SOME PROPERTIES FOR THE MARKOV PROCESSES OF ORNSTEIN-UHLENBECK TYPE*

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ABSTRACT
In this paper, we investigate the first passage problem and establish a general criterion for recurrence and transience of a Markov process of Ornstein-Uhlenbeck type $\{X_t, t \in R^+\}$.

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1. INTRODUCTION

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, and $M(R^d)$ be the totality of real $d \times d$ matrices whose all eigenvalues have positive real parts. The starting at x Markov process of Ornstein-Uhlenbeck (O-U) type $X = \{X_t, t \in R^+, P^x\}$ over $(\Omega, \mathfrak{F}, P)$ is a Feller process with infinitesimal generator

$$A = G - \sum_{j=1}^{d}\sum_{k=1}^{d} Q_{jk} x_k \frac{\partial}{\partial x_j},$$

where G is the infinitesimal generator of a Lévy process $Z = \{Z_t, t \in R^+\}$ taking values in R^d, $Q \in M(R^d)$ and $x \in R^d$. An equivalent definition of this process X is given by the unique solution of the equation

$$X_t = x - \int_0^t QX_s ds + Z_t,$$

which can be expressed as

$$X_t = e^{-tQ} x + \int_0^t e^{(s-t)Q} dZ_s, (1.1)$$

where the stochastic integral with respect to the Lévy process Z is defined by convergence in probability from integrals of simple functions. When Z is the Brownian motion taking values in R^d, X is the ordinary O-U process.

The study of Markov processes of O-U type keeps receiving much attention both in the physical and the mathematical literature, for example, in climate models to explain the so-called Dansgaard-Oeschger events--see [4] and the references therein.

Since a Markov process of O-U type X is determined by the Lévy process $\{Z_t, t \in R_+\}$ and the matrix Q, it is natural to ask how the properties of X are related to those of Z and Q. Several authors have studied the distributional properties of X. For example, Nourdin [8] considered the problem of absolute continuity for X taking values in R. Sato [6] has established criteria for positive recurrence and transience of X. A criterion for recurrence and transience of X was given by Shiga [7] as follows: **Theorem A.** Let $X = \{X_t, t \in R^+, P^x\}$ be a one dimensional Markov process of O-U type defined by (1.1), it is recurrent or transient according as

$$\int_0^1 \frac{1}{z} \exp\left(-\int_z^1 \frac{\lambda_\rho(y)}{Qy} dy\right) dz = \infty \ or \ < +\infty, (1.2)$$

where

$$\lambda_\rho(y) = \int_{|u|\ge 1} (1 - e^{-y|u|}) \rho(du),$$

and ρ is the Lévy measure. The criterion (1.2) depended only on ρ. This result may be right. Unfortunately, the proof of Theorem A in [7] seems to be wrong (see Section 3). Watanabe [9] on R^d when $Q \in M(R^d)$, where

$M(R^d)$ be the totality of real $d\times d$ matrices whose all eigenvalues have positive real parts. The main purpose of this paper is to investigate the first passage problem and establish a general criterion for recurrence and transience of a Markov process of Ornstein-Uhlenbeck type $\{X_t, t\in R^+\}$ on R^d .

The paper is organized as follows as follows. we will give some basic results about the Lévy process and X In section 2. In section 3, we study the first passage cross a lever. In Section 4, we establish completely general criteria for recurrence and transience of X in terms of Fourier analysis following Chung and Fuchs.

Throughout this paper, x is the starting point of the Markov process of O-U type. C always stands for a positive constant, whose value is irrelevant. The ordinary scalar product and the norm in R^d is denoted by $<\cdot,\cdot>$ and $\|\cdot\|$.

2. PRELIMINARIES

In this section, we collect some basic results about the Lévy process and the Markov process of O-U type which will be used throughout this paper.

