ENDPOINT ESTIMATES FOR MULTILINEAR COMMUTATOR OF LITTLEWOOD-PALEY OPERATOR

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ABSTRACT
In this paper, we prove the endpoint estimates for the multilinear commutator of Littlewood-Paley operator.

keywords: *Littlewood-Paley operator; Multilinear commutator; Hardy spaces;* $BMO(R^n)$.
MR Subject Classification: *42B20, 42B25.*

1. INTRODUCTION

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund operator, the commutator $[b,T]$ generated by b and T is defined by

$$[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberb and Weiss (see [4]) proved that the commutator $[b,T]$ is bounded on $L^p(R^n), (1 < p < \infty)$. In [3][6], the boundedness properties of the commutators for the extreme values of p are obtained. In this paper, we will introduce the multilinear commutator of Littlewood-Paley operator and prove the boundedness properties of the operator for the extreme cases.

First let us introduce some notations (see [2][5][10][11]). In this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a function b, let $b_Q = |Q|^{-1}\int_Q b(x)dx$ and $b(Q) = \int_Q b(x)dx$, the sharp function of b is defined by

$$b^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |b(y) - b_Q|\,dy.$$

It is well-known that (see [4])

$$b^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|}\int_Q |b(y) - c|\,dy.$$

We say that b belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^\infty(R^n)$ and define $\|b\|_{BMO} = \|b^{\#}\|_{L^\infty}$. We also define the central BMO space by $CMO(R^n)$, which is the space of those functions $b \in L_{loc}(R^n)$ such that

$$\|b\|_{CMO} = \sup_{r>1} |Q(0,r)|^{-1}\int_Q |b(y) - b_Q|\,dy < \infty.$$

It is well-known that

$$\|b\|_{CMO} \approx \sup_{r>1} \inf_{c \in C} |Q(0,r)|^{-1}\int_Q |b(y) - c|\,dy.$$

Definition 1. A function a is called an $H^1(R^n)-$atom, if there exists a cube Q, such that

1) supp $a \subset Q = Q(x_0, r)$,
2) $\|a\|_{L^\infty} \le |Q|^{-1}$,
3) $\int_{R^n} a(x)dx = 0.$

It is well known that the Hardy space $H^1(R^n)$ has the atomic decomposition characterization (see [4][8]).

Definition 2. Let $0<\delta<n$ and $1<p<n/\delta$. We shall call $B_p^\delta(R^n)$ the space of those functions f on R^n such that

$$\| f\|_{B_p^\delta}=\sup_{r>1} r^{-n(1/p-\delta/n)}\| f\chi_{Q(0,r)}\|_{L^p}<\infty.$$

Definition 3. Let $0<\delta<n$ and ψ be a fixed function satisfies the following properties:

1) $\int\psi(x)dx=0,$

2) $|\psi(x)|\le C(1+|x|)^{-(n+1-\delta)},$

3) $|\psi(x+y)-\psi(x)|\le C|y|(1+|x|)^{-(n+2-\delta)}$ when $2|y|<|x|$.

Let $\vec{b}=(b_1,\cdots,b_m)$ with $b_j\in BMO(R^n)$ for $1\le j\le m.$ Then the multilinear commutator of Littlewood-Paley operator is defined by

$$g_{\mu,\delta}^{\vec{b}}(f)(x)=\left[\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|F_t^{\vec{b}}(f)(x,y)|^2\frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x,y)=\int_{R^n}\left[\prod_{j=1}^m(b_j(x)-b_j(z))\right]\psi_t(y-z)f(z)dz.$$

When $m=1$, set

$$g_{\mu,\delta}^{b_1}(f)(x)=\left[\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|F_t^{b_1}(f)(x,y)|^2\frac{dydt}{t^{n+1}}\right]^{1/2},$$

where

$$F_t^{b_1}(f)(x,y)=\int_{R^n}(b_1(x)-b_1(z))\psi_t(y-z)f(z)dz$$

and $\psi_t(x)=t^{-n+\delta}\psi(x/t)$ for $t>0.$ Set $F_t(f)(x)=f*\psi_t(x),$ we also define that

$$g_{\mu,\delta}(f)(x)=\left[\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|F_t(f)(y)|^2\frac{dydt}{t^{n+1}}\right]^{1/2},$$

which is the Littlewood-Paley operator (see [1][7-9][12]).

Let H be the space $H=\left\{h:\|h\|=(\iint_{R_+^{n+1}}|h(y,t)|^2\,dydt/t^{n+1})^{1/2}<\infty\right\}$. Then for each fixed $x\in R^n$, $F_t(f)(x)$ may be viewed as a mapping from $[0,+\infty)$ to H. It is clear that

$$g_{\mu,\delta}(f)(x)=\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t(f)(y)\right\| \text{ and } g_{\mu,\delta}^{\vec{b}}(f)(x)=\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t^{\vec{b}}(f)(x,y)\right\|.$$

Given a positive integer m and $1\le j\le m$, we denote by C_j^m the family of all finite subsets $\sigma=\{\sigma(1),\cdots,\sigma(j)\}$ of $\{1,\cdots,m\}$ of j different elements. For $\sigma\in C_j^m$, set $\sigma^c=\{1,\cdots,m\}\setminus\sigma$. For $\vec{b}=(b_1,\cdots,b_m)$ and $\sigma=\{\sigma(1),\cdots,\sigma(j)\}\in C_j^m$, set $\vec{b}_\sigma=(b_{\sigma(1)},\cdots,b_{\sigma(j)})$, $b_\sigma=b_{\sigma(1)}\cdots b_{\sigma(j)}$ and $\|\vec{b}_\sigma\|_{BMO}=\|b_{\sigma(1)}\|_{BMO}\cdots\|b_{\sigma(j)}\|_{BMO}$.

