almost similar. Two quasi-similar operators having equivalent quasi-affinities on a finite dimensional Hilbert space which are unitary are also almost similar.

We investigate unitary equivalence of completely non-unitary operators and quasitriangular operators in relation to almost similarity. Evidently, Quasi-triangularity of operators is not preserved under similarity. For $A \in B(H)$ such that $A \overset{a.s}{\sim} T$ where T is an isometry implies that the direct summands of A are isometric. This does not mean that $A_1 \sim U$ and $A_2 \sim S_+$. But if almost similarity is replaced with unitary equivalence, then the direct summands are preserved. Two operators $A, B \in B(H)$ are both completely non-unitary if they are contractions which are almost similar to each other.

2. SOME RESULTS ON ALMOST SIMILARITY

Recall that two operators A and B are said to be almost similar (denoted by $A \overset{a.s}{\sim} B$) if there exists an invertible operator N such that the following two conditions are satisfied:

$$A^*A = N^{-1}(B^*B)N$$

$$A^* + A = N^{-1}(B^* + B)N.$$

Theorem 2.1: Almost similarity of operators is an equivalence relation.

Proof: (i) Let $A \in B(H)$. Then $A^*A = N^{-1}(A^*A)N$, where N is an invertible operator. Also, $A^* + A = N^{-1}(A^{*+}A)$. Hence $A \overset{a.s}{\sim} A$.

(ii) Now suppose that $A \overset{a.s}{\sim} B$, there exists an invertible operator N such that

$$A^*A = N^{-1}(B^*B)N \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

$$\text{and } A^* + A = N^{-1}(B^* + B)N \dots\dots\dots\dots\dots\dots (2).$$

Since N is invertible, upon pre-multiplication of (1) and (2) by N and post multiplication of (1) and (2) by N^{-1} and applying the adjoint operation, we have

$$A^*A = M^{-1}(B^*B)M$$

$A^* + A = M^{-1}(B^* + B)M$ where $N = M^{-1}$ which is an invertible operator, since N^{-1} is invertible. Hence $B \overset{a.s}{\sim} A$.

(iii) Let A, B, C be in $B(H)$. Suppose that $A \overset{a.s}{\sim} B$ and $B \overset{a.s}{\sim} C$. Then we have

$$A^*A = N^{-1}(B^*B)N, \quad A^* + A = N^{-1}(B^* + B)N \dots\dots\dots (3)$$

$$\text{and } A^*A = M^{-1}(B^*B)M, \quad A^* + A = M^{-1}(B^* + B)M \dots\dots\dots (4),$$

where M and N are invertible operators. Using (3) and (4) we have that

$A^*A = N^{-1}[M^{-1}(C^*C)M]N = (MN)^{-1}C^*C(MN) = S^{-1}(C^*C)S$ and

$A^* + A = N^{-1}[M^{-1}(C^* + C)M]N = (MN)^{-1}(C^* + C)(MN) = S^{-1}(C^* + C)S$ where $S = MN$, is invertible since M and N are invertible. It then follows that $A \overset{a.s}{\sim} C$.

Example 2.2

We illustrate part (i) of Theorem 2.1 above. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be such that A is hermitian and

$N = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be operators on a two dimensional space $\mathbb{C}^2$. Then $A \overset{a.s}{\sim} A$. That is

$$A^*A = N^{-1}(A^*A)N \dots\dots\dots\dots (i)$$

$$\text{and } A + A = N^{-1}(A^* + A)N \dots\dots (ii)$$

From (i) we have $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ i.e. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

From (ii) we have $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ i.e. $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$.

Hence $A \overset{a.s}{\sim} A$. $i.e.$ almost similarity is symmetric.

Proposition 2.3: *Let* $A, B \in B(H)$. *Then*

(i) If $A \overset{a.s}{\sim} 0$. *then* $A = O$

(ii) If $A \overset{a.s}{\sim} B$ *and* B *is isometric, then* A *is isometric.*

Proof: (i) $A \overset{a.s}{\sim} 0$ means that $A^*A = N^{-1}(0)N$ and $A^* + A = N^{-1}(0)N$, which implies that $= 0$.