Let $\{Z_t, t\in R^+\}$ be a Lévy process taking values in R^d, whose characteristic function is given by

$$E(e^{i<\theta,Z_t>}) = \exp\{-t\psi(\theta)\},$$

where

$$\psi(\theta) = \frac{<\sigma\theta,\theta>}{2} - i<\theta,m> - \int_{R^d}(e^{i<\theta,u>} - 1 - \frac{i<\theta,u>}{1+|u|^2})\rho(du)(2.1)$$

is called the Lévy exponent, $\sigma=(\sigma_{ij})_{1\le i,j\le d}$ is a symmetric, non-negative definite and real matrix, $m\in R^d$, and ρ is a measure on R^d satisfying that $\rho(\{0\})=0$ and the integrability condition

$$\int_{R^d}(1\wedge|u|^2)\rho(du)<\infty.$$

Certainly, the process Z is characterized by the triplet $(m,\sigma,\rho(\cdot))$.

Let $X=\{X_t, t\in R^+, P^x\}$ be a Markov process of O-U type defined by (1.1). The following result is due to [3].

Proposition 2.1. The characteristic function of X_t is

$$E^x(e^{i<\theta,X_t>}) = \exp\{i<x,e^{-tQ^*}\theta> - \int_0^t\psi(e^{-(t-s)Q^*}\theta)ds\},(2.3)$$

where ψ is given in (2.1) and Q^* stands for the transposed matrix of Q.

To begin, we introduce some definitions following [2].

Definition 2.1. (i) For every measurable function $f\ge 0$, define

$$U^q f(x) = E^x(\int_0^\infty e^{-qt}f(X_t)dt).(2.5)$$

We say that U^q is the resolvent operator of X .

(ii) For every $x\in R^d$, $A\in \mathcal{B}(R^d)$, define

$$U(x,A) = \int_0^\infty P^x(X_t\in A)dt = E^x(\int_0^\infty I_{\{X_t\in A\}}dt).(2.6)$$

We say that $U(x,A)$ is a potential measure of A.

Definition 2.2. (i) We say that the process X is recurrence if $U(0,B)=\infty$ for every open ball B centered at the origin.

(ii) We say that X is transience if for every compact set K,

$$U(x,K)<\infty,\quad x\in R^d.$$

A process can not be both transient and recurrent, hence if there exists $\varepsilon>0$ such that $U(0,B)<\infty$ where B is the open ball centered at 0 with radius ε, then X is transient.

3. THE FIRST PASSAGE CROSS A LEVER

Let $X=\{X_t, t\in R^+, P^x\}$ be one dimensional Markov process of O-U type defined by (1.1) taking values in R and $Q>0$. Given a real number $a>x$, let us introduce the first passage time strictly above a, $T_a=\inf\{t\geq 0: X_t>a\}$, and let $\sigma_a=\inf\{t\geq 0: X_t=a\}$ provided that the sets in braces is not empty, and $+\infty$ otherwise.

When Z is a Lévy process with non-positive jumps, $\Delta X_t=\Delta Z_t\leq 0$. If $T_a<\infty$, one gets immediately

$$X_{T_a}=a.(3.1)$$

Using martingle technique, Hadjiev [3] proved that

$$E\exp\{-\theta T_a\}=\frac{\int_0^\infty y^{\theta/Q-1}\exp\{xy+g(y)\}dy}{\int_0^\infty y^{\theta/Q-1}\exp\{ay+g(y)\}dy},\ \theta>0,(3.2)$$

where

$$g(y)=Q^{-1}\int_1^y u^{-1}\psi(iu)du,\quad y>0.$$

When Z is a Lévy process with positive jumps, does the similar property (3.1) hold? We will prove that the answer is negative.