2.THEOREMS AND PROOFS We begin with some preliminaries lemmas.

Lemma 1. Let $1<r<\infty$, $b_j \in BMO$ for $j=1,\cdots,k$ and $k\in N$. Then, we have

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^{k}|b_j(y)-(b_j)_Q|\,dy \le C\prod_{j=1}^{k}\|b_j\|_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^{k}|b_j(y)-(b_j)_Q|^r\,dy\right)^{1/r} \le C\prod_{j=1}^{k}\|b_j\|_{BMO}.$$

Lemma 2. Let $w\in A_1$, $0<\delta<n$, $1<p<n/\delta$ and $1/q=1/p-\delta/n$. Then $g_{\mu,\delta}$ is bounded from $L^p(w)$ to $L^q(w)$.

Lemma 3. Let $w\in A_p$, $1<p<\infty$, then $w\chi_Q \in A_p$ for any cube Q.

Theorem 1. Let $\mu>3+1/n-2\delta/n$ and $0<\delta<n, \vec{b}=(b_1,\cdots,b_m)$ with $b_j\in BMO(R^n)$ for $1\le j\le m$. Then $g_{\mu,\delta}^{\vec{b}}$ is bounded from $L^{n/\delta}$ to $BMO(R^n)$.

Proof. It is only to prove that there exist a constant C_Q such that

$$\frac{1}{|Q|}\int_Q |g_{\mu,\delta}^{\vec{b}}(f)(x)-C_Q|\,dx \le C\|f\|_{L^{n/\delta}}.$$

Fix a cube Q, $Q=Q(x_0,r)$, we decompose f into $f=f_1+f_2$ with $f_1=f\chi_Q$, $f_2=f\chi_{R^n\setminus Q}$.

When $m=1$, set $(b_1)_Q=|Q|^{-1}\int_Q b_1(y)dy$, we have

$$F_t^{b_1}(f)(x,y)=(b_1(x)-(b_1)_Q)F_t(f)(y)-F_t((b_1-(b_1)_Q)f_1)(y)-F_t((b_1-(b_1)_Q)f_2)(y).$$

Then

$$\begin{aligned}
&|g_{\mu,\delta}^{b_1}(f)(x)-g_{\mu,\delta}(((b_1)_Q-b_1)f_2)(x_0)|\\
&=\left\|\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t^{b_1}(f)(x,y)\right\|-\left\|\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}F_t(((b_1)_Q-b_1)f_2)(y)\right\|\right\|\\
&\le\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t^{b_1}(f)(x,y)-\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}F_t(((b_1)_Q-b_1)f_2)(y)\right\|\\
&\le\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}(b_1(x)-(b_1)_Q)F_t(f)(y)\right\|+\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t((b_1-(b_1)_Q)f_1)(y)\right\|\\
&+\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t((b_1-(b_1)_Q)f_2)(y)-\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}F_t((b_1-(b_1)_Q)f_2)(y)\right\|\\
&=A(x)+B(x)+C(x).
\end{aligned}$$

For $A(x)$, set $1<p<n/\delta$, $1/q=1/p-\delta/n$ and $1/q+1/q'=1$, by the Hölder's inequality and Lemma 2,3, we get

$$\frac{1}{|Q|}\int_Q |A(x)|\,dx=\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_Q\|g_{\mu,\delta}(f)(x)|\,dx$$

$$\le \left(\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_Q|^{q'}\,dx \right)^{1/q'} \left(\frac{1}{|Q|}\int_{R^n} |g_{\mu,\delta}(f)(x)|^{q}\,\chi_Q(x)dx \right)^{1/q}$$

$$\le C\|b_1\|_{BMO}|Q|^{-1/q}\left(\int_{R^n}|f(x)|^p\,\chi_Q(x)dx\right)^{1/p}$$

$$\le C\|b_1\|_{BMO}|Q|^{-1/q}\left[\left(\int_{R^n}|f(x)|^{n/\delta}\,dx\right)^{\delta p/n}\left(\int_Q\chi_Q(x)dx\right)^{1-\delta p/n}\right]^{1/p}$$

$$\le C\|b_1\|_{BMO}|Q|^{-1/q}\|f\|_{L^{n/\delta}}|Q|^{(1-(\delta p/n))/p}$$

$$\le C\|b_1\|_{BMO}\|f\|_{L^{n/\delta}}.$$

For $B(x)$, taking $1<r<n/\delta$ and $1/s=1/r-\delta/n$, by the Hölder's inequality, we have

$$\frac{1}{|Q|}\int_Q|B(x)|\,dx=\frac{1}{|Q|}\int_Q|g_{\mu,\delta}((b_1-(b_1)_Q)f_1)(x)|\,dx$$

$$\le\left(\frac{1}{|Q|}\int_{R^n}(g_{\mu,\delta}((b_1(x)-(b_1)_Q)f_1)(x))^s\,dx\right)^{1/s}$$

$$\le C|Q|^{-1/s}\int_{R^n}|(b_1-(b_1)_Q)|^r|f(x)|^r\,\chi_Q(x)dx)^{1/r}$$

$$\le C\left(\frac{1}{|Q|}\int_Q|b_1(x)-(b_1)_Q|^s\,dx\right)^{1/s}\|f\|_{L^{n/\delta}}$$

$$\le C\|b_1\|_{BMO}\|f\|_{L^{n/\delta}}.$$

For $C(x)$, by the inequality $a^{1/2}-b^{1/2}\le(a-b)^{1/2}$ for $a>b$, we have

$$C(x)=\left[\iint_{R_+^{n+1}}\left(\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}-\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}\right|F_t((b_1-(b_1)_Q)f_2)(y)\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}$$

$$\le\left[\iint_{R_+^{n+1}}\left(\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}-\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}\right|\right.\right.$$

