THE ANTI-REFLEXIVE SOLUTIONS OF THE MATRIX EQUATION *AXB=C* IN MINKOWSKI SPACE $\mathcal{M}$

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ABSTRACT
In this paper we study the existence of anti-reflexive with respect to the generalized reflection anti-symmetric matrix $P^{\sim}$ and solution of the Matrix equation $AXB = C$ in Minkowski Space $\mathcal{M}$.

Keywords and Phrases: *Anti-Reflexive Matrix, Generalized Reflection Matrix, Matrix Equation, Generalized Inverse, Minkowski Inverse.*

AMS Subject Classifications: 65F30, 15A09.

1. INTRODUCTION

Let C^n be the space of complex n-tuples. We shall index the components of a complex vector in C^n from o to $n-1$. That is $u = (u_0, u_1, u_2, ...u_{n-1})$. Let G be the Minkowski metric matrix

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix}$$

and $G^2 = I_n$. Minkowski inner product on C^n is defined by $(u,v) = < u, Gv >$ where $< .,. >$ denotes the conventional Hilbert space inner product. A space with Minkowski inner product is called a Minkowski space denoted as $\mathcal{M}$. With respect to the Minkowski inner product the adjoint of a matrix $P \in C^{n\times n}$ is given by $P^{\sim} = GP^*G,$ where P^* is the usual Hermition adjoint.

Let $C^{n\times m}$ denote the set of all complex $n \times m$ matrices. I_n denotes the unit matrix of order n. A^* denotes the conjugate transpose matrix A of order $n \times m.$ The symbol A^g stands for an arbitrary generalized inverse of A.

2. PRELIMINARIES

Definition 2.1 [1] A^g is said to be a generalized inverse(g-inverse) of A if

$$AA^g A = A. \quad (1)$$

Definition 2.2 [4] A matrix $P \in C^{n\times n}$ is called a generalized reflection matrix if $P^* = P$ *and* $P^2 = I.$

Definition 2.3 [7] A matrix $X \in C^{n\times m}$ is said to be reflexive with respect to P if

$$X \in C^{n\times m}_{rp} = \{X/PX = X, X \in C^{n\times m}\},$$

and a matrix $X \in C^{n\times m}$ is said to be anti-reflexive with respect to P if

$$X \in C^{n\times m}_{ap} = \{X/PX = -X, X \in C^{n\times m}\}.$$

Definition 2.4 A matrix $P \in C^{n\times n}$ is called a generalized reflection anti-symmetric matrix in Minkowski space $\mathcal{M}$ if $GPG = -P^*$ *and* $(GPG)^2 = I.$

Chen and Sameh B, H.C.Chen $C1$ introduced the following two special classes of subspaces in $C^{n\times n}$ as

$$C^{n\times n}_r(P) = \{A \in C^{n\times n} : A = PAP\},$$

$$C^{n\times n}_a(P) = \{A \in C^{n\times n} : A = -PAP\},$$

where $P \in C^{n\times n}$ is a generalized reflection matrix in complex.

Here we introduce the following two special classes of subspaces in $C^{n\times n}$ as

$$C_r^{n\times n}(P^{\sim}) = \{A \in C^{n\times n} : A = P^{\sim} A P^{\sim}\},$$
$$C_a^{n\times n}(P^{\sim}) = \{A \in C^{n\times n} : A = -P^{\sim} A P^{\sim}\},$$

where $P^{\sim} \in C^{n\times n}$ is a generalized reflection anti-symmetric matrix in Minkowski space $\mathcal{M}$ and $P^{\sim} = GP^{*}G$. By definition (2.3) $GPG = -P^{*}$, this implies that $GP^{*}G = -P.$ Therefore $P^{\sim} = -P.$

A matrix $X \in C_a^{n\times n}(p^{\sim})\,(X \in C_r^{n\times n}(p^{\sim}))$ is anti-reflexive (reflexive) matrix, with respect to the generalized reflection anti-symmetric matrix $P^{\sim}$.

The basic aim is to find the necessary and sufficient conditions of E, F for the existence of a solution to the matrix equation

$$AX = B. \tag{2}$$

such that X belongs to some special class of matrices.

