ON GENERALIZED VECTOR VARIATIONAL-LIKE INEQUALITIES AND VECTOR OPTIMIZATION PROBLEMS

Dapeng Gao[1] & Shiqiang Feng[2]
School of Mathematics and Information, China West Normal University, Nanchong, Sichuan 637002, China
E-mail: [1]gaodapeng126@126.com, [2]cwnufsq@163.com
Foundation item: Supported by the National Natural Science Foundation of China (60804065) and the Foundation of China West Normal University(11A029).

ABSTRACT

In this paper, some solution relationships between nonsmooth vector optimization problems and generalized vector variational-like inequalities are established under pseudoinvexity or invariant pseudomonotonicity. A perturbed generalized weak Stampacchia vector variational-like inequality problem and its relation with generalized weak Minty vector variational-like inequality problem are also presented.

keywords: *Nonsmooth vector optimization. Generalized vector variational-like inequality. Pseudoinvexity. Invariant pseudomonotonicity. Clarke's generalized subdifferential.*
Mathematics Subject Classification(2000): *58E35. 49J52. 90C29.*

1. INTRODUCTION

The concept of vector variational inequality(in short, VVI) was first introduced and studied by Giannessi[1] in finite-dimensional spaces. Since then, VVIs have received much attention by many authors due to its potential application in vector optimization problem(in short, VOP), vector traffic equilibrium problem, economics and management science. A large number of results have appeared in the literature. For the details, we can refer to [1-9,17,18,20] and the references therein.

In recent years, VVIs have been used as a tool for studying VOPs(see[3-7,9,17] and the references therein). In 1998, Giannessi[2] first used, so called, Minty type vector variational inequality(in short, MVVI) to establish the necessary and sufficient conditions for a point to be an efficient solution of a VOP for differentiable and convex function. Since then, several researchers have studied VOP by using different kinds of MVVI under different assumptions. Yang et al. [3] generalized the results of [2] to pseudoconvexity or pseudomononicity. On the other hand, the VVI has been extended to the vector variational-like inequality(in short, VVLI). Recently, Ruiz-Garzon et al.[4] obtained some relations between Stampacchia VVLI and the VOP. Yang and Yang[5] gave some relations between a solution of a Minty VVLI and an efficient solution or a weakly efficient solution to the VOP under the assumptions of pseudoinvexity or invariant pseudomonotonicity. Al-Homidan and Ansari[6] studied the relationship between the generalized Minty VVLI, generalized Stampacchia VVLI and VOPs involving nonsmooth invex functions. They also considered the generalized weak Minty VVLI and the generalized weak Stampacchia VVLI and obtained some relations between the solution of these problems and a weakly efficient solution of the nondifferentiable VOP.

Very recently, Ansari et al.[7] considered generalized Minty VVLIPs, generalized Stampacchia VVLIPs and nonsmooth VOPs under nonsmooth pseudoinvexity assumption. They also considered the weak formilations of generalized Minty VVLIPs and generalized Stampacchia VVLIPs in a very general setting and established the existence for their solutions. Long et al.[8] established a relation between a solution of the generalized VVLI and the nonsmooth VOP under the assumptions of pseudoinvexity or invariant pseudomonotonixity. Motivated by the work reported in [3-9], the purpose of this article is to discuss solution relationships among generalized Minty VVLIPs, generalized Stampacchia VVLIPs and nonsmooth VOPs under pseudoinvexity or invariant pseudomonotonicity assumption. Furthermore, the relations among the weak formulations of generalized Minty VVLIPs, generalized Stampacchia VVLIPs and nonsmooth VOP are established.

The rest of the paper is organized as follows. In Sect.2, we recall some basic definitions and preliminary results. In Sect.3, we first introduce four classes of generalized VVLIPs and investigate some relations between a solution of the generalized Minty or Stampacchia VVLI and an efficient solution of the nonsmooth VOP under the assumptions of pseudoinvexity or invariant pseudomonotonicity and the weak version of these results are presented in Sect.4. In Sect.5, the relation between the perturbed form of generalized weak Stampacchia VVLIP and generalized weak Minty VVLIP are obtained. In Sect.6, solutions for the generalized weak VOP are provided.

2. PRELIMINARIES

In this section, we recall some definitions and some results needed in the following sections. Throughout this article, unless otherwise specified, we assume that K is a nonempty subset of R^n and $\eta: K\times K\to R^n$ is a given mapping, $J=\{1,2,\cdots,p\}$. The interior of K is denoted by int K.

Let $f=(f_1,f_2,\cdots,f_p):K\to R^p$ be a vector-valued mapping. In this article, we consider the following vector optimization problem:

$$(VOP)\qquad Minimize \quad f(x)=(f_1(x),f_2(x),\cdots,f_p(x)) \quad s.t. \quad x\in K.$$

Definition 2.1 A point $x\in K$ is said to be an efficient solution(respectively, a weakly efficient solution) of VOP if

$$f(y)-f(x)=(f_1(y)-f_1(x),f_2(y)-f_2(x),\cdots,f_p(y)-f_p(x))\notin -R_+^p\setminus\{0\}, for\ all \quad y\in K.$$

(respectively, $f(y)-f(x)=(f_1(y)-f_1(x),f_2(y)-f_2(x),\cdots,f_p(y)-f_p(x))\notin -intR_+^p$ for all $y\in K$), where R_+^p is the nonnegative orthant of R^p and 0 is the zero vector of R^p.