$$\left.\left.\times\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||\psi_t(y-z)|\,dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}$$

$$\le C\int_{Q^c}\left[\iint_{R_+^{n+1}}\left(\frac{t^{n\mu/2}|x-x_0|^{1/2}|b_1(z)-(b_1)_Q||\psi_t(y-z)||f(z)|}{(t+|x-y|)^{(n\mu+1)/2}}\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}dz$$

$$\le C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||x-x_0|^{1/2}\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu+1}\frac{t^{-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}\right)^{1/2}dz,$$

set $B=B(x,t)$, note that

$$t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\mu+1}\frac{dy}{(t+|y-z|)^{2n+2-2\delta}}$$

$$\leq t^{-n}\left(\int_{B(x,t)}\left(\frac{t}{t+|x-y|}\right)^{n\mu+1}\frac{dy}{(t+|y-z|)^{2n+2-2\delta}}\right)$$
$$+t^{-n}\left(\sum_{k=1}^{\infty}\int_{2^kB\setminus 2^{k-1}B}\left(\frac{t}{t+|x-y|}\right)^{n\mu+1}\frac{dy}{(t+|y-z|)^{2n+2-2\delta}}\right)$$

$$\leq Ct^{-n}\left(\int_{B(x,t)}\frac{2^{2n+2-2\delta}dy}{(2t+|y-z|)^{2n+2-2\delta}}+\sum_{k=1}^{\infty}\int_{2^kB\setminus 2^{k-1}B}\left(\frac{t}{t+2^{k-1}t}\right)^{n\mu+1}\frac{2^{(k+1)(2n+2-2\delta)}dy}{(2^{k+1}t+|y-z|)^{2n+2-2\delta}}\right)$$
$$\leq Ct^{-n}\left(\int_{B(x,t)}\frac{dy}{(t+|x-z|)^{2n+2-2\delta}}+\sum_{k=1}^{\infty}2^{(1-k)(n\mu+1)}\int_{2^kB}\frac{2^{(k+1)(2n+2-2\delta)}dy}{(t+|x-z|)^{2n+2-2\delta}}\right)$$
$$\leq Ct^{-n}\left(t^n+\sum_{k=1}^{\infty}2^{-k(n\mu+1)}2^{k(2n+2-2\delta)}(2^kt)^n\right)\frac{1}{(t+|x-z|)^{2n+2-2\delta}}$$
$$\leq \frac{C}{(t+|x-z|)^{2n+2-2\delta}}$$

and

$$\int_0^{\infty}\frac{dt}{(t+|x-z|)^{2n+2-2\delta}}=C|x-z|^{-(2n+1-\delta)}.$$

Note that $|x-z|\approx|x_0-z|$ for $x\in Q$ and $z\in R^n\setminus Q$. We obtain

$$\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||x-x_0|^{1/2}\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu+1}\frac{t^{-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}\right)^{1/2}dz$$
$$\leq C\int_{Q^c}|b_1(z)-(b_1)_Q||f(z)||x-x_0|^{1/2}\left(\int_0^{\infty}\frac{dt}{(t+|x-z|)^{2n+2-2\delta}}\right)^{1/2}dz$$
$$\leq C\int_{Q^c}\frac{|b_1(z)-(b_1)_Q||f(z)||x-x_0|^{1/2}}{|x_0-z|^{n+1/2-\delta}}dz$$
$$\leq C\sum_{k=1}^{\infty}\int_{2^kQ\setminus 2^{k-1}Q}\frac{|Q|^{1/2n}|b_1(z)-(b_1)_Q||f(z)|dz}{|x_0-z|^{n+1/2-\delta}}$$
$$\leq C\sum_{k=1}^{\infty}2^{-k/2}\frac{1}{|2^kQ|^{1-\delta/n}}\int_{2^kQ}|b_1(z)-(b_1)_Q||f(z)|dz$$
$$\leq C\sum_{k=1}^{\infty}2^{-k/2}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|b_1(z)-(b_1)_Q|^{n/(n-\delta)}dz\right)^{(n-\delta)/n}\left(\int_{2^kQ}|f(z)|^{n/\delta}dz\right)^{\delta/n}$$
$$\leq C\|b_1\|_{BMO}\sum_{k=1}^{\infty}k\cdot 2^{-k/2}\|f\|_{L^{n/\delta}}$$
$$\leq C\|b_1\|_{BMO}\|f\|_{L^{n/\delta}},$$

so

$$\frac{1}{|Q|}\int_Q|C(x)|dx\leq C\|b_1\|_{BMO}\|f\|_{L^{n/\delta}}.$$

This completes the proof of the case $m=1$.

When $m>1$, set $\vec{b}_Q=((b_1)_Q,\cdots,(b_m)_Q)\in R^n$, where

$$(b_j)_Q=|Q|^{-1}\int_Q b_j(y)dy, 1\le j\le m,$$

we have

$$\begin{aligned}
F_t^{\vec{b}}(f)(x,y)&=\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma\int_{R^n}(\vec{b}(z)-\vec{b}_Q)_{\sigma^c}\psi_t(y-z)f(z)dz\\
&=(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)\\
&\quad+(-1)^m F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f)(y)\\
&\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma\int_{R^n}(\vec{b}(z)-\vec{b}_Q)_{\sigma^c}\psi_t(y-z)f(z)dz\\
&=(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)\\
&\quad+(-1)^m F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f)(y)\\
&\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}c_{m,j}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x,y),
\end{aligned}$$

thus

$$\begin{aligned}
&|g_{\mu,\delta}^{\vec{b}}(f)(x)-g_{\mu,\delta}(((b_1)_{2Q}-b_1)\cdots((b_m)_Q-b_m))f_2)(x_0)|\\
&\le\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t^{\vec{b}}(f)(x,y)-\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}F_t(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(y)\right\|\\
&\le\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)\right\|\\
&\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x,y)\right\|\\
&\quad+\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(y)\right\|\\
&\quad+\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_2)(y)-\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}F_t(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_2)(y)\right\|\\
&=S_1(x)+S_2(x)+S_3(x)+S_4(x).
\end{aligned}$$