(ii) $A \overset{a.s}{\sim} B$ means that $A^*A = N^{-1}(B^*B)N$ and $A^* + A = N^{-1}(B^* + B)N$. Since B is an isometry $B^*B = I$. So $B^*B = I$ which means that $A^*A = N^{-1}(I)N$. Thus A is isometric.

Proposition 2.4: *If* $A \in B(H)$ *and* $A \overset{a.s}{\sim} I$, *then* $A = I$.

Proof: Since $A \overset{a.s}{\sim} I$, there is an invertible operator N such that

$$I^*I = N^{-1}(A^*A)N \dots\dots\dots\dots (i)$$

$$\text{and } I^* + I = N^{-1}(A^* + A)N \dots\dots\dots\dots\dots (ii).$$

From (i) and (ii) above, we conclude that $A^*A = I$ and $A^* + A = 2I$. This implies that $A^*A + A^2 = 2A$. As $A^*A = I$, we get

$$A^2 - 2A + I = 0 \dots\dots\dots (*).$$

Next we show that the solution to $(*)$ is I.

Let $x \in H$, then $(A^2 - 2A + I)x = (A - I)(A - I)x = 0$. Put $(A - I)x = y$. Thus we get $(A - I)y = 0$ and hence $Ay = y$ and $Ax = x + y$. By iteration we get $x = x + ny$ for any natural number. Hence

$$n \| y \| = \| ny \| = \| A^n - x \| \leq \| A^n x \| + \| x \| = \| x \| + \| x \| = 2 \| x \|$$

so that $n \| y \| \leq 2 \| x \|$ for all natural numbers n.

Thus $\| y \| \leq \frac{2}{n} \| x \| \to 0$ as $n \to \infty$ and hence $= 0$, and consequently $y(A - I)x = 0 \ \forall x \in H$. This implies that $Ax = x \quad \forall x$ and hence $A = I$.

Theorem 2.5 [3]: *Let H be a Hilbert space, $A \in B(H)$ be a bounded linear operator and A^* the Hilbert space adjoint operator of A. Then A is compact if and only if A^*A is compact.*

Proposition 2.6: *If $A, B \in B(H)$ such that $A \overset{a.s}{\sim} B$, and if A is compact, then so is B.*
Proof: By assumption there exists an invertible operator N such that $B^*B = N^{-1}(A^*A)N$. Since A is compact $N^{-1}(A^*A)N$ is compact which implies that B^*B is compact. Thus by Theorem 2.5 above, B is compact.

Definition 2.7 [1]: *An operator $A \in B(H)$ is called a θ −operator if $A^* + A$ commutes with A^*A. The class of all θ −operators in $B(H)$ is denoted by θ i.e. $\theta = \{A \in B(H): [A^*A, \ A^* + A] = 0\}$.*

Proposition 2.8: *If $A, B \in B(H)$ such that $\theta \in B$ and $A \overset{a.s}{\sim} B$, then $\theta \in A$.*
Proof: By assumption there exists an invertible operator N such that $A^*A = N^{-1}(B^*B)N$ and $A^* + A = N^{-1}(B^* + B)N$. Thus we have

$$[N^{-1}(B^*B)N][N^{-1}(B^* + B)N] = A^*A(A^* + A) \ldots\ldots (1)$$

and $[N^{-1}(B^* + B)N][N^{-1}(B^*B)N] = (A^* + A)A^*A) \ldots\ldots (2)$

From (1) we get

$$N^{-1}B^*B(B^* + B)N = A^*A(A^* + A) \ldots\ldots\ldots\ldots\ldots (3)$$

and from (2) we get

$$N^{-1}(B^* + B)B^*B)N = (A^* + A)A^*A) \ldots\ldots\ldots\ldots\ldots (4).$$

Since $B \in \theta$, the left hand sides of (3) and (4) are equal, which implies that the right hand sides of (3) and (4) are equal. Thus $A \in \theta$.