Lemma 3.1. Let $X=\{X_t, t\in R^+, P^x\}$ be a Markov process of O-U type defined by (1.1). Then for every $x\neq 0$ and $y\in R^d$, the potential measure of X is diffuse, that is,

$$U(x,\{y\})=0.$$

Proof. Since

$$X_t=e^{-tQ}x+\int_0^t e^{(s-t)Q}dZ_t$$

and the distribution of Z is a diffuse except when Z is a compound Poisson process (cf. [5]), for every $x\neq 0$,

$$P^x\{X_t=y\}=0,$$

which implies

$$U(x,\{y\})=\int_0^\infty P^x\{X_t=y\}dt=0.$$

Theorem 3.1. Let $X=\{X_t, t\in R^+, P^x\}$ be a Markov process of O-U type defined by (1.1). If $\rho(-\infty,0)=0$, we have

$$P^x\{X_{T(a)-}<a=X_{T(a)}\}=0.$$

Proof. Let $f,g\geq 0$ be two Borel functions with $f(a)=0$. Applying the compensation formula and recalling $\Delta X_t=\Delta Z_t\geq 0$, we have

$$\begin{aligned}
&E^x(f(X_{T(a)-})g(X_{T(a)}))\\
&=E^x(\sum_{t\geq 0}f(X_{t-})g(X_{t-}+\Delta X_t)I_{\{a-\Delta X_t\leq X_{t-}<a\}})\\
&=\int_0^\infty dtE^x(f(X_{t-})I_{\{X_{t-}<a\}}\int_0^\infty g(X_{t-}+s)I_{\{s\geq a-X_{t-}\}}\rho(ds))\\
&=\int_0^\infty dt\int_{0\leq y<a, s\geq a-y}f(y)g(y+s)P^x\{X_t\in dy\}\rho(ds)
\end{aligned}$$

$$= \int_{0\le y<a\le z} f(y)g(z)U(x,dy)\rho(dz-y).$$

Taking $f = I_{[0,a)}$ and $g = I_{\{a\}}$, we obtain

$$P^x\{X_{T(a)-} < a = X_{T(a)}\} = \int_{[0,a)} U(x,dy)\rho(\{a-y\}).(3.3)$$

There are at most countably many $y \in [0,a)$ with $\rho(\{a-y\}) > 0$ and $\rho(\{0\}) = 0$. Moreover the potential measure is diffuse by Lemma 3.1. Hence the right-hand side of (3.3) is zero.

We deduce from Theorem 3.1 that X is a.s. continuous at time $T(a)$ on the event $X_{T(a)} = a$, so $P\{X_{T(a)-} = a \mid X_{T(a)} = a\} = 1$ on $P\{X_{T(a)} = a\} > 0$ and $P\{X_{T(a)} > a \mid X_{T(a)-} < a\} = 1$ on $P\{X_{T(a)-} < a\} > 0$.

It is well known from [1] that we can write $Z_t = bt + \sigma W_t + Z_s^1$, where bt is a drift, W_t is the Brownian motion and Z_s^1 is a Lévy process of pure jumps type. Hence X has the following decomposition:

$$X_t = e^{-tQ}x + b\int_0^t e^{(s-t)Q}ds + \sigma\int_0^t e^{(s-t)Q}dW_s + \int_0^t e^{(s-t)Q}dZ_s^1.(3.4)$$

Theorem 3.2. Assume that $x = b = \sigma = 0$ in (3.4), if

$$\rho(-\infty,0) = 0 \; and \; \int_0^1 x\rho(dx) = C < +\infty, (3.5)$$

then $X_{T(a)} > a$ a.s.

Proof. Note that $0 < e^{(s-t)Q} < 1$ for $s < t$, X gets its supremum just by jumping, that is, $P\{X_{T(a)-} < a\} > 0$ by Theorem 3.1, the theorem is proved.