In this paper we will consider the anti-reflexive solutions of the matrix equation

$$AXB = C. \tag{3}$$

From now on, by anti-reflexive solution we mean anti-reflexive with respect to a generalized reflection anti-symmetric matrix $P^{\sim}$. The generalization from the equation (2) to (3) is non-trivial.

3. MAIN RESULTS

For generalized reflection anti-symmetric matrix $P^{\sim}$ there exists unitary matrix $U = [U_1 \;\; U_2]$, where $U_1 \in C^{n\times r}, U_2 \in C^{n\times n-r}$ such that

$$P^{\sim} = U\begin{bmatrix} -I_r & 0 \\ 0 & I_{n-r} \end{bmatrix}U^{*}. \tag{4}$$

Throughout this paper we will assume that a generalized reflection anti-symmetric matrix $P^{\sim}$ is represented by (4). The next lemma gives a necessary and sufficient condition for X to be in $C_a^{n\times n}(P^{\sim})$.

Lemma 3.1 The matrix $X \in C_a^{n\times n}(P^{\sim})$ if and only if X can be expressed as

$$X = U\begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix}U^{*}$$

where $M \in C^{r\times(n-r)}, N \in C^{(n-r)\times r}$.

proof:

$$Let \;\; X = U\begin{bmatrix} E & M \\ N & F \end{bmatrix}U^{*} \in C_a^{n\times n}(P^{\sim}), \tag{5}$$

where $M \in C^{r\times(n-r)}$ *and* $N \in C^{(n-r)\times r}$.

We claim that $P^{\sim}XP^{\sim} = -X$.

$$P^{\sim}XP^{\sim} = U\begin{bmatrix} -I_r & 0 \\ 0 & I_{n-r} \end{bmatrix}U^{*}U\begin{bmatrix} E & M \\ N & F \end{bmatrix}U^{*}U\begin{bmatrix} -I_r & 0 \\ 0 & I_{n-r} \end{bmatrix}U^{*} = U\begin{bmatrix} E & -M \\ -N & F \end{bmatrix}U^{*}$$

$$That\ is\ \ P^{\sim}XP^{\sim} = U\begin{bmatrix} E & -M \\ -N & F \end{bmatrix}U^{*}.$$

It follows that

$$\begin{bmatrix} E & -M \\ -N & F \end{bmatrix} = \begin{bmatrix} E & M \\ N & F \end{bmatrix}$$

which implies that $E=0$ and $F=0$. The other direction is trivial.

Lemma 3.2 The matrix $X \in C_r^{n\times n}(P^{\sim})$ if and only if X can be expressed as

$$X = U\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}U^*,$$

where $M \in C^{r\times r}, N \in C^{(n-r)\times(n-r)}$ *and* U is the same as (4).

Without loss of generality we will suppose that the matrices $A \in C^{m\times n}$, $B \in C^{n\times p}$ and $C \in C^{m\times p}$ have the following decompositions

$$A = U\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}U^*, B = U\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}U^*, C = U\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}U^* \qquad (6)$$

where $A_1, B_1, C_1 \in C^{r\times r}$ *and* $A_4 \in C^{(m-r)\times(n-r)}, B_4 \in C^{(n-r)\times(p-r)}, C_4 \in C^{(m-r)\times(p-r)}$.

Theorem 3.3 The matrix equation $AXB = C$ has a solution $X \in C_a^{n\times n}(P^{\sim})$ if and only if the following system of the matrix equation has a solution

$$A_2NB_1 + A_1MB_3 = C_1,\ A_2NB_2 + A_1MB_4 = C_2,$$
$$A_4NB_1 + A_3MB_3 = C_3,\ A_4NB_2 + A_3MB_4 = C_4. \qquad (7)$$

In this case, the solution is represented by

$$X = U\begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix}U^*.$$

Proof: First suppose that the matrix equation (3) has a solution $X \in C_a^{n\times n}(P^{\sim})$. By lemma 3.1 there exists $M \in C^{r\times(n-r)}, N \in C^{(n-r)\times r}$, such that

$$X = U\begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix}U^*$$

Using (6) and the decompositions of the matrices A, B *and* C, we get

$$\begin{bmatrix} A_2NB_1 + A_1MB_3 & A_2NB_2 + A_1MB_4 \\ A_4NB_1 + A_3MB_3 & A_4NB_2 + A_3MB_4 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