For $S_1(x)$, taking $1<p<n/\delta$, and $1/q=1/p-\delta/n$, by the Hölder's inequality and Lemma 1,2,3, we have

$$\frac{1}{|Q|}\int_Q S_1(x)dx\le\left(\frac{1}{|Q|}\int_Q|\prod_{j=1}^{m}(b_j(x)-(b_j)_Q)|^{q'}dx\right)^{1/q'}\left(\frac{1}{|Q|}\int_Q|g_{\mu,\delta}(f)(x)|^q dx\right)^{1/q}$$

$$\le C\|\vec{b}\|_{BMO}|Q|^{-1/q}\left(\int_Q|f(x)|^p\,dx\right)^{1/p}$$

$$\le C\|\vec{b}\|_{BMO}|Q|^{-1/q}\left(\int_Q|f(x)|^{n/\delta}\,dx\right)^{\delta/n}|Q|^{(1-(\delta p/n))/p}$$

$$\le C\|\vec{b}\|_{BMO}\|f\|_{L^{n/\delta}}.$$

For $S_2(x)$, taking $1<p<n/\delta$ and $1/q=1/p-\delta/n$, then

$$\frac{1}{|Q|}\int_Q S_2(x)dx$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|Q|}\int_Q|(\vec{b}(x)-\vec{b}_Q)_\sigma|^{q'}\,dx\right)^{1/q'}\left(\frac{1}{|Q|}\int_Q|g_{\mu,\delta}((\vec{b}-\vec{b}_Q)_{\sigma^c})f)(x)|^q\,dx\right)^{1/q}$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}|Q|^{-1/q}\left(\int_{R^n}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c})f(x)|^p\,\chi_Q(x)dx\right)^{1/p}$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}\left(\frac{1}{|Q|}\int_Q|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|^q\,dx\right)^{1/q}\|f\|_{L^{n/\delta}}$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}\|\vec{b}_{\sigma^c}\|_{BMO}\|f\|_{L^{n/\delta}}$$

$$\le C\|\vec{b}\|_{BMO}\|f\|_{L^{n/\delta}}.$$

For $S_3(x)$, taking $1<p<n/\delta$ and $1/q=1/p-\delta/n$, we get

$$\frac{1}{|Q|}\int_Q S_3(x)dx$$

$$\le\left(\frac{1}{|Q|}\int_Q|g_{\mu,\delta}((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|^q\,dx\right)^{1/q}$$

$$\le C|Q|^{-1/q}\|((b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f_1(x)\|_{L^p}$$

$$\le C\left(\frac{1}{|Q|}\int_Q|(b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)|^q\,dx\right)^{1/q}\|f\|_{L^{n/\delta}}$$

$$\le C\|\vec{b}\|_{BMO}\|f\|_{L^{n/\delta}}.$$

For $S_4(x)$, we have

$$S_4(x)\le C\int_{Q^c}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}|\prod_{j=1}^m(b_j(z)-(b_j)_Q)\|f(z)|dz$$

$$\le C\sum_{k=1}^\infty\int_{2^kQ\setminus 2^{k-1}Q}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}|\prod_{j=1}^m(b_j(z)-(b_j)_Q)\|f(z)|dz$$

$$\le C\sum_{k=1}^\infty\int_{2^kQ\setminus 2^{k-1}Q}\frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2-\delta}}|\prod_{j=1}^m(b_j(z)-(b_j)_Q)\|f(z)|dz$$

$$\le C\sum_{k=1}^{\infty}2^{-k/2}\frac{1}{|2^kQ|^{1-\delta/n}}\int_{2^kQ}|\prod_{j=1}^{m}(b_j(z)-(b_j)_Q)\|f(z)|dz$$

$$\le C\sum_{k=1}^{\infty}2^{-k/2}\left(\frac{1}{|2^kQ|}\int_{2^kQ}|\prod_{j=1}^{m}(b_j(z)-(b_j)_{2Q})|^{n/(n-\delta)}dz\right)^{(n-\delta)/n}\left(\int_{2^kQ}|f(z)|^{n/\delta}dz\right)^{\delta/n}$$

$$\le C\sum_{k=1}^{\infty}2^{-k/2}k^m\prod_{j=1}^{m}\|b_j\|_{BMO}\|f\|_{L^{n/\delta}}$$

$$\le C\|\vec{b}\|_{BMO}\|f\|_{L^{n/\delta}},$$

so

$$\frac{1}{|Q|}\int_Q|S_4(x)|dx\le C\|\vec{b}\|_{BMO}\|f\|_{L^{n/\delta}}.$$

This completes the total proof of Theorem 1.

Theorem 2. Let $\mu>3+(1-2\delta)/n$ and $0<\delta<n$, $1<p<n/\delta$ and $\vec{b}=(b_1,\cdots,b_m)$ with $b_j\in BMO(R^n)$ for $1\le j\le m$. Then $g_{\mu,\delta}^{\vec{b}}$ is bounded from $B_p^\delta(R^n)$ to $CMO(R^n)$.