Theorem 2.9 [1]: *An operator $T \in B(H)$ is hermitian if and only if $(T + T^*)^2 \geq 4T^*T$.*
Remark 2.10: In the proof of the next Proposition, we may assume the equality sign in Theorem 2.9 above i.e. $T \in B(H)$ is hermitian if and only if $(T + T^*)^2 = 4T^*T$ and prove the results as follows:
If T is hermitian, then $(T + T^*)^2 = (T + T)^2 = (2T)^2 = 4T^2$ and also

$$4T^*T = (T + T)^2 = (2T)^2 = 4T^2.$$

Now suppose that $(T + T^*)^2 = 4T^*T$ and let $T = A + IB$ be the Cartesian decomposition of T. Then

$$(T + T^*)^2 = (A + iB + A - iB) = (2A)^2 = 4A^2 \text{ and}$$
$$4T^*T = [(A + iB)(A - iB)] = 4[A^2 + B^2 + i(AB - BA)].$$

Thus we have $4A^2 = 4A^2 + 4B^2$ which implies that $A^2 = 0$. Since B is hermitian, $B = 0$, which implies that T is hermitian.

Proposition 2.11: *If $A, B \in B(H)$ such that $A \overset{a.s}{\sim} B$ and B is hermitian, then A is hermitian.*

Proof: Since $A \overset{a.s}{\sim} B$ there is an invertible operator N such that $A^*A = N^{-1}(B^*B)N$ which implies that

$$4A^*A = N^{-1}(4B^*B)N \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Also, $A \overset{a.s}{\sim} B \Rightarrow A^* + A = N^{-1}(B^* + B)N$
which implies that $[N^{-1}(B^* + B)N][N^{-1}(B^* + B)N] = (A + A^*)^2$. Thus

$$N^{-1}(B + B^*)^2N = (A + A^*)^2 \ldots\ldots\ldots\ldots\ldots (2).$$

Since B is hermitian, we have that $(B + B^*)^2 = (2B)^2 = 4B^2 = 4B^*B$ and substituting this in (2) we get

$$N^{-1}4B^*BN = (A + A^*)^2 \ldots\ldots\ldots\ldots\ldots (3).$$

From (1) and (3) we have $4A^*A = (A + A^*)^2$ which implies by the above remark that A is hermitian.

Definition 2.12 [2, Definition 1.2]: *An operator $T \in B(H)$ is said to be partially isometric in case T^*T is a projection. Equivalently, $TT^*T = T$ i.e. $(T^*T)^2 = T^*T$ and $(T^*T)^* = T^*T$.*

Proposition 2.13: *If $A, B \in B(H)$ such that $A \overset{a.s}{\sim} B$ and A is partially isometric then so is B.*
Proof: $A \overset{a.s}{\sim} B$ implies that there exists an invertible operator N such that $N^{-1}(B^*B)N = A^*A$. Since A is partially isometric, A^*A is a projection (i.e. $(A^*A)^2 = A^*A$), which implies that $[N^{-1}(B^*B)N][N^{-1}(B^*B)N] = N^{-1}(B^*B)N$. We thus have that $N^{-1}B^*BB^*BN = N^{-1}(B^*B)N$ which implies that $(B^*B)^2 = B^*B$. Thus B^*B is a projection, which implies that B is partially isometric.

Proposition 2.14 *: If $A, B \in B(H)$ such tha t $A \overset{a.s}{\sim} B$, and A is a projection then so is B.*
Proof: $A \overset{a.s}{\sim} B$ implies that there exists an invertible operator N such that

$$A^*A = N^{-1}(B^*B)N \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and $$A^* + A = N^{-1}(B^{*+}B)N \ldots\ldots\ldots\ldots\ldots (2).$$

Since A is a projection, it is hermitian i.e. $A^* = A$ and this implies (by Proposition 2.11) that B is hermitian. From (1), we get $A^2 = A = N^{-1}B^2N$ and from(2) we get $2A = N^{-1}2BN$ i.e. $A = N^{-1}BN$. This implies that $N^{-1}B^2N = N^{-1}BN$ which implies that B is a projection.