Remark 3.1. Let $(v,+\infty)$ be the state space of X, under the condition (3.5), for every $\theta > 0$ and every $x \ge a > v$, Shiga [7] showed a formula similar to (3.2) (P_{434} of [7])

$$E^x \exp\{-\theta\sigma_a\} = \frac{\int_0^\infty z^{\theta-1}\exp\{(v-x)z + \int_1^z \frac{h(y)}{y}dy\}dz}{\int_0^\infty z^{\theta-1}\exp\{(v-a)z + \int_1^z \frac{h(y)}{y}dy\}dz}, (3.6)$$

where

$$h(y) = \int_0^\infty (1-e^{-yu})\rho(du).$$

Unfortunately, the proof of (3.6) in [7] has some problem. First, one can not prove $\sigma_a < \infty$, which is necessary. Second, if σ_a in (3.6) be replaced by T_a, (3.6) can not be proved under the conditions of Theorem 3.2, because the following properties for T_a are needed in the proof:

I $T_a < \infty$,

II $X_{T_a} = a.$

But $X_{T_a} > a$ by Theorem 3.2.

4. THE RECURRENCE OF THE MARKOV PROCESS OF O-U TYPE IN R^d

In this section, we will give a completely general criterion for transience and recurrence in terms of the characteristic function of X. The proof is based on Fourier analysis method due to Chung and Fuchs, that is quite standard. At first, we calculate Fourier transform of the resolvent operators of X.

Denote the Fourier transform of a function g in $L^1(R^d)$ by

$$\mathfrak{F}g(\xi) = \int_{R^d} e^{i<\xi,x>} g(x)dx \quad (\xi \in R^d).$$

It is well-known that $\det e^A = e^{trA}$ where A is a $d \times d$ matrices, recalling (1.1), (2.5) and (2.4), for every $f \in L^1 \cap L^\infty$, we have for every $\xi \in R^d$

$$\begin{aligned}
\mathfrak{F}(U^q f)(\xi) &= \int_{R^d} e^{i<\xi,x>} E^x (\int_0^\infty e^{-qt} f(X_t)dt)dx \\
&= \int_{R^d} e^{i<\xi,x>} E^0 (\int_0^\infty e^{-qt} f(X_t + e^{-Qt}x)dt)dx \\
&= \int_0^\infty [e^{-qt} \int_{R^d} E^0 e^{i<\xi,x>} f(X_t + e^{-Qt}x)dx)]dt \\
&= \int_0^\infty [e^{-qt} \int_{R^d} E^0 e^{i<\xi,(y-X_t)e^{Qt}>} f(y) e^{trQ\cdot t} dy)]dt \\
&= \int_0^\infty [e^{-(q-trQ)t} E^0 e^{i<\xi,-e^{Qt}X_t>} \int_{R^d} e^{i<\xi,e^{Qt}y>} f(y)dy)]dt \\
&= \int_0^\infty e^{-(q-trQ)t} \exp\{-\int_0^t \psi(-e^{sQ}\xi)ds\} \mathfrak{F}f(e^{tQ}\xi)dt. \qquad (1)
\end{aligned}$$

Theorem 4.1 Let X be defined by (1.1) and ψ is the characteristic exponent of Z. X is transient if and only if for some $r > 0$ small enough,

$$\limsup_{q\to 0+} \int_{B_r} \int_0^\infty e^{-qt} \exp\{-\int_0^t \psi(-e^{-(t-s)Q}\xi)ds\} dt d\xi < \infty,$$

where B_r is the ball with radius r centered at the origin.

Proof. For the sake of simplicity, we will assume that the dimension of the space $d = 1$. Pick $r > 0$ arbitrarily small and let $f = I_{[-r,r]} * I_{[-r,r]}$, where $*$ denotes the convolution operator. Clearly $f \geq 0$, and f is a continuous function with support $[-2r,2r]$, and its Fourier transform is given by

$$\mathfrak{F}f(\xi) = [2\xi^{-1}\sin(r\xi)]^2 \quad (\xi \neq 0), \qquad \mathfrak{F}f(0) = 4r^2. \qquad (2)$$