Proof. It suffices to prove that there exist constant C_Q, such that

$$\frac{1}{|Q|}\int_Q|g_{\mu,\delta}^{\vec{b}}(f)(x)-C_Q|dx\le C\|f\|_{B_p^\delta}$$

holds for any cube $Q=Q(0,d)$ with $d>1$. Fix a cube $Q=Q(0,d)$ with $d>1$. Set $f_1=f\chi_Q$, $f_2=f\chi_{R^n\setminus Q}$ and $\vec{b}_Q=((b_1)_Q,\cdots,(b_m)_Q)$, where $(b_j)_Q=|Q|^{-1}\int_Q|b_j(y)|dy$, $1\le j\le m$, we have

$$F_t^{\vec{b}}(f)(x,y)=(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)$$
$$+(-1)^mF_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(y)$$
$$+(-1)^mF_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(y)$$
$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x,y),$$

thus

$$|g_{\mu,\delta}^{\vec{b}}(f)(x)-g_{\mu,\delta}(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(0)|$$

$$\le\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t^{\vec{b}}(f)(x,y)-\left(\frac{t}{t+|y|}\right)^{n\mu/2}F_t(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(y)\right\|$$

$$\le\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(y)\right\|$$

$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x,y)\right\|$$

$$+\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(y)\right\|$$

$$+\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t(\prod_{j=1}^m (b_j-(b_j)_Q)f_2)(y)-\left(\frac{t}{t+|y|}\right)^{n\mu/2} F_t(\prod_{j=1}^m (b_j-(b_j)_Q)f_2)(y)\right\|$$

$$=H_1(x)+H_2(x)+H_3(x)+H_4(x).$$

Taking $1<p<n/\delta,\ 1/q=1/p-\delta/n$, by the Hölder's inequality and Lemma 1,2,3, we have

$$\frac{1}{|Q|}\int_Q H_1(x)dx$$

$$\le\left(\frac{1}{|Q|}\int_Q|\prod_{j=1}^m (b_j(x)-(b_j)_Q)|^{q'}dx\right)^{1/q'}\left(\frac{1}{|Q|}\int_Q|g_{\mu,\delta}(f)(x)|^q dx\right)^{1/q}$$

$$\le C\|\vec{b}\|_{BMO}|Q|^{-1/q}\left(\int_{R^n}|f(x)|^p\chi_Q(x)dx\right)^{1/p}$$

$$\le C\|\vec{b}\|_{BMO}\, d^{-n(1/p-\delta/n)}\|f\chi_Q\|_{L^p}$$

$$\le C\|\vec{b}\|_{BMO}\|f\|_{B_p^\delta}.$$

For $H_2(x)$, taking $1<p<n/\delta$, $1/s=1/r-\delta/n$, and $1/s'+1/s=1$, then

$$\frac{1}{|Q|}\int_Q H_2(x)dx$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|Q|}\int_Q|(\vec{b}(x)-\vec{b}_Q)_\sigma|^{s'}dx\right)^{1/s'}\left(\frac{1}{|Q|}\int_Q|g_{\mu,\delta}((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)|^s dx\right)^{1/s}$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}|Q|^{-1/s}\left(\int_{R^n}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c})f(x)|^r\chi_Q(x)dx\right)^{1/r}$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}\left(\frac{1}{|Q|}\int_Q|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|^{pr/(p-r)}dx\right)^{(p-r)/pr}|Q|^{(\delta/n-1/p)}\|f\chi_Q\|_{L^p}$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{BMO}\|\vec{b}_{\sigma^c}\|_{BMO}\, d^{-n(1/p-\delta/n)}\|f\chi_Q\|_{L^p}$$

$$\le C\|\vec{b}\|_{BMO}\|f\|_{B_p^\delta}.$$

For $H_3(x)$, taking $1<p<n/\delta$, $1/s=1/r-\delta/n$ and $1/s'+1/s=1$, we get

$$\frac{1}{|Q|}\int_Q H_3(x)dx$$

$$\le\left(\frac{1}{|Q|}\int_Q|g_{\mu,\delta}((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|^s dx\right)^{1/s}$$

$$\le C|Q|^{-1/s}\left(\int_Q|(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f_1(x)|^r dx\right)^{1/r}$$

$$\le C|Q|^{-1/s}\|(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f\chi_Q\|_{L^r}$$

$$\le C\left(\frac{1}{|Q|}\int_Q|(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)|^{pr/(p-r)}\,dx\right)^{(p-r)/pr} d^{-n(1/p-\delta/n)}\|f\chi_Q\|_{L^p}$$

$$\le C\|\vec{b}\|_{BMO}\|f\|_{B_p^\delta}.$$

For $H_4(x)$, Taking $1<p<n/\delta$, , by the Hölder's inequality and Lemma 1,2,3, we have

$$H_4(x)\le\left[\iint_{R_+^{n+1}}\left(\left|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}-\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}\right|\right.\right.$$

$$\left.\left.\times\int_{Q^c}\prod_{j=1}^m|b_j(z)-(b_j)_Q\|\psi_t(y-z)\|f(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right]^{1/2}$$

$$\le C\int_{Q^c}\prod_{j=1}^m|b_j(z)-(b_j)_Q\|f(z)\|x-x_0|^{1/2}$$

$$\times\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu+1}\frac{t^{-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}\right)^{1/2}dz$$

$$\le C\int_{Q^c}\prod_{j=1}^m|b_j(z)-(b_j)_Q\|f(z)\|x-x_0|^{1/2}\left(\int_0^\infty\frac{dt}{(t+|x-z|)^{2n+2-2\delta})}\right)^{1/2}dz$$

$$\le C|x-x_0|^{1/2}\sum_{k=0}^\infty\int_{2^{k+1}Q\setminus2^kQ}|x_0-z|^{-(n+1/2-\delta)}\prod_{j=1}^m|b_j(z)-(b_j)_Q\|f(z)|dz$$

$$\le C|x-x_0|^{1/2}\sum_{k=1}^\infty\int_{2^kQ}|x_0-z|^{-(n+1/2-\delta)}\prod_{j=1}^m|b_j(z)-(b_j)_Q\|f(z)|dz$$