Definition 2.2[10] Let $f:K\to R$ be locally Lipschitz at a given point $x\in K$. The Clarke's generalized directional derivative of f at $x\in K$ in the direction of a vector $v\in K$, denoted by $f^0(x;v)$, is defined by

$$f^0(x;v)=\lim_{y\to x}\sup_{t\downarrow 0}\frac{f(y+tv)-f(y)}{t}.$$

Definition 2.3[10] Let $f:K\to R$ be locally Lipschitz at a given point $x\in K$. The Clarke's generalized subdifferential of f at $x\in K$, denoted by $\partial f(x)$, is defined by

$$\partial f(x)=\{\xi\in R^n \mid f^0(x;v)\geq\langle\xi,v\rangle, \quad for\ all \quad v\in R^n\}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in R^n.

The definition of Clarke's generalized derivative can be extended to a locally Lipschitz vector-valued function $f:K\to R^n$. In fact the Clarke's generalized derivative of f at x in direction v is $f^0(x;v)=f_1^0(x;v)\times f_2^0(x;v)\times\cdots\times f_n^0(x;v)$, where $f_i, i=1,2,\cdots,n$ is the components of f. The Clarke's generalized subdifferential of f at $x\in K$ is the set $\partial f(x)=\partial f_1(x)\times\partial f_2(x)\times\cdots\times\partial f_n(x)$.

Definition 2.4[11] The set K is said to be invex with respect to η iff, for any $x,y\in K$ and $\lambda\in[0,1]$, we have

$$y+\lambda\eta(x,y)\in K.$$

Remark 2.1 If $\eta(x,y)=x-y$ for any $x,y\in K$, then Definition 2.4 reduces to the convexity of the set K.

Definition 2.5[4] The vector-valued function $\eta:K\times K\to R^n$ is said to be skew iff, for any $x,y\in K$,

$$\eta(x,y)+\eta(y,x)=0.$$

Definition 2.6 Let K be invex with respect to η and $f:K\to R$. f is said to be

(i)prequasiinvex [12] with respect to η on K iff, for any $x,y\in K$ and $\lambda\in[0,1]$,

$$f(y+\lambda\eta(x,y))\leq max\{f(x),f(y)\};$$

(ii)semi-strictly prequasiinvex [13] with respect to η on K iff, for any $x,y\in K$ and $\lambda\in(0,1)$ with

$f(x) \neq f(y)$,

$$f(y+\lambda\eta(x,y)) < max\{f(x), f(y)\}.$$

Definition 2.7[14] Let K be invex with respect to η and $f:K \to R$ be locally Lipschitz on K. f is said to be

(i)pseudoinvex with respect to η on K iff, for any $x, y \in K$ and any $\xi \in \partial f(y)$,

$$\langle \xi, \eta(x,y) \rangle \geq 0 \Rightarrow f(x) \geq f(y);$$

(ii)quasiinvex with respect to η on K iff, for any $x, y \in K$ and $\xi \in \partial f(y)$,

$$f(x) \leq f(y) \Rightarrow \langle \xi, \eta(x,y) \rangle \leq 0.$$

Definition 2.8[14] Let K be invex with respect to η and $f:K \to R$ be locally Lipscistz on K. ∂f is said to be invariant pseudomonotone with respect to η on K iff, for any $x, y \in K$ and any $\xi \in \partial f(x), \zeta \in \partial f(y)$,

$$\langle \xi, \eta(y,x) \rangle \geq 0 \Rightarrow \langle \zeta, \eta(x,y) \rangle \leq 0.$$

Mohan and Neogy[12] introduced Condition C defined as follows.

Condition C. Let $K \subseteq R^n$ be an invex set with respect to $\eta: K \times K \to R^n$. Then, η is said to satisfy Condition C iff,for any $x, y \in K$ and $\lambda \in [0,1]$,

$$\eta(y, y+\lambda\eta(x,y)) = -\lambda\eta(x,y),$$
$$\eta(x, y+\lambda\eta(x,y)) = (1-\lambda)\eta(x,y).$$

Remark 2.2 (i)The function η in example 2.1 [15] satisfies Condition C but is not a skew function. The example of the mapping η which is a skew function but does not satisfy Condition C is given in [8].

(ii)Yang et al.[19] have shown that if $\eta: K \times K \to R^n$ satisfies Condition C,then $\eta(y+\lambda\eta(x,y), y) = \lambda\eta(x,y)$.

Gang and Liu [16] considered the following Condition C^*.

Condition C * Let $K \subseteq R^n$ be an invex set with respect to $\eta: K \times K \to R^n$. We say that the mapping $\eta: K \times K \to R^n$ satisfies the Condition C * iff, for any $x, y \in K$ and for all $\lambda \in [0,1]$

$$\eta(y, y+\lambda\eta(x,y)) = -\alpha(\lambda)\eta(x,y),$$

and

$$\eta(x, y+\lambda\eta(x,y)) = \beta(\lambda)\eta(x,y),$$

where $\alpha(\lambda) > 0, \beta(\lambda) > 0$ for all $\lambda \in (0,1)$.

Remark 2.3 We note that if η satisfies the Condition C, then it satisfies the Condition C *. However, the example in[16] shows that the converse is not true in general.

Condition A[13] Let K be invex with respect to η, and let $f:K \to R$. f is said to satisfy Condition A iff, for any $x, y \in K$,

$$f(y+\eta(x,y)) \leq f(x).$$

The following lemmas will be used in the sequel.