If the system of the matrix equation (6) has a solution then

$$X = U\begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix}U^* \in C_a^{n\times n}(P^{\sim}) \text{ and } AXB = C.$$

Lemma 3.4 The matrix equation $AXB = C$ has a solution $X \in C_r^{n\times n}(P^{\sim})$ if and only if the following system of the matrix equation has a solution

$$A_1MB_1 + A_2NB_3 = C_1,\ A_1MB_2 + A_2NB_4 = C_2,$$
$$A_3MB_1 + A_4NB_3 = C_3,\ A_3MB_2 + A_4NB_4 = C_4.$$

In this case, the solution is represented by

$$X = U\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}U^*.$$

Lemma 3.5 [1] (1) If $B_1 = 0\ and\ B_2 = 0$, then the matrix equation $AXB = C$ has a solution $X \in C_a^{n\times n}(P^{\sim})$ if and only if $A^{'}A^{'g}C^{'}B^{'}B^{'g} = C^{'}, where\ A^{'} = [A_1\ \ A_3]^T, B^{'} = [B_3\ \ B_4], U^*CU = C^{'}$. In this case the general solution is represented by

$$X = U\begin{bmatrix} 0 & M \\ A^{'g}C^{'}B^{'g} + Y - A^{'g}A^{'}YB^{'}B^{'}g & 0 \end{bmatrix}U^*,$$

where $M \in C^{r\times(n-r)}\ and\ Y \in C^{(n-r)\times(n-r)}$ are arbitrary matrices.

(2) If $B_2 = 0\ and\ B_3 = 0,$ then the matrix equation $AXB = C$ has a solution $X \in C_a^{n\times n}(P^{\sim})$ if and only if $A^{'}A^{'g}C^{'}B^{'}B^{'g} = C^{'}\ and\ A^{''}A^{''g}C^{''}B^{''}B^{''g} = C^{''}$, where $A^{'} = [A_2\ \ A_4]^T, A^{''} = [A_1\ \ A_3]^T, C^{'} = [C_2\ \ C_4]^T, C^{''} = [C_1\ \ C_3]^T$. In this case the general solution is represented by

$$X = \begin{bmatrix} 0 & A^{'g}C^{'}B_1^g + Y - A^{'g}A^{'}YB_1B_1^g \\ A^{''g}C^{''}B_4^g + W - A^{''g}A^{''}WB_4B_4^g & 0 \end{bmatrix}U^*,$$

where $Y \in C^{r\times r}\ and\ W \in C^{(n-r)\times(n-r)}$ are arbitrary matrices.

Proof (1): Let $B_1 = 0, B_2 = 0, and\ X \in C_a^{n\times n}(P^{\sim})$ the solution of equation(3). We can assume that X is represented by(4). Now from $AXB = C$ it follows that

$$\begin{bmatrix} A_2NB_1 + A_1MB_3 & A_2NB_2 + A_1MB_4 \\ A_4NB_1 + A_3MB_3 & A_4NB_2 + A_3MB_4 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

$$Here\ \ B_1 = 0, B_2 = 0\ \ impliesthat \begin{bmatrix} A_1MB_3 & A_1MB_4 \\ A_3MB_3 & A_3MB_4 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

$$[A_1\ \ A_3]^T M[B_3\ \ B_4] = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

It is denoted by

$$A^{'} = [A_1\ \ A_3]^T, B^{'} = [B_1\ \ B_3]\ \ and\ \ U^*CU = U^*U\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}U^*U = C^{'}.$$

$$That\ \ is\ \ C^{'} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix},\ \ Therefore\ \ U^*CU = C^{'}.$$

It is well known that equation $A^{'}MB^{'} = C'$ has a solution if and only if $A^{'}A^{'g}C^{'}B^{'g}B^{'} = C'$. In this case the general solution is represented by $M = A^{'g}C^{'}B^{'g} + Y - A^{'g}A^{'}YB'B^{'g}$ for arbitrary $Y \in C^{(n-r)\times(n-r)}$.