$$\le C\sum_{k=1}^\infty2^{-k/2}\frac{1}{|2^kQ|^{1-\delta/n}}\int_{2^kQ}\left|\prod_{j=1}^m(b_j(z)-(b_j)_{2Q})\right||f(z)|dz$$

$$\le C\sum_{k=1}^\infty2^{-k/2}\frac{|2^kQ|^{1-1/p}}{|2^kQ|^{1-\delta/n}}\left(\frac{1}{|2^kQ|}\int_{2^kQ}\left|\prod_{j=1}^m(b_j(z)-(b_j)_{2Q})\right|^{p/(p-1)}dz\right)^{(p-1)/p}\left(\int_{2^kQ}|f(z)|^p\,dz\right)^{1/p}$$

$$\le C\sum_{k=1}^\infty2^{-k/2}|2^kQ|^{-(1/p-\delta/n)}\|\vec{b}\|_{BMO}\|f\chi_{2^kQ}\|_{L^p}$$

$$\le C\|\vec{b}\|_{BMO}\|f\|_{B_p^\delta},$$

so

$$\frac{1}{|Q|}\int_Q|H_4(x)|dx\le C\|\vec{b}\|_{BMO}\|f\|_{B_p^\delta}.$$

This completes the total proof of Theorem 2.

Theorem 3. Let $0<\delta<n$ and $\mu>3+(4-2\delta)/n$, $\vec{b}=(b_1,\cdots,b_m)$ with $b_j\in BMO(R^n)$ for

$1 \le j \le m$. If for any $H^1(R^n)-$atom a supported on certain cube Q and $u \in Q$, there is

$$\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\int_{(2Q)^c}\left[|\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left|\int_Q(\vec{b}(z)-\vec{b}_Q)_\sigma a(z)dz\right|^2\right.\right.$$
$$\left.\left.\times|\psi_t(y-u)|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\right]^{n(n-\delta)}dx\le C,$$

then $g_{\mu,\delta}^{\vec{b}}$ is bounded from $H^1(R^n)$ to $L^{n/(n-\delta)}(R^n)$.

Proof. Let a be an atom supported in some cube Q. We write

$$\int_{R^n}|g_{\mu,\delta}^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx=\int_{2Q}|g_{\mu,\delta}^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx+\int_{(2Q)^c}|g_{\mu,\delta}^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx=I+II.$$

For I, taking $1<p<n/\delta$ and $1/q=1/p-\delta/n$, we have

$$I\le\|g_{\mu,\delta}^{\vec{b}}(a)\|_{L^q}^{n/(n-\delta)}|2Q|^{1-n/((n-\delta)q)}\le C\|a\|_{L^p}^{n/(n-\delta)}|Q|^{1-n/((n-\delta)q)}\le C\|a\|_{L^\infty}^{n(n-\delta)}|Q|^{n(n-\delta)p}|Q|^{1-n(n-\delta)q}\le C$$

For II, we first calculate $F_t^{\vec{b}}(a)(x)$, we have

$$|F_t^{b_1}(a)(x,y)|=\left|\int_Q\psi_t(y-z)a(z)b_1(x)dz-\int_Q\psi_t(y-z)a(z)b_1(z)dz\right|$$
$$\le\left|\int_Q\psi_t(y-z)a(z)(b_1(x)-(b_1)_Q)dz\right|$$
$$+\left|\int_Q(\psi_t(y-z)-\psi_t(y-u))a(z)(b_1(z)-(b_1)_Q)dz\right|$$
$$+\left|\int_Q\psi_t(y-u)(b_1(z)-(b_1)_Q)a(z)dz\right|=v_1'+v_2'+v_3',$$

so

$$g_{\mu,\delta}^{b_1}(a)(x)\le\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|v_1'|^2\frac{dydt}{t^{n+1}}\right)^{1/2}+\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|v_2'|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$
$$+\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}|v_3'|^2\frac{dydt}{t^{n+1}}\right)^{1/2}=A'(x)+B'(x)+C'(x).$$

For $A'(x),$ we have

$$A'(x)=\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left|\int_Q\psi_t(y-z)a(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}|b_1(x)-(b_1)_Q|$$
$$=g_{\mu,\delta}(a)(x)|b_1(x)-(b_1)_Q|$$
$$=|b_1(x)-(b_1)_Q|\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left|\int_Q(\psi_t(y-z)-\psi_t(y-\mu))a(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$
$$\le C|b_1(x)-(b_1)_Q|\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left|\int_Q\frac{t|u-z\|a(z)|dz}{(t+|y-u|)^{n+2-\delta}}\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$

$$\leq C\,|\,b_1(x)-(b_1)_Q\,|\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{t^{1-n}dydt}{(t+|y-u|)^{2(n+2-\delta)}}\right)^{1/2}\left(\int_Q|u-z\,\|\,a(z)\,|\,dz\right).$$