Lemma 2.1[9] Let K be invex with respect to η. Let $f:K \to R$ be locally Lipschitz on K. If f is pseudoinvex with respect to η on K, then f is semi-strictly prequasiinvex with respect to the same η on K.

Lemma 2.2[5] Let K be invex with respect to η such that η satisfies Condition C. If $f: K \to R$ is lower semicontinuous and semi-strictly prequasiinvex with respect to η on K, then f is prequasiinvex with respect to the same η on K.

Lemma 2.3[9] Let K be invex with respect to η such that η satisfies Condition C, and $f: K \to R$ be locally Lipschitz on K.
(i)If f is quasiinvex with respect to η on K, then f is prequasiinvex with respect to η on K.
(ii)Conversely, if f is prequasiinvex with respect to η on K and η is continuous with respect to the second argument, then f is quasiinvex with respect to the same η on K.

Remark 2.4 When $f: K \to R$ is pseudoinvex with respect to η, then by Lemma 2.1, f is semi-strictly prequasiinvex with respect to the same η, and hence, Lemma 2.2 implies that f is prequasiinvex with respect to η if it is lower semicontinuous.

Lemma 2.5[8] Let K be invex with respect to η such that η satisfies Condition C and is continuous with respect to the second argument, and let $f: K \to R$ be locally Lipschitz on K. If f is pseudoinvex with respect to η on K, then ∂f is invariant pseudomonotone with respect to the same η on K.

Theorem 2.1(Lebourg Mean Value Theorem)[10] Let x and y be points in $K \subseteq R^n$ and suppose that $f: K \to R$ is locally Lipschitz on an open set containing the line segment $[x, y]$. Then there exists a point $u \in (x, y)$ such that

$$f(x) - f(y) \in \langle \partial f(u), x - y \rangle,$$

where (x, y) denotes the line segment joining x and y excluding the end points x and y.

3. GENERALIZED VVLIs

Let K be a nonempty subset of R^n and let $\eta: K \times K \to R^n$ be a given mapping. Let $f = (f_1, f_2, \cdots, f_p): K \to R^n$ be a vector-valued mapping. In this section, we consider the following four types of generalized VVLI problems:

(I)type 1 generalized Minty vector variational-like inequality problem(GMVVLIP)$_1$:find $x \in K$ such that, for any $y \in K$ and any $\xi_i \in \partial f_i(y), i = 1,2,\cdots, p$,

$$(\langle \xi_1, \eta(y,x)\rangle, \cdots, \langle \xi_p, \eta(y,x)\rangle) \notin -R_+^p \setminus \{0\}.$$

(II)type 2 generalized Minty vector variational-like inequality problem(GMVVLIP)$_2$:find $x \in K$ such that, for any $y \in K$ and any $\xi_i \in \partial f_i(y), i = 1,2,\cdots, p$,

$$(\langle \xi_1, \eta(x,y)\rangle, \cdots, \langle \xi_p, \eta(x,y)\rangle) \notin R_+^p \setminus \{0\}.$$

(III)type 1 generalized Stampacchia vector variational-like inequality problem(GSVVLIP)$_1$:find $x \in K$ such that, for any $y \in K$, there exists $\zeta_i \in \partial f_i(x), i = 1,2,\cdots, p$,

$$(\langle \zeta_1, \eta(y,x)\rangle, \cdots, \langle \zeta_p, \eta(y,x)\rangle) \notin -R_+^p \setminus \{0\}.$$

(IV)type 2 generalized Stampacchia vector variational-like inequality problem(GSVVLIP)$_2$:find $x \in K$ such that, for any $y \in K$, there exists $\zeta_i \in \partial f_i(y), i = 1,2,\cdots, p$,

$$(\langle \zeta_1, \eta(y,x)\rangle, \cdots, \langle \zeta_p, \eta(y,x)\rangle) \notin -R_+^p \setminus \{0\}.$$

It is obviously that every solution of (GMVVLIP)$_1$, ia a solution of (GSVVLIP)$_2$.

It is worth noting that (GMVVLIP)$_1$ and (GSVVLIP)$_1$ are introduced and studied by Al-Homidan and Ansari[6] under the condition that each f_i is invex and the skewness of η. Long et al.[8] generalized and improved the conclusion of Al-Homidan and Ansari[6] since the invexity of f_i has been weakened by pseudoinvexity of f_i and the condition η is skew has been removed. Ansari et al.[7] established that every efficient solution of (VOP) is a solution of (GMVVLIP)$_1$ for pseudoinvex functions while it is proved in [6] for invex funtions with some extra conditions.

Now, we investigate the relation between a solution of generalized vector variational-like inequalities and an efficient solution to the nonsmooth VOP under suitable conditions.

Theorem 3.1 Let $K \subseteq R^n$ be a nonempty invex set with respect to $\eta : K \times K \to R^n$ such that η is continuous with respect to the second argument and satisfies Condition C. For each $i \in J$, let f_i be locally Lipschitz pseudoinvex function with respect to the same η on K.

(a)If $x_0 \in K$ is a solution of (GMVVLIP)$_1$, then it is an efficient solution of (VOP).

(b)If $x_0 \in K$ is an efficient solution of (VOP), then it is a solution of (GMVVLIP)$_2$. In addition, if η is skew, then x_0 is a solution of (GSVVLIP)$_2$.