Proof(2): Suppose that $B_1 = 0, B_2 = 0,$ *and* $X \in C_a^{n\times n}(P^{\sim})$ is the solution of equation(3). Then X is represented by(4). From $AXB = C,$ we obtain that

$$\begin{bmatrix} A_2 \\ A_4 \end{bmatrix} NB_1 = [C_1 \quad C_3]$$

$$[A_2 \quad A_4]^T NB_1 = [C_1 \quad C_3]$$

$$\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} MB_4 = [C_2 \quad C_4]$$

$$[A_1 \quad A_3]^T MB_4 = [C_2 \quad C_4]$$

Let $A^{'} = [A_2 \quad A_4]^T,\ A^{''} = [A_1 \quad A_3]^T,\ C^{'} = [C_1 \quad C_3],\ C^{''} = [C_2 \quad C_4]^T,$ solutions of these equation exists if and only if $A^{'}A^{'g}C^{'}B_1^{g}B_1^{g} = C^{'}$ *and* $A^{''}A^{''g}C^{''}B_4^{g}B_4 = C^{''}.$

$$A^{'}NB_1 = C^{'}A^{'g}A^{'}NB_1B_1^{g} = A^{'g}CB_1^{'}.$$

$$N = A^{'g}CB_1^{g}$$

$$That\ is\ \ N = A^{'g}CB_1^{g} + Y - A^{'g}A^{'}YB_1B_1^{g}.$$

$$A^{''}MB_4 = C^{''}$$

$$M = A^{''g}C^{''g}B_4^{g}$$

$$M = A^{''g}C^{''g}B_4^{g} + W - A^{''g}A^{''}WB_4B_4^{g}.$$

In this case the general solutions are represented by

$$N = A^{'g}C^{'}B^{'g} + Y - A^{'g}A^{'}YB'B'g,\quad M = A^{''g}C^{''g}B_4^{g} + W - A^{''g}A^{''}WB_4B_4^{g}.$$

for arbitrary $Y \in C^{(n-r)\times(n-r)}$ *and* $W \in C^{(n-r)\times(n-r)}.$

Now in the following theorem we will discuss about the anti-reflexive solutions of the matrix equation

$$AXB + CYD = E. \tag{8}$$

Theorem 3.6 Given $A,B,C,D,E \in C^{n\times n}$ and a generalized anti- reflection matrix $P^{\sim}$ of size n. Then the following conditions are equivalent.

1.The matrix equation $AXB + CYD = E$ has the anti-reflexive solutions

$$X, Y \in C_a^{n\times n}(P^{\sim}),$$

2.The following matrix equation has a solution

$A^{''}X_3B^{'} + A^{'}X_2B^{''} + C^{''}Y_3D^{'} + C^{'}Y_2D^{''} = E^{'}$ where

$$A^{'} = (A_1^t, A_3^t)^t, B^{'} = (B_1, B_2),\ A^{''} = (A_2^t, A_4^t)^t, B^{''} = (B_3, B_4),$$

$$C^{'} = (C_1^t, C_3^t)^t, D^{'} = (D_1, D_2),\ C^{''} = (C_2^t, C_4^t)^t, D^{''} = (D_3, D_4). \tag{9}$$

3.The following system of matrix equations has a solution

$$\begin{aligned} A_2X_3B_1 + A_1X_2B_3 + C_2Y_3D_1 + C_1Y_2D_3 &= E_1, \\ A_2X_3B_2 + A_1X_2B_4 + C_2Y_3D_2 + C_1Y_2D_1 &= E_2, \\ A_4X_3B_1 + A_3X_2B_3 + C_4Y_3D_1 + C_3Y_2D_3 &= E_3, \\ A_4X_3B_2 + A_3X_2B_4 + C_4Y_3D_2 + C_3Y_2D_4 &= E_4, \end{aligned} \tag{10}$$