Set $B=B(x,t)$, note that

$$t^{-n}\int_{R^n}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y-u|)^{2(n+2-\delta)}}$$
$$\leq t^{-n}\left(\int_{B(x,t)}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y-u|)^{2(n+2-\delta)}}\right.$$
$$\left.+\sum_{k=1}^{\infty}\int_{2^kB\backslash 2^{k-1}B}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{dy}{(t+|y-u|)^{2(n+2-\delta)}}\right)$$
$$\leq Ct^{-n}\left(t^n+\sum_{k=1}^{\infty}2^{-kn\mu}2^{2k(n+2-\delta)}(2^kt)^n\right)\frac{1}{(t+|x-u|)^{2(n+2-\delta)}}$$
$$\leq C\frac{1}{(t+|x-u|)^{2(n+2-\delta)}}$$

and

$$\int_0^{\infty}\frac{tdt}{(t+|x-\mu|)^{2(n+2-\delta)}}=C\,|\,x-\mu\,|^{-2(n+1-\delta)},$$

we obtain

$$A'(x)\leq C\,|\,b_1(x)-(b_1)_Q\,|\left(\int_0^{\infty}\frac{tdt}{(t+|x-u|)^{2(n+2-\delta)}}\right)^{1/2}\cdot|\,Q\,|^{1+1/n}\|\,a\,\|_{L^{\infty}}$$

$$\leq C\,|\,b_1(x)-(b_1)_Q\,\|\,x-u\,|^{-(n+1-\delta)}\cdot|\,Q\,|^{1+1/n}\|\,a\,\|_{L^{\infty}},$$

thus

$$\left(\int_{(2Q)^c}(A'(x))^{n/(n-\delta)}dx\right)^{(n-\delta)/n}$$
$$\leq C\,|\,Q\,|^{1+1/n}\|\,a\,\|_{L^{\infty}}\sum_{k=1}^{\infty}\left[\int_{2^{k+1}Q\backslash 2^kQ}\left(|\,b_1(x)-(b_1)_Q\,\|\,x-u\,|^{-(n+1-\delta)}\right)^{nn-\delta}dx\right]^{n-\delta n}\cdot$$
$$\leq C\,|\,Q\,|^{-1}|\,Q\,|^{1+1/n}\sum_{k=1}^{\infty}|\,2^kQ\,|^{-(1+1/n-\delta/n)}\left(\int_{2^{k+1}Q}|\,b_1(x)-(b_1)_Q\,|^{n/(n-\delta)}\,dx\right)^{(n-\delta)/n}\cdot$$
$$\leq C\,\|\,b_1\,\|_{BMO}\sum_{k=1}^{\infty}k2^{-k}$$
$$\leq C\,\|\,b_1\,\|_{BMO}\,.$$

For $B'(x)$, we have

$$B'(x)=\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left|\int_Q(\psi_t(y-z)-\psi_t(y-u))a(z)(b_1(z)-(b_1)_Q)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$

$$\le C\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left(\int_Q\frac{t|u-z|}{(t+|y-u|)^{n+2-\delta}}|a(z)||b_1(z)-(b_1)_Q|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}$$

$$\le C\int_Q|u-z||a(z)||b_1(z)-(b_1)_Q|dz\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{t^{1-n}dydt}{(t+|y-u|)^{2(n+2-\delta)}}\right)^{1/2}$$

$$\le C\|b_1\|_{BMO}|Q|^{1+1/n}\|a\|_{L^\infty}\left(\int_0^\infty\frac{tdt}{(t+|x-u|)^{2(n+2-\delta)}}\right)^{1/2}$$

$$\le C\|b_1\|_{BMO}|x-u|^{-(n+1-\delta)}|Q|^{1+1/n}\|a\|_{L^\infty},$$

thus

$$\left(\int_{(2Q)^c}(B'(x))^{n/(n-\delta)}dx\right)^{(n-\delta)/n}$$

$$\le C\|b_1\|_{BMO}|Q|^{1+1/n}\|a\|_{L^\infty}\sum_{k=1}^\infty\left[\int_{2^{k+1}Q\setminus 2^kQ}\left(|x-u|^{-(n+1-\delta)}\right)^{nn-\delta}dx\right]^{n-\delta n}$$

$$\le C\|b_1\|_{BMO}|Q|^{1+1/n}\|a\|_{L^\infty}\sum_{k=1}^\infty|2^kQ|^{-(1+1/n-\delta/n)}\left(\int_{2^{k+1}Q}dx\right)^{n-\delta n}$$

$$\le C\|b_1\|_{BMO}|Q|^{1+1/n}|Q|^{-1}\sum_{k=1}^\infty 2^{-k}|Q|^{-1/n}$$

$$\le C\|b_1\|_{BMO}\sum_{k=1}^\infty 2^{-k}$$

$$\le C\|b_1\|_{BMO}.$$

From that we know, if

$$\int_{(2Q)^c}\left[\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left|\int_Q(b_1(z)-(b_1)_Q)a(z)dz\right|^2|\psi_t(y-u)|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\right]^{n(n-\delta)}dx\le C,$$

then

$$\int_{R^n}|g_{\mu,\delta}^{b_1}(a)(x)|^{n/(n-\delta)}dx\le C.$$

This completes the proof of the case $m=1$.
When $m>1$, we have

$$|F_t^{\vec b}(a)(x,y)|\le\left|\prod_{j=1}^m(b_j(x)-(b_j)_Q)\int_Q\psi_t(y-z)a(z)dz\right|$$

$$+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\left|(\vec b(x)-\vec b_Q)_{\sigma^c}\int_Q(\psi_t(y-z)-\psi_t(y-u))(\vec b(z)-\vec b_Q)_\sigma a(z)dz\right|$$

$$+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\left|(\vec b(x)-\vec b_Q)_{\sigma^c}\int_Q\psi_t(y-u)(\vec b(z)-\vec b_Q)_\sigma a(z)dz\right|$$

$$=\nu_1+\nu_2+\nu_3,$$

so

$$g_{\mu,\delta}^{\vec{b}}(a)(x) \le \left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu} |\nu_1|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} + \left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu} |\nu_2|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

$$+\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu} |\nu_3|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} = A(x)+B(x)+C(x).$$