Proof. (a) Let $x_0 \in K$ be a solution of (GMVVLIP)$_1$. Suppose by contradiction that x_0 is not an efficient solution of VOP. Then there exists $y \in K$ such that

$$f(x_0) - f(y) = (f_1(x_0) - f_1(y), \cdots, f_p(x_0) - f_p(y)) \in R_+^p \setminus \{0\}. \tag{1}$$

Let $y(\lambda) = x_0 + \lambda\eta(y, x_0)$, for any $\lambda \in [0,1]$. Since K is invex, $y(\lambda) \in K$ for any $\lambda \in [0,1]$.

Since each f_i is pseudoinvex with respect to η on K, it follows from Lemma 2.1 and Lemma 2.2 and Remark 2.4 that each f_i is both prequasiinvex and semi-strictly prequasiinvex with to the same η. Then by using prequasiinvexity, semi-strictly prequasiinvexity and (3.1), we have

$$f(x_0) - f(y(\lambda)) \in R_+^p \setminus \{0\}, \quad \textit{for any} \quad \lambda \in (0,1)$$

that is

$$f(y(0)) - f(y(\lambda)) \in R_+^p \setminus \{0\}, \quad \textit{for any} \quad \lambda \in (0,1) \tag{2}$$

By Mean Valve Theorem 2.1, there exists $\lambda_i \in (0,1)$ and $\xi_i \in \partial f_i(y(\lambda_i))$ such that

$$f_i(y(0)) - f_i(y(\lambda)) = \langle \xi_i, -\lambda\eta(y, x_0)\rangle,$$

for all $i \in J$.

By using (3.2), we obtain

$$\langle \xi_i, \eta(y, x_0)\rangle \le 0, \tag{3}$$

for all $i \in J$, and one of which becomes strict inequality.

From Condition C, we have

$$\eta(y(\lambda_i), x_0) = \eta(x_0 + \lambda_i\eta(y, x_0), x_0) = \lambda_i\eta(y, x_0),$$

for all $i \in J$, which together with (3.3) yields

$$\langle \xi_i, \eta(y(\lambda_i), x_0)\rangle \le 0, \tag{4}$$

for all $i \in J$, where strict inequality holds for some $i \in J$. Without loss of generality, we can assume that the strict inequality holds for $i = 2$.

Suppose $\lambda_1, \lambda_2, \cdots, \lambda_p$ that are all equal. Then, it follows from (3.4) that

$$(\langle \xi_1, \eta(y(\lambda_1), x_0)\rangle, \cdots, \langle \xi_p, \eta(y(\lambda_p), x_0)\rangle) \in -R_+^p \setminus \{0\},$$

which contradicts the fact that x_0 is a solution of (GMVVLIP)$_1$.

Consider the case when $\lambda_1, \lambda_2, \cdots, \lambda_p$ are all not equal. Let $\lambda_1 \neq \lambda_2$. From Condition C, we have

$$\eta(y(\lambda_1), y(\lambda_2)) = \frac{\lambda_1 - \lambda_2}{\lambda_1}\eta(y(\lambda_1), x_0) = \frac{\lambda_1 - \lambda_2}{\lambda_2}\eta(y(\lambda_2), x_0), \tag{5}$$

$$\eta(y(\lambda_2), y(\lambda_1)) = \frac{\lambda_2 - \lambda_1}{\lambda_1}\eta(y(\lambda_1), x_0) = \frac{\lambda_2 - \lambda_1}{\lambda_2}\eta(y(\lambda_2), x_0). \tag{6}$$

If $\lambda_1 > \lambda_2$, then from (3.5) and (3.6), we know that there exists $\xi_1 \in \partial f_1(y(\lambda_1))$ such that

$$\langle \xi_1, \eta(y(\lambda_2), y(\lambda_1))\rangle \geq 0.$$

since ∂f_i is invariant pseudomonotone for each $i \in J$ according to Lemma 2.5, we know that for all $\zeta_1 \in \partial f_1(y(\lambda_2))$,

$$\langle \zeta_1, \eta(y(\lambda_1), y(\lambda_2))\rangle \leq 0.$$

The fact together with (3.5) yields

$$\langle \zeta_1, \eta(y(\lambda_2), x_0)\rangle \leq 0, \quad \textit{for all} \quad \zeta_1 \in \partial f_1(y(\lambda_2)).$$

If $\lambda_1 < \lambda_2$, then from (3.4) and (3.5), we know that there exists $\xi_2 \in \partial f_2(y(\lambda_2))$ such that

$$\langle \xi_2, \eta(y(\lambda_1)), y(\lambda_2))\rangle > 0. \tag{7}$$

By the invariant pseudomonotone of ∂f_2 and (3.7), we know that for all $\zeta_2 \in \partial f_2(y(\lambda_1))$,

$$\langle \zeta_2, \eta(y(\lambda_2), y(\lambda_1))\rangle < 0.$$

It follows from (3.7) that for all $\zeta_2 \in \partial f_2(y(\lambda_1))$,

$$\langle \zeta_2, \eta(y(\lambda_1), x_0)\rangle < 0.$$

Therefore, for the case $\lambda_1 \neq \lambda_2$, let $\bar{\lambda} = min\{\lambda_1, \lambda_2\}$. There exists $\bar{\zeta}_i \in \partial f_i(y(\bar{\lambda}))$ such that

$$\langle \bar{\zeta}_i, \eta(y(\bar{\lambda}), x_0)\rangle \leq 0, i = 1,2,$$

where strict inequality holds for $i = 2$.