3. CHARACTRERIZAION OF UNITARY EQUIVALENCE OPERATORS IN TERMS OF ALMOTST SIMILARITY

Proposition 3.1: *If $A, B \in B(H)$ such that A and B are unitarily equivalent, then $A \overset{a.s}{\sim} B$.*
Proof: By assumption, there exists a unitary operator U such that $A = U^*BU$ which implies that $A^* = U^*B^*U$.
Thus $A^*A = U^*B^*UU^*BU = U^*B^*BU = U^{-1}B^*BU$, and

$$A^* + A = U^*B^*U + U^*BU = U^*(B^* + B)U = U^{-1}(B^* + B)U.$$

Corollary 3.2: *If $A, B \in B(H)$ where H is a finite dimensional Hilbert space such that A and B are quasi-similar, then $A \overset{a.s}{\sim} B$.*
Proof: Since $A, B \in B(H)$ are quasi-similar, there exists quasi-affinities $X \in B(H, K)$ and $Y \in B(K, H)$ such that $XA = BX$ and $BY = YA$. Assume that $X = Y$ is unitary. Then by definition
$X^*X = XX^* = I \Rightarrow X^* = X^{-1}$. But $A = X^{-1}BX$ which implies that $A^* = X^*B^*(X^{-1})^* = X^*B^*X$.

Now $A^*A = (X^*B^*X)(X^{-1}BX) = X^*B^*BX = X^{-1}B^*BX$ and

$A^* + A = (X^*B^*X) + (X^{-1}BX) = X^*(B^* + B)X = X^{-1}(B^* + B)X$. This implies that $A \overset{a.s}{\sim} B$.

This corollary gives a condition under which quasisimilarity implies almost similarity i.e. only if the quasi-affinities are unitary and are equal.

Proposition 3.3*: If $A, B \in B(H)$ such that $A \overset{a.s}{\sim} B$, and if A is hermitian, then A and B are unitarily equivalent.*
Proof: By assumption, there exists an invertible operator N such that $A^* + A = N^{-1}(B^{*+}B)N$. Since A is hermitian and $A \overset{a.s}{\sim} B$ by Proposition 2.8, B is hermitian. Thus we have $2A = N^{-1}2BN$ which implies that $A = N^{-1}BN$.This implies that A and B are similar ($i.e. A \sim B$) and since both operators are normal (both A and B are hermitian), they are unitarily equivalent.

Remark 3.4: The above Proposition gives a condition under which almost similarity of operators implies similarity.

Proposition 3.5 [2, Proposition 2.3]: *If $A \in B(H)$ is normal, then $A \overset{a.s}{\sim} A^*$.*
Proof: Since A is normal, then $A^*A = AA^*$. Thus $A^*A = AA^* = (A^*)^*A^* = I^{-1}(A^*)^*A^*I$.
Also $A^* + A = A + A^* \Rightarrow (A^*)^* + A^* = I^{-1}((A^*)^* + A^*)I$. Thus $A \overset{a.s}{\sim} A^*$.

Remark 3.6: The converse to Proposition 3.5 is not true in general, for consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By matrix computation, $A^*A = N^{-1}(AA^*)N$ and $A^* + A = N^{-1}(A + A^*)N$. That is $A \overset{a.s}{\sim} A^*$ although A is not normal.

Definition 3.7[1]: *If $A \in \theta$, then $4A^*A - (A^* + A)^2 \geq 0$. Define $B = A + A^* + i\sqrt{4A^*A - (A^* + A)^2}/2$. Then B is normal, $\sigma(A)$ is contained in the closed upper half plane, $B^*B = AA^*$ and $B^* + B = A + A^*$.*

In particular, $(\lambda I - A^*)(\lambda I - A) =, (\lambda I - B^*)(\lambda I - B) \ \ \forall\ \lambda$.

Proposition 3.8: *If* $A \in B(H)$ *then* $A \in \theta$ *if and only if* $A \overset{a.s}{\sim} B$ *for some normal operator* B.
Proof: Let $A \in \theta$, then $4A^*A - (A^* + A)^2 \geq 0$ and the operator
$B = A + A^* + i\sqrt{4A^*A - (A^* + A)^2}/2$ is normal with $(A^*A) = (B^*B)$ and
$(A^* + A) = (B^* + B)$ (by Definition 3.7). Thus $A^*A = I^{-1}B^*BI$ and $A^* + A = I^{-1}(B^* + B)I$. Hence $A \overset{a.s}{\sim} B$.
Conversely, let $A \overset{a.s}{\sim} B$ for some normal operator B. Then there exists an invertible operator N such that $A^*A = N^{-1}(B^*B)N$ and $A^* + A = N^{-1}(B^* + B)N$

$$A^*A(A^* + A) = N^{-1}B^*B(B^* + B)N \ldots\ldots(i)$$

$$(A^* + A)A^*A = N^{-1}(B^* + B)B^*BN \ldots\ldots(ii)$$

Since B is normal, $B \in \theta$. Thus the right hand sides (i) and (ii) are equal which implies that

$(A^* + A)A^*A = A^*A(A^* + A)$. Thus $A \in \theta$.