For $A(x),$ we have

$$A(x) = \left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu} \prod_{j=1}^{m} |b_j(x)-(b_j)_Q|^2 \left|\int_Q \psi_t(y-z)a(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

$$= \prod_{j=1}^{m} |b_j(x)-(b_j)_Q| \, g_{\mu,\delta}(a)(x)$$

$$= \left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu} \left|\int_Q (\psi_t(y-z)-\psi_t(y-u))a(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} \prod_{j=1}^{m} |b_j(x)-(b_j)_Q|^2$$

$$\le C\prod_{j=1}^{m} |b_j(x)-(b_j)_Q| \, |x-u|^{-(n+1-\delta)} |Q|^{1+1/n} \|a\|_{L^\infty}$$

thus, similar to the proof of $A'(x)$,

$$\left(\int_{(2Q)^c} (A(x))^{n/(n-\delta)} dx\right)^{(n-\delta)/n}$$

$$\le C\|a\|_{L^\infty} |Q|^{1+1n} \sum_{k=1}^{\infty}\left[\int_{2^{k+1}Q\setminus 2^k Q}\left(\frac{1}{|x-u|^{n+1-\delta}}\prod_{j=1}^{m}|b_j(x)-(b_j)_Q|\right)^{n/(n-\delta)} dx\right]^{(n-\delta)/n}$$

$$\le C|Q|^{1+1/n}\|a\|_{L^\infty} \sum_{k=1}^{\infty} |2^k Q|^{-(1+1-\delta n)} \left(\int_{2^{k+1}Q}\left(\prod_{j=1}^{m}|b_j(x)-(b_j)_Q|\right)^{n/(n-\delta)} dx\right)^{(n-\delta)/n}$$

$$\le C|Q|^{1+1/n}|Q|^{-1} \sum_{k=1}^{\infty} 2^{-k} |Q|^{-1/n} \left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left(\prod_{j=1}^{m}|b_j(x)-(b_j)_Q|\right)^{n/(n-\delta)} dx\right)^{(n-\delta)/n}$$

$$\le C\sum_{k=1}^{\infty} 2^{-k} k^m \|\vec{b}\|_{BMO}$$

$$\le C\|\vec{b}\|_{BMO} \, .$$

For $B(x),$ we have

$$B(x) \le C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m} |(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|$$

$$\times\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left(\int_Q \frac{t|u-z|}{(t+|y-u|)^{n+2-\delta}} |(\vec{b}(z)-\vec{b}_Q)_\sigma| \, |a(z)| \, dz\right)^2 \frac{dydt}{t^{n+1}}\right)^{1/2}$$

$$\le C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec b(x)-\vec b_Q)_{\sigma^c}|$$

$$\times\int_Q|u-z\|a(z)\|(\vec b(z)-\vec b_Q)_\sigma|dz\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\frac{t^{1-n}dydt}{(t+|y-u|)^{2(n+2-\delta)}}\right)^{1/2}$$

$$\le C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec b(x)-\vec b_Q)_{\sigma^c}\|Q|^{1+1/n}\|a\|_{L^\infty}\|\vec b_\sigma\|_{BMO}\left(\int_0^\infty\frac{tdt}{(t+|x-u|)^{2(n+2-\delta)}}\right)^{1/2}$$

$$\le C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec b(x)-\vec b_Q)_{\sigma^c}|\cdot|x-u|^{-(n+1-\delta)}|Q|^{1+1/n}\|a\|_{L^\infty}\|\vec b_\sigma\|_{BMO},$$

thus

$$\left(\int_{(2Q)^c}(B(x))^{n/(n-\delta)}dx\right)^{(n-\delta)/n}$$

$$\le C\left[\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left(\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec b(x)-\vec b_Q)_{\sigma^c}|\cdot|x-u|^{-(n+1-\delta)}\right.\right.$$

$$\left.\left.\times|Q|^{1+1/n}\|a\|_{L^\infty}\|\vec b_\sigma\|_{BMO}\right)^{n/(n-\delta)}dx\right]^{(n-\delta)/n}$$

$$\le C|Q|^{1+1/n}\|a\|_{L^\infty}\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\|\vec b_\sigma\|_{BMO}$$

$$\times\sum_{k=1}^{\infty}\left(\int_{2^{k+1}Q\setminus 2^kQ}\left(|x-u|^{-(n+1-\delta)}|(\vec b(x)-\vec b_Q)_{\sigma^c}|\right)^{n/(n-\delta)}dx\right)^{(n-\delta)/n}$$

$$\le C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\|\vec b_\sigma\|_{BMO}|Q|^{1+1/n}\|a\|_{L^\infty}\sum_{k=1}^{\infty}|2^kQ|^{-(1+1-\delta n)}\left(\int_{2^{k+1}Q}\left(|(\vec b(x)-\vec b_Q)_{\sigma^c}|\right)^{n/(n-\delta)}dx\right)^{(n-\delta)/n}$$

$$\le C\|\vec b\|_{BMO}\sum_{k=1}^{\infty}k^m2^{-k}$$

$$\le C\|\vec b\|_{BMO},$$

so, if

$$\int_{(2Q)^c}(C(x))^{n/(n-\delta)}dx$$

$$\le\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\int_{(2Q)^c}\left[|\vec b(x)-\vec b_Q)_{\sigma^c}|\left(\iint_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu}\left|\int_Q(\vec b(z)-\vec b_Q)_\sigma a(z)dz\right|^2\right.\right.$$

$$\left.\left.\times|\psi_t(y-u)|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\right]^{n(n-\delta)}dx\le C,$$

then

$$\int_{R^n}|g_{\mu,\delta}^{\vec b}(a)(x)|^{n/(n-\delta)}dx\le C.$$

This completes the total proof of the Theorem 3.

3. ACKNOWLEDGEMENTS
The first author is grateful to Prof. Lanzhe Liu at Hunan University for his comments and suggestions. The second author was supported by Doctor Foundation of University of South China (Grant No. 2011XQD53) and Hunan Provincial Natural Science Foundation of China (Grant No. 13JJ4075).

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