By continuing this process, we can find $\lambda^* \in (0,1)$ and $\zeta_i^* \in \partial f_i(y(\lambda^*))$ such that $\lambda^* = min\{\lambda_1, \lambda_2, \cdots, \lambda_p\}$ and $\langle \zeta_i^*, \eta(y(\lambda^*), x_0)\rangle \leq 0, i = 1,2,\cdots, p$, where strict inequality holds for some $i \in J$. This contradicts the fact that x_0 is a solution to the (GMVVLIP)$_1$. This completes the proof.

(b) With some modifications in the proof of Theorem 6 of [7] and Theorem 3.2 of [8], one can obtain the results. Hence the proof is omitted.

Remark 3.1 (i)Theorem 3.1 generalized and improved Theorem 3.1 of Al-Homidan and Ansari [6] since the invexity of f_i has been weakened by pseudoinvexity of f_i and the condition $f_i(i \in J)$ satisfies Condition A has been removed from Al-Homidan and Ansari [6] and Long et al.[8].

(ii)In the proof of Theorem 3.1, we used simple mean value theorem for Clarke's generalized subdifferentials which is different from the proof of Theorem 3.1 in [6] and Theorem 3.1 in [8].

(iii)The condition that η is skew in Ansari et al.[7] has been cancelled the proof of part (a).

(iv)It is clear that Theorem 3.1 also generalized and extended Proposition 1 of Giannessi [18], Theorem 2.1 of Lee

[17], Theorem 3.1 of Yang et al.[3], Theorem 3.1(i) of Yang and Yang [5].

Theorem 3.2 Let K be a nonempty invex set with respect to η. For each $i \in J$, let ∂f_i be invariant pseudomonotone with respect to η on K.

(a)If η is skew, then $x_0 \in K$ is a solution of (GSVVLIP)$_1$ implies that x_0 is a solution of (GMVVLIP)$_1$.

(b)If $x_0 \in K$ is a solution of (GSVVLIP)$_1$, then x_0 is a solution to the (GMVVLIP)$_2$.

Proof. (a) Let x_0 be a solution of (GSVVLIP)$_1$. Suppose to the contrary that x_0 is not a solution of (GMVVLIP)$_1$. Then, there exist $y \in K, \xi_i \in \partial f_i(y)$ such that

$$(\langle \xi_1, \eta(y, x_0)\rangle, \cdots, \langle \xi_p, \eta(y, x_0)\rangle) \in -R_+^p \setminus \{0\}.$$

Since η is skew, we have $\langle \xi_i, \eta(x_0, y)\rangle \ge 0, i = 1,2,\cdots, p$, where strict inequality holds for some $i_0 \in J$. By the invariant pseudomonotonicity of each $f_i, i = 1,2,\cdots, p$,

$$\langle \zeta_i, \eta(y, x_0)\rangle \le 0, i = 1,2,\cdots, p, \zeta_i \in \partial f_i(x_0)$$

where strict inequality holds for the above mentioned $i_0 \in J$. It follows that

$$(\langle \zeta_1, \eta(y, x_0)\rangle, \cdots, \langle \zeta_p, \eta(y, x_0)\rangle) \in -R_+^p \setminus \{0\}.$$

This contradiction leads to the results.

(b) The fact that x_0 is a solution to the (GMVVLIP)$_2$ is the conclusion of Theorem 3.3 of [8],so we omit it.

Remark 3.2 Theorem 3.1 and 3.2, respectively, generalized and extended Theorem 3.1(ii) and Theorem 3.3 of Yang and Yang [5] from smooth case to the nonsmooth case.

4. GENERALIZED WEAK VVLIs

Let K be a nonempty subset of R^n and let $\eta: K \times K \to R^n$ be a given mapping. Let $f = (f_1, f_2, \cdots, f_p): K \to R^p$ be a vector-valued mapping. In the section, we consider the following four types of generalized weak VVLI problems:

(I)type 1 generalized weak Minty vector variational-like inequality problem(GWMVVLIP)$_1$:find $x \in K$ such that, for any $y \in K$ and any $\xi_i \in \partial f_i(y), i = 1,2,\cdots, p$,

$$(\langle \xi_1, \eta(y, x)\rangle, \cdots, \langle \xi_p, \eta(y, x)\rangle) \notin -intR_+^p.$$

(II)type 2 generalized weak Minty vector variational-like inequality problem(GWMVVLIP)$_2$:find $x \in K$ such that, for any $y \in K$ and any $\xi_i \in \partial f_i(x), i = 1,2,\cdots, p$,

$$(\langle \xi_1, \eta(x, y)\rangle, \cdots, \langle \xi_p, \eta(x, y)\rangle) \notin -intR_+^p.$$

(III)type 1 generalized weak Stampacchia vector variational-like inequality problem(GWSVVLIP)$_1$:find $x \in K$ such that, for any $y \in K$, there exists $\zeta_i \in \partial f_i(x), i = 1,2,\cdots, p$,

$$(\langle \zeta_1, \eta(y, x)\rangle, \cdots, \langle \zeta_p, \eta(y, x)\rangle) \notin -intR_+^p.$$

(IV)type 2 generalized weak Stampacchia vector variational-like inequality problem(GWSVVLIP)$_2$:find $x \in K$ such that, for any $y \in K$, there exists $\zeta_i \in \partial f_i(y), i = 1,2,\cdots, p$,

$$(\langle \zeta_1, \eta(x, y)\rangle, \cdots, \langle \zeta_p, \eta(x, y)\rangle) \notin -intR_+^p.$$

It is worth noting that (GWMVVLIP)$_1$ and (GWSVVLIP)$_1$ are introduced and studied by Al-Homidan and Ansari[6]. Now, we present some results which show the relationship among the solutions of these four types of generalized weak VVLI problems and a weakly efficient solution of VOP.