Proposition 3.9: *If* $T \in B(H)$ *is invertible and* $T \overset{a.s}{\sim} U$ *for some unitary operator* $U \in B(H)$ *then* T *is unitary.*
Proof: Since $T \overset{a.s}{\sim} U$, there exists an invertible operator N such that $T^*T = N^{-1}(U^*U)N = I$. This implies that $T^{*-1}T^{-1}TT^{-1} = T^{*-1}T^{-1}$. Since $T^{*-1}T^{-1}TT^{-1} = I, T^{*-1}T^{-1} = (TT^*)^{-1} = I$ which implies that $TT^* = I$. Thus $T^*T = TT^* = I$ i.e. T is unitary.

4. CHARACTERIZATION OF ISOMETRIC OPERATORS

Proposition 4.1: *An operator* $A \in B(H)$ *is isometric if and only if* $A \overset{a.s}{\sim} U$ *for some unitary operator* U.
Proof: Let A be isometric, then $A \in \theta$ .Thus by Proposition 3.8, there is a normal operator N with $A \overset{a.s}{\sim} N$. Since if $A \overset{a.s}{\sim} N, N$ is isometric by Proposition 2.8 (ii). Thus N is unitary.
Now suppose that $A \overset{a.s}{\sim} U$ for some unitary operator U then there exists an invertible operator with $N^{-1}(A^*A)N = U^*U = I$. This implies that $A^*A = N^{-1}N = I$. Thus A is isometric.
Note: Let $T \in B(H)$ be unitary, then $T^*T = TT^* = I$. Thus, $T^*T = I^{-1}(T^*)^*TI$. Also $T^* + T = T + T^*$ implies that
$T^* + T = I^{-1}((T^*)^* + T)I$. Hence $T \overset{a.s}{\sim} T^* = T^{-1}$. However, if $T \in B(H)$ and $T \overset{a.s}{\sim} T^{-1}$ then T is not necessarily unitary as illustrated in the example below:

Example 4.2: Consider the operator $T = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}$ on the two dimensional space $\mathbb{C}^2$. Then
$T^2 = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ (is an involution) which implies that $T^{-1} = T$. Thus $T \overset{a.s}{\sim} T^{-1}$. However $\|T\| > 1$ which means that T is not unitary.

Proposition 4.3 [2, Proposition 2.7]: *If* $A, B \in B(H)$ *such that* $A \overset{a.s}{\sim} B$, *then* $(A + \lambda I) \overset{a.s}{\sim} (B + \lambda I)$ *for all real* λ.
Proof: By assumption, there exists an invertible operator N such that

$$A^* + A = N^{-1}(B^* + B)N \ldots\ldots\ldots\ldots(i)$$

$$A^*A = N^{-1}(B^*B)N \ldots\ldots\ldots\ldots(ii).$$

From (i) we have $A^* + A = N^{-1}B^*N + N^{-1}BN$ which implies that
$A^* + A + 2\lambda = N^{-1}B^*N + N^{-1}BN + 2\lambda$. Thus we have
$(A^* + \lambda I) + (A + \lambda I) = N^{-1}(B^* + \lambda I)N + N^{-1}(B + \lambda I)N = N^{-1}[(B + \lambda I)^* + (B + \lambda I)]N$,
which implies that
$(A^* + \lambda I) + (A + \lambda I) = N^{-1}(B^* + \lambda I)N + N^{-1}(B + \lambda I)N = N^{-1}[(B^* + \lambda I) + (B + \lambda I)]N \ldots\ldots(iii)$.
From (iii)

$\lambda A^* + A\lambda + \lambda^2 = N^{-1}\lambda B^* N + N^{-1}\lambda BN + N^{-1}\lambda^2 BN \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(iv)$.