In that case, the anti-reflexive solutions of the matrix equation $AXB + CYD = E$ can be expressed by the following

$$X = U\begin{bmatrix} 0 & X_2 \\ X_3 & 0 \end{bmatrix}U^* \ \ and \ \ Y = U\begin{bmatrix} 0 & Y_2 \\ Y_3 & 0 \end{bmatrix}U^*.$$

Proof : First we show that (2) implies (3).

$$A^{''}X_3B^{'} + A^{'}X_2B^{''} + C^{''}Y_3D^{'} + C^{'}Y_2D^{''} = E^{'}.$$

$$(A_2^t, A_4^t)^t X_3(B_1, B_2) + (A_1^t, A_3^t)^t X_2(B_3, B_4) + (C_2^t, C_4^t)^t Y_3(D_1, D_2) + (C_1^t, C_3^t)^t Y_2(D_3, D_4) = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}$$

$$\begin{aligned} A_2X_3B_1 + A_1X_2B_3 + C_2Y_3D_1 + C_1Y_2D_3 &= E_1, \\ A_2X_3B_2 + A_1X_2B_4 + C_2Y_3D_2 + C_1Y_2D_4 &= E_2, \\ A_4X_3B_1 + A_3X_2B_3 + C_4Y_3D_1 + C_3Y_2D_3 &= E_3, \\ A_4X_3B_2 + A_3X_2B_4 + C_4Y_3D_2 + C_3Y_2D_4 &= E_4. \end{aligned}$$

Therefore (2) implies (3). Now to show that (1) implies (3). (1) implies that the matrix equation $AXB + CYD = E$ has the anti-reflexive solutions $X, Y \in C_a^{n\times n}(P^{\sim})$. (8) implies that $AXB + CYD = E$. By a lemma, The matrix equation $A \in C_a^{n\times n}(P^{\sim})$ if and only if A can be expressed as

$$A = U\begin{bmatrix} 0 & A_2 \\ A_3 & 0 \end{bmatrix}U^*$$

where $A_2 \in C^{r\times(n-r)}, A_3 \in C^{(n-r)\times r}$. Suppose that the matrix equation $AXB + CYD = E$ has the anti-reflexive solutions $X \in C_a^{n\times n}(P^{\sim})$ and $Y \in C_a^{n\times n}(P^{\sim})$. By Lemma (3.1) there exists $X_2, Y_2 \in C^{(n-r)\times(n-r)}$ and $X_3, Y_3 \in C^{(n-r)\times(n-r)}$ such that

$$X = U\begin{bmatrix} 0 & X_2 \\ X_3 & 0 \end{bmatrix}U^* \text{ and } Y = U\begin{bmatrix} 0 & Y_2 \\ Y_3 & 0 \end{bmatrix}U^*.$$

Now using the decompositions (6) from $AXB + CYD = E,$ we can get

$$\begin{bmatrix} A_2X_3B_1 + A_1X_2B_3 + C_2Y_3D_1 + C_1Y_2D_3 & A_2X_3B_2 + A_1X_2B_4 + C_2Y_3D_2 + C_1Y_2D_4 \\ A_4X_3B_1 + A_3X_2B_3 + C_4Y_3D_1 + C_3Y_2D_3 & A_4X_3B_2 + A_3X_2B_4 + C_4Y_3D_2 + C_3Y_2D_4 \end{bmatrix} = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}.$$

If the system of matrix equations (9) has a solution, then

$$X = U\begin{bmatrix} 0 & X_2 \\ X_3 & 0 \end{bmatrix}U^*$$

$$, Y = U\begin{bmatrix} 0 & Y_2 \\ Y_3 & 0 \end{bmatrix}U^* \in C_a^{n\times n}(P^{\sim}) \text{ and } AXB + CYD = D.$$

4. CONCLUSION

In this paper we considered some special cases and exhibited a complete characterization of the set of anti-reflexive solutions. Also we obtained the necessary and sufficient conditions for the existence of anti-reflexive solution of equation(3).

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