Theorem 4.1 Let K be a nonempty invex set with respect to η. For each $i \in J$, let f_i be locally Lipschitz, pseudoinvex function with respect to the same η on K. If $x_0 \in K$ is a solution of $(\text{GWSVVLIP})_1$, then x_0 is a weakly efficient solution to the VOP.

Proof. The assertion is clear from Long et al.[8].

Theorem 4.2 Let K be a nonempty invex set with respect to η such that η is skew, continuous with respect to the second argument and satisfies Condition C. For each $i \in J$, let f_i be locally Lipschitz, pseudoinvex function with respect to the same η on K. If $x_0 \in K$ is a weakly efficient solution of VOP, then x_0 is a solution to the $(\text{GWMVVLIP})_1$.

Proof. Let $x_0 \in K$ be a weakly efficient solution of VOP. Suppose by contradiction that x_0 is not a solution of $(\text{GWMVVLIP})_1$. Then, there exist $y \in K$ and $\xi_i \in \partial f_i(y)$ such that

$$(\langle \xi_1, \eta(y, x_0)\rangle, \cdots, \langle \xi_p, \eta(y, x_0)\rangle) \in -intR_+^p.$$

Since η is skew, we have $\langle \xi_i, \eta(x_0, y)\rangle > 0, i = 1,2,\cdots,p$. By the pseudoinvexity of $f_i, i \in J$ and lemmas 2.1-2.3, f_i is quasiinvex. It follows that

$$f_i(x_0) > f_i(y), i = 1,2,\cdots,p.$$

Therefore

$$(f_1(y) - f_1(x_0)), \cdots, f_p(y) - f_p(x_0) \in -intR_+^p,$$

which contradicts that x_0 is a weakly efficient solution of VOP. This completes the proof.

Theorem 4.3 Let K be a nonempty invex set with respect to η such that η is skew. For each $i \in J$, let ∂f_i be invariant pseudomonotone with respect to η on K.

(a)If $x_0 \in K$ is a solution of $(\text{GWSVVLIP})_1$, then x_0 is a solution to the $(\text{GWMVVLIP})_1$.

(b)If $x_0 \in K$ is a solution of $(\text{GWSVVLIP})_2$, then x_0 is a solution to the $(\text{GWMVVLIP})_2$.

Proof. (a) Let $x_0 \in K$ be a solution of $(\text{GWSVVLIP})_1$. If x_0 is not a solution of $(\text{GWMVVLIP})_1$, then there exist $y \in K, \xi_i \in \partial f_i(y), i = 1,2,\cdots,p$ such that

$$(\langle \xi_1, \eta(y, x_0)\rangle, \cdots, \langle \xi_p, \eta(y, x_0)\rangle) \in -intR_+^p.$$

It follows that $\langle \xi_i, \eta(y, x_0)\rangle < 0, i = 1,2,\cdots,p$. Since η is a skew function, we obtain $\langle \xi_i, \eta(x_0, y)\rangle > 0, i = 1,2,\cdots,p$. By the invariant pseudomonotonicity of $\partial f_i, i = 1,2,\cdots,p$,

$$\langle \zeta_i, \eta(y, x_0)\rangle < 0, \quad \text{for all} \quad \zeta_i \in \partial f_i(x_0),$$

which contradicts the fact that x_0 is a solution of $(\text{GWSVVLIP})_1$. This completes the proof.

(b) By using the proof techniques of part (a), one can obtain the results. Hence the proof is omitted.

Remark 4.1 (i)Theorem 4.3 includes Theorem 4.1 of Long et al.[8] as a special case.

(ii)Theorem 4.1-4.3, respectively, generalized and improved Propositions 4.1,4.3 and 4.4 of Al-Homidan and Ansari[6] since invexity of f_i has been weakend by pseudoinvexity of f_i or invariant pseudomonotonixity of ∂f_i.

(iii)Theorem 4.1 and 4.3, respectively, generalized and extended Theorems 4.1 and 4.3 of Yang and Yang [5] from smooth case to nonsmooth case.

5 . THE PERTURBED FORM OF GENERALIZED WEAK STAMPACCHIA VVLI PROBLEM

In this section, let $f = (f_1, f_2, \cdots, f_p) : K \to R^p$ be a locally Lipschitz function and $\{C(x) : x \in K\}$ be a family of convex and pointed cones of R^p such that $C(x) \subseteq R_+^p \setminus \{0\}, \forall x \in K$, with $intC(x) \neq \varnothing$.

Now we consider the following two types of generalized VVLIs.

(I)the perturbed form of generalized weak Stampacchia vector variational-like inequality problem (PGWSVVLIP):Find $y \in K$ for which there exists $t_0 \in (0,1)$ such that

$$\langle \partial f(y+t\eta(x,y)), \eta(x,y)\rangle \not\subseteq -intC(y), \quad \textit{for any} \quad x \in K \quad \textit{and any} \quad t \in (0,t_0].$$

(II)generalized weak Minty vector variational-like inequality problem (GWMVVLIP): Find $y \in K$ such that

$$\langle \partial f(x), \eta(x,y)\rangle \not\subseteq -intC(y), \forall x \in K.$$

The following result provides the relationship between a solution of (PGWSVVLIP) and (GWMVVLIP).