Adding (ii) and (iv) we get

$A^*A + \lambda A^* + A\lambda + \lambda^2 = N^{-1}\lambda B^* N + N^{-1}\lambda BN + N^{-1}\lambda^2 BN + N^{-1}(B^*B)N$ which implies that

$(A^* + \lambda I)(A + \lambda I) = N^{-1}[(B^* + \lambda I)(B + \lambda I)]N$. Thus

$$(A + \lambda I)^*(A + \lambda I) = N^{-1}[(B + \lambda I)^*(B + \lambda I)]N \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(v).$$

From(iii) and (v) we conclude that $(A + \lambda I) \overset{a.s}{\sim} (B + \lambda I)$.

Remark 4.4: For a certain class of operators for example hermitian or projection operators, results proved in proposition 4.3 above shows that if $A \overset{a.s}{\sim} B$, then A and B have equal spectrum as illustrated in the corollary that follows:

Corollary 4.5: *If $A, B \in B(H)$ are projection operators such that $A \overset{a.s}{\sim} B$ and $(A + \lambda I) \overset{a.s}{\sim} (B + \lambda I)$ for all real λ, then $\sigma_p(A) = \sigma_p(B)$.*

Proof: Since $A \overset{a.s}{\sim} B$, then there exists an invertible operator N such that

$$A^* + A = N^{-1}(B^* + B)N \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

$$A^*A = N^{-1}(B^*B)N \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots..(ii).$$

Since $A^* = A, B^* = B$ then (i) becomes $2A = N^{-1}2BN$ i.e. $A = N^{-1}BN$ i.e.

$NA = BN$ i.e. $\sigma_p(A) = \sigma_p(B)$. Similarly since $A^* = A = A^2$ and $B^* = B = B^2$ (ii) becomes $A^2 = N^{-1}B^2N$ i.e.

$A = N^{-1}BN$ and so $\sigma_p(A) = \sigma_p(B)$.

5. ALMOST SIMILARITY AND COMPLETELY NON UNITARY OPERATORS

Definition 5.1[5, Chap. 6 Sec. 6.3]: *An operator $T \in B(H)$ is said to be quasitriangular (or quasidiagonal) if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of projections of finite rank such that $P_n \to 1$ weakly and $\|P_nTP_n - TP_n\| \to 0$ as $n \to \infty$.*

We write $QT(H)$ for the set of all quasidiagonal operators in $B(H)$.

The class of biquasitrangular operators, denoted by (BQT) is defined as

$(BQT) = \{T \in B(H)\text{: such that } T \text{ and } T^* \text{ are quasitriangular}\}$

Compact operators are quasitriangular. Indeed, if P_n is a projection such that $P_n \to 1$ weakly and K is compact then $\|P_nKP_n - K\| \to 0$. So

$\|P_nKP_n - KP_n\| = |P_n(KP_n)P_n - (KP_n)\| = \|P_nK'P_n - K'\| \to 0.$

A trivial example of a quasitriangular operator is an upper triangular operator: Indeed if P_n denotes the orthogonal projection onto $V\{e_1, e_2, \ldots, e_n\}$ then $P_nH \subset P_nH$, so $P_nTP_n = TP_n$.

We further illustrate quasitriangularity as follows: An operator $Q = (q_{ij})$ is quasitrangulra if $h_{ij} = 0$ whenever $i \geq j + 1$. That is, Q is a Hessenberg matrix if all entries below the subdiagonal of Q are zero.

Corollary 5.2 [6, Corollary 2.3]: *Let $A \in B(H)$ and suppose that $A \overset{a.s}{\sim} S_+$ where S_+denotes the unilateral shift of finite multiplicity. Then A is a completely non-unitary contraction such that $Re(A) \sim Q$ where Q is a quasidiagonal operator and $Re(A)$ denotes the real part of A.*

Proof: Since $A \overset{a.s}{\sim} S_+$, $A^*A = N^{-1}(S_+^*S_+)N$ and $A^* + A = N^{-1}(S_+^* + S_+)N$ where N is an invertible operator. Since $(S_+^*S_+) = I$, then by Proposition 2.3, A is an isometry (indeed a c.n.u. isometry). A simple matrix computation shows that $S_+^* + S_+$ is a quasidiagonal operator Q. Hence $Re(A) \sim Q$.