Since the real part of $\int_0^t \psi(-e^{-sQ}\xi)ds$, $\mathfrak{Re}\int_0^t \psi(-e^{-sQ}\xi)ds \geq 0$ from (4.1), we have

$$|\mathfrak{F}(U^q f)(\xi)| \in L^1.$$

Applying the Fourier inversion and Fubini's theorem, by (1), we deduce that for every $q > 0$

$$\begin{aligned}
U^q f(0) &= \frac{1}{2\pi} \int_{-\infty}^\infty d\xi \int_0^\infty e^{-(q-Q)t} \exp\{-\int_0^t \psi(-e^{sQ}\xi)ds\} \mathfrak{F}f(e^{tQ}\xi)dt \\
&= \frac{1}{2\pi} \int_0^\infty e^{-(q-Q)t} dt \int_{-\infty}^\infty \exp\{-\int_0^t \psi(-e^{sQ}\xi)ds\} \mathfrak{F}f(e^{tQ}\xi)d\xi \\
&= \frac{1}{2\pi} \int_0^\infty e^{-qt} dt \int_{-\infty}^\infty \exp\{-\int_0^t \psi(-e^{-(t-s)Q}\theta)ds\} \mathfrak{F}f(\theta)d\theta.
\end{aligned}$$

Observe that $f \leq 2rI_{[-2r,2r]}$ and $U^q f(0)$ is a real number, by monotone convergence, we have

$$\begin{aligned}
2rU(0,[-2r,2r]) &\geq \lim_{q\to 0+} U^q f(0) \\
&= \frac{1}{2\pi} \lim_{q\to 0+} \int_0^\infty e^{-qt} dt \int_{-\infty}^\infty \exp\{-\int_0^t \psi(-e^{-(t-s)Q}\theta)ds\} \mathfrak{F}f(\theta)d\theta.
\end{aligned}$$

By (4.2) the latter quantity is infinite whenever

$$\lim_{q\to 0+}\int_{-r}^{r}d\theta\int_{0}^{\infty}e^{-qt}\exp\{-\int_{0}^{t}\psi(-e^{-(t-s)Q}\theta)ds\}dt=\infty,$$

and then X is recurrent. On the other hand, assume that for some $r>0$

$$\frac{1}{2\pi}\int_{0}^{\infty}e^{-qt}dt\int_{-2r}^{2r}\exp\{-\int_{0}^{t}\psi(-e^{-(t-s)Q}\theta)ds\}\mathfrak{F}f(\theta)d\theta$$

remains bounded as $q\to 0+$. Then consider the function

$$g(u)=[2u^{-1}\sin(ru)]^{2}\quad(u\neq 0),\qquad g(0)=4r^{2}.$$

By the preceding calculations, its Fourier transform is

$$\mathfrak{F}g(\xi)=2\pi I_{[-r,r]}\star I_{[-r,r]}.$$

Again by Fourier inversion and (1), we get

$$U^{q}g(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{-(q-Q)t}\exp\{-\int_{0}^{t}\psi(-e^{sQ}\xi)ds\}\mathfrak{F}g(e^{tQ}\xi)dtd\xi.$$

Using the inequality $g(u)\geq r^{2}$ whenever $|u|\leq\frac{\pi}{3r}$, we see by monotone convergence that

$$U(0,[-\frac{\pi}{3r},\frac{\pi}{3r}])<\infty.$$

Hence X is transient.

Is X recurrent when Z is recurrent? The answer is negative. We see an interesting example which can be found in [6].

For $c>0$, let ρ^{c} be a bounded measure on R defined by

$$\rho^{c}(du)=\frac{c}{u(\log u)^{2}}du\quad for\ |u|\geq 2,$$

$$=0\qquad otherwise.$$

Let (X_{t}^{c},P^{x}) be a one dimensional process of O-U type on R associated with $Q>0$ and a pure jump Lévy process Z with the Lévy measure ρ^{c}. Then it is known in [6] that

(i) if $2c\leq Q$, then the process X^{c} is null recurrent, but

(ii) if $2c>Q$, then the process X^{c} is transient.

Of course, we can show Z is recurrence for all $c>0$ and $Q>0$.

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