Theorem 5.1 Let K be an invex set with respect to $\eta : K \times K \to R^n$ such that η is skew and satisfies Condition C*. If $y_0 \in K$ is a solution of (GWMVVLIP), then y_0 is a solution of (PGWSVVLIP). Conversely, if ∂f, the Clarke's generalized subdifferential of f, is C-strictly quasimonotone with respect to η on K, that is,

$$\langle \partial f(x), \eta(y,x)\rangle \subseteq intC(y) \Rightarrow \langle \partial f(y), \eta(x,y)\rangle \subseteq -intC(y),$$

for any $x, y \in K$. Then, $y_0 \in K$ is a solution of (PGWSVVLIP) implies that it is a solution of (GWMVVLIP).

Proof. Let $y_0 \in K$ be a solution of (GWMVVLIP). The invexity of K with respect to η implies $x(t) = y_0 + t\eta(x, y_0) \in K$, for any $x \in K$, $t \in (0,t_0], t_0 \in (0,1)$. It follows that

$$\langle \partial f(y_0 + t\eta(x,y_0)), \eta(y_0 + t\eta(x,y_0), y_0)\rangle \not\subseteq -intC(y_0), \forall t \in (0,t_0].$$

Since η is skew, we have

$$\langle \partial f(y_0 + t\eta(x,y_0)), \eta(y_0, y_0 + t\eta(x,y_0))\rangle \not\subseteq intC(y_0), \tag{1}$$

for any $x \in K$ and any $t \in (0,t_0]$. By Condition C*, we have

$$\eta(y_0, y_0 + t\eta(x,y_0)) = -\alpha(t)\eta(x,y_0),$$

where $\alpha(t) > 0$, for all $t \in (0,1)$. It follows from (5.1) that

$$\langle \partial f(y_0 + t\eta(x,y_0)), \eta(x,y_0)\rangle \not\subseteq -intC(y_0),$$

for any $x \in K$ and any $t \in (0,t_0]$. Thus, $y_0 \in K$ is a solution of (PGWSVVLIP).

Conversely, let $y_0 \in K$ be a solution of (PGWSVVLIP). Then there exists $t_0 \in (0,1)$ such that

$$\langle \partial f(y_0 + t\eta(x,y_0)), \eta(x,y_0)\rangle \not\subseteq -intC, \tag{2}$$

for any $x \in K$ and any $t \in (0,t_0]$. By the Condition C*, we have

$$\eta(x, y_0 + t\eta(x,y_0)) = \beta(t)\eta(x,y_0), \tag{3}$$

where $\beta(t) > 0$, for any $t \in (0,1)$. It follows from (5.2) that

$$\langle \partial f(y_0 + t\eta(x,y_0)), \eta(x, y_0 + t\eta(x,y_0))\rangle \not\subseteq -intC,$$

for any $x \in K$ and any $t \in (0,t_0]$. By C-strictly quasimonotonicity of ∂f, we have

$$\langle \partial f(x), \eta(y_0 + t\eta(x,y_0), x)\rangle \not\subseteq intC.$$

By (5.3) and skewness of η, we obtain

$$\langle \partial f(x), \eta(x,y_0)\rangle \not\subseteq -intC.$$

Hence, $y_0 \in K$ is a solution of (GWMVVLIP).

Remark 5.1 Theorem 5.1 implies Theorem 4.4 of [9] since the Condition C has been weakened by Condition C* and the proof techniques are different from [9].

6. SOLUTION FOR THE GENERALIZED WEAK VOP

In this section, let K be a convex subset of a Banach space X, $f : K \to R^n$, $f(x) = (f_1(x), f_2(x), \cdots, f_n(x))$

for $x \in K$, and $\{C(x): x \in K\}$ be a family of closed, convex and point cones of R^n such that $R^n_+ \subseteq C(x), \forall x \in K$.

A point $y \in K$ is called a solution for the generalzied weak VOP of f iff

$$f(y) - f(x) \notin intC(y), \forall x \in K.$$

Theorem 6.1 Let $f: K \to R^n$ be a pseudoinvex function with respect to η on K. Consider the following problems:

(p1)There exists $y_0 \in K$ such that $\langle \partial f(y_0), \eta(x, y_0)\rangle \not\subseteq -intC(y_0), \forall x \in K$.

(p2) $y_0 \in K$ is a solution for the generalized weak VOP.

(p3)There exists $y_0 \in K$ such that $\langle \partial f(x), \eta(x, y_0)\rangle \not\subseteq -intC(y_0), \forall x \in K$.

Then, (p1)$\Rightarrow$(p2). If η is skew, (p2)$\Rightarrow$(p3). If the pseudoinvexity of f is replaced by invexity and η is both affine in the first argument and $\eta(x, x) = 0$, then (p3)$\Rightarrow$(p1).