Remark 5.3: Corollary 5.2 above says indirectly that quasitriangularity is not preserved under similarity. (QT) and (BQT) classes are invariant under similarity.

Lemma 5.4 [5, Lemma 5.4]: An operator is a unilateral shift if and only if it is a completely non-unitary isometry.

Theorem 5.5 [5, Corollary 5.6]: ***(Von-Neuman-Wold Decomposition for Isometries).***
If T is an isometry on a Hilbert space H, then $Ker(I - A^*)$ is a reducing subspace for T. Moreover, the decomposition $T = S_+ \oplus U$ on $H = Ker(I - A^*)^\perp \oplus Ker(I - A^*)$ is such that $S_+ := T|_{Ker(I-A^*)}$ is a unilateral shift and $U := T|_{Ker(I-A^*)}$ is unitary.
Proof: If T is an isometry ($i.e. A = I$), then T is a contraction for which $\text{Ker}(I - A) = H$, so that $\text{Ker}(I - A) \cap \text{Ker}(I - A^*)$ applying the Nagy-Foias-Langer decomposition for contractions with $\mathfrak{U} = \text{Ker}(I - A^*)$; and note that $T|_{\mathfrak{U}^\perp}$ is a c.n.u. isometry on $\mathfrak{U}^\perp$, which by the Lemma 5.4, means that it is a unilateral shift.

Proposition 5.6: *Let $A \in B(H)$ be such that $A \overset{a.s}{\sim} T$, where T is an isometry. Then the direct summands of A are isometric.*
Proof: Since T is an isometry, by Theorem 5.5 above $T = S_+ \oplus U$ where U is unitary and S_+ is the unilateral shift. Since $A \overset{a.s}{\sim} T$ then there exists an operator N such that
$A^*A = N^{-1}[(S_+ \oplus U)^*(S_+ \oplus U)]N = N^{-1}(S_+^*S_+ \oplus U^*U)N = N^{-1}(I \oplus I)N$
Letting $A = A_1 \oplus A_2$, then $A^*A = (A_1^*A_1 \oplus A_2^*A_2)$. This shows that $A_i^*A_i \sim I$ $i = 1,2$. This means that there exists an operator N such that $A_i^*A_i = N^{-1}IN = I$. Thus $A_i^*A_i = I$. This proves that the direct summands of A are isometric.

Remark 5.7: The above proposition does not mean that $A_1 \sim U$ and $A_2 \sim S_+$. If the relation of almost similarity is replaced with unitary equivalence in the above proposition, then the direct sums and summands are preserved.

Theorem 5.8 [4, Theorem 5.1] ***:(Nagy-Foias-Langer Decomposition Theorem)***
Let T be a contraction on a Hilbert space H and set $\mathfrak{U} = Ker(I - A) \cap Ker(I - A_*)$. $\mathfrak{U}$ is a reducing subspace for T. Moreover the decomposition $T = C \oplus U$ on $H = \mathfrak{U}^\perp \oplus \mathfrak{U}$ is such that $C := T|_{\mathfrak{U}^\perp}$ is a c.n.u. contraction and $C := T|_{\mathfrak{U}}$ is unitary.
Proof: (See [4]).

Proposition 5.9: *If $A, B \in B(H)$ are contractions such that $A \overset{a.s}{\sim} B$ and B is c.n.u., then A is c.n.u.*
Proof: By Theorem 5.8 above, $B = U \oplus C$ on $H = H_1 \oplus H_2$ where $U = B|_{H_1}$ is the unitary part of B and $U = B|_{H_2}$ is the c.n.u. part of B. Since B is c.n.u, the unitary part is missing on $H_1 = \{0\}$. Without loss of generality, we suppose that $B = C$. Then $A^*A = N^{-1}(B^*B)N = N^{-1}(C^*C)N$.This shows that $A^*A \sim C^*C$. Now suppose that $A = A_1 \oplus A_2$ where A_1 is the unitary part and A_2 c.n.u. part of A. Then $(A_1^*A_1 \oplus A_2^*A_2) \sim C^*C$. That is $(I \oplus A_2^*A_2) \sim C^*C$. This holds true if and only if the direct summand is missing. That is $A = A_2$ and so A is completely non unitary.

6. REFERENCES

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