Proof. (p1)$\Rightarrow$(p2). Suppose that y_0 is not a solution for the generalized weak VOP. Then there exists $x \in K$ such that

$$f(y_0) - f(x) \in intC(y_0).$$

On the other hand, by the pseudoinvexity of f, we have $\langle \xi_i, \eta(x, y_0)\rangle < 0, \forall i = 1,2,\cdots,n, \xi_i \in \partial f_i(y_0)$. Therefore,

$$(\langle \xi_1, \eta(x, y_0)\rangle, \cdots, \langle \xi_n, \eta(x, y_0)\rangle) \subseteq -intR^n_+ \subseteq -intC(y_0),$$

Hence, $\langle \partial f(y_0), \eta(x, y_0)\rangle \subseteq -intC(y_0)$, which contradicts with (p1).

(p2)$\Rightarrow$(p3). Suppose that (p3) is not true, Then, for each $y \in K$, there exists $x_0 \in K$ such that

$$\langle \partial f(x_0), \eta(x_0, y)\rangle \subseteq -intC(y).$$

Since η is skew, we have

$$\langle \partial f(x_0), \eta(y, x_0)\rangle \subseteq intC(y).$$

That is, for any $i = 1,2,\cdots,n, \xi_i \in \partial f_i(x_0)$, we have $\langle \xi_i, \eta(y, x_0)\rangle \in intC(y)$. By the pseudoinvexity of $f_i, i = 1,2,\cdots,n$, we have obtain $f(y) - f(x_0) \in intC(y)$, which contradicts with (p2).

(p3)$\Rightarrow$(p1). The proof follows the lines of the proof of Theorem 4.1 in [20].

Remark 6.1 (i)We obtained the half part of Theorem 6.1 under the assumption of pseudoinvex functions while it is proved in [20] for invex functions.

(ii)All the results of this paper can be extended to Banach space setting. However, for the sake of simplicity, we consider finite dimensional space R^n.

REFERENCES

[1]. Giannessi, F.:(1980) Theorem of alternative, quadratic problems and complementarity problems, in Variational Inequality and Complementarity Problem, In: Cottle, R.W., Giannessi, F., Lions J.L.(eds.) John Wiley and Sons, pp151-186. Chichester, UK

[2]. Giannessi, F.:(1998) On Minty variational principle, in New Trends in Mathematical Programming. In:Giannessi,F., Komlo'si,S., Tapcsack, T.(eds.) Kluwer Academic, pp.93-99.Dordrechet

[3]. Yang,X.M., Yang,X.Q., Teo,K.L.: Some remarks on the minty vector variational inequality. J. Optim. Theory Appl. **121**,193-201(2004)

[4]. Ruiz-Garzo'n,G., Osuna-Go'mez,R., Rufia'n-Lizana,A.:Relationships between vector variational-like inequality and optimization problems.Eur.J.Oper.Res.**157**,113-119(2004).

[5]. Yang,X.M., Yang,X.Q.: Vector variational-like inequality with pseudoinvexity. Optimization **55**,157-170(2006)

[6]. Al-Homidan,S., Ansari,Q.H.: Generalized Minty vector variational-like inequalities and vector optimization

problems.J.Optim.Theory Appl.**144**,1-11(2010)

[7]. Ansari,Q.H.,Rezaie,M.,Zafarani,J.:Generalized vector variational-like inequalities and vector optimization. J.Glob.Optim. DOI:10.1007/s10898-011-9686-1

[8]. Long,X.J.,Peng,J.W.,Wu,S.Y.:Generalized vector variational-like inequalities and nonsmooth vector optimization problems. Optimization. DOI:10.1080/02331934.2010.538056

[9]. Rezaie,M.,Zafarani,J.:Vector optimization and variational-like inequalities. J.Glob.Optim.43,47-66(2009)

[10]. Clarke,F.H.:Optimization and Nonsmooth Analysis. SIAM, Philadelphia,Pennsylvania(1990)

[11]. Weir,T.,Mond,B.:Preinvex functions in multiple-objective optimization.J.Math.Anal.Appl.136,29-38(1988)

[12]. Mohan,S.R.,Neogy,S.K.:On invex sets and preinvex functions.J.Math.Anal.Appl.189,901-908(1995)

[13]. Yang,X.M., Yang,X.Q., Teo,K.L.: Characterizations and applications of prequasiinvex function. J. Optim. Theory Appl. **110**,645-668(2001)

[14]. Jabarootian,T.,Zafarani,J.:Generalized invariant monotonicity and invexity of nondifferentiable functions.J.Glob.Optim.**36**,537-564(2006)

[15]. Peng,J.W.:Criteria for generalized invex monotonicities without Condition C.Eur.J.Oper.Res.**170**,667-671(2006)

[16]. Gang,X.,Liu,S.:On Minty vector variational-like inequality.Comput.Math.Appl.**56**,311-323(2008)

[17]. Lee,G.M.:(2000) On relations between vector variational inequality and vector optimization problem. In:Yang,X.Q.,Mees,A.I.,Fisher, M.E.,Jennings,L.S.(eds.)Progress in Optimization, II:Contributions from Australasia, pp.167-179.Kluwer Academic Publisher, Dordrecht,Holland

[18]. Giannessi,F(ed.).:(2000) Vector Variational Inequalities and vector Equilibria:Mathematical Theories,Kluwer Academic,Dordrechet

[19]. Yang,X.M., Yang,X.Q., Teo,K.L.: Criteria for generalized invex monotonocity. Eur. J. Oper.Res. **164**,115-119(2005)

[20]. Jabarootian,T.,Zafarani,J.:Gneralized vector variational-like inequalities.J.Optim.Theory Appl.136,15-30(2008).