DYNAMIC EQUATIONS WITH RATIONAL EXPECTATIONS ON TIME SCALES

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ABSTRACT
In this paper, we introduce a class of dynamic equations with rational expectations on time scales. We use the basics of time scale calculus and stochastic calculus to derive the general solution of the equation. We discuss existence of a unique solution for linear dynamic equations with rational expectations including an initial condition.

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1. INTRODUCTION

The theory of rational expectations is based on the idea that individuals or economic agents use currently available information to predict prices, interest rates, and even government policies. Dynamic models in economics which assume that agents form expectations rationally are commonly used in the literature [2, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18]. The idea of rational expectations were first proposed by John F. Muth in the early 1960s [3]. Then the theory has been developed intensively over the last three decades through articles and books by Sargent [4], Lucas and Sargent [5], and Hansen and Sargent [6].

We are motivated by the model called the Cagan's (1956) hyper-inflation model [9]. Cagan's model is given by the stochastic difference equation

$$y_t = aE[y_{t+1} \mid I_t] + cz_t, \tag{1}$$

where y_t is an endogenous variable and z_t is an exogenous variable, a and c are constant parameters and $E[y_{t+1} \mid I_t] \equiv E_t[y_{t+1}]$ is the subjective expectation formed by an economic agent. Following the rational expectation hypothesis, it is assumed that this expectation is identical to the conditional mathematical expectation of y_{t+1} with respect to all the information available at time t and included in I_t. The information set contains the observations on y_t, z_t and their past values. Consequently, it represents an increasing set with the property that $I_t \supset I_{t-1} \supset I_{t-2} \supset \cdots$. This implies that the economic agent has infinite memory.

Let $\mathbb{T}$ be a time scale, which is defined as a nonempty closed subset of the reals $\mathbb{R}$. Time scale calculus was first introduced by Hilger [7] and then developed by many mathematicians in order to unify and generalize the continuous and discrete calculus. In this paper, we assume that the reader is familiar with the time scale calculus. For further reading we refer the reader to an excellent book [1].

In this paper, we introduce the following dynamic equation with rational expectations on time scales ($\mathbb{T}$)

$$E_t[y_t^{\Delta}] = p(t)y_t + f(t, z_t), \tag{2}$$

where $t \in \mathbb{T}$ and $E[y_t^{\Delta} \mid I_t] \equiv E_t[y_t^{\Delta}]$ is the conditional expectation given all information I_t available at time t, and p is an rd-continuous function and regressive (i.e. $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$) and f is also an rd-continuous function.

The plan of the paper follows: In Section 2, we shall formalize the general solution of the dynamic equation (2). Then we shall illustrate the solution of the Cagan's model (1) as an example of our formulation. In Section 3, we shall give an example where one can see the non-uniqueness of an initial value problem in discrete time. We shall close this paper with a short conclusion section.

2. THE GENERAL SOLUTION

Let y_t be a random variable such that its Δ - derivative exists with respect to t. We define

$$A_t := \int E_t[y_t^\Delta]\Delta t - y_t.$$

Lemma 2.1 A_t is a stochastic process which satisfies the stochastic dynamic equation

$$E_t(A_t^\Delta) = 0.$$

Proof. Δ - derivative of A_t is

$$A_t^\Delta = E_t[y_t^\Delta] - y_t^\Delta.$$

Then applying the conditional expectation over the information set I_t on both sides, we have

$$E_t(A_t^\Delta) = E_t(E_t(y_t^\Delta)) - E_t(y_t^\Delta) = 0.$$

Theorem 2.1 y_t is the solution of the model (2) if and only if

$$y_t = e_p(t,0)X(t) + e_p(t,0)\int e_{\ominus p}(\sigma(t),0) f(t,z_t)\Delta t$$

where $t \in \mathbb{T}$ and X is an arbitrary stochastic process such that $E_t(X^\Delta(t)) = 0.$

Proof. Let y_t be the solution of the model (2). We multiply both sides of the equation (2) by the exponential function $e_{\ominus p}(\sigma(t),0)$, so we have

$$e_{\ominus p}(\sigma(t),0)E_t[y_t^\Delta] = e_{\ominus p}(\sigma(t),0)p(t)y_t + e_{\ominus p}(\sigma(t),0)) f(t,z_t).$$

This implies that

$$E_t[(e_{\ominus p}(t,0)y_t)^\Delta] = e_{\ominus p}(\sigma(t),0) f(t,z_t).$$

By integrating each side of the last equation we have

$$\int E_t[(e_{\ominus p}(t,0)y_t)^\Delta]\Delta t = \int e_{\ominus p}(\sigma(t),0) f(t,z_t)\Delta t.$$

Applying the result of Lemma 2.1 to the left hand side of the above equation we obtain

$$e_{\ominus p}(t,0)y_t = X_t + \int e_{\ominus p}(\sigma(t),0) f(t,z_t)\Delta t,$$

where X_t is a stochastic process such that $E_t(X^\Delta(t)) = 0.$

We divide each side of the above equality by $e_{\ominus p}(t,0)$. Hence we obtain the desired result.

Conversely, we show that

$$y_t = e_p(t,0)X(t) + e_p(t,0)\int e_{\ominus p}(\sigma(t),0)f(t,z_t)\Delta t$$

satisfies the equation (2). In fact, applying Δ derivative to each side of the above equation, we have

$$y^{\Delta}(t) = e_p(\sigma(t),0)X^{\Delta}(t) + p(t)e_p(t,0)X(t) + e_p(\sigma(t),0)e_{\ominus p}(\sigma(t),0)f(t,z_t)$$

$$+ p(t)e_p(t,0)\int e_{\ominus p}(\sigma(t),0)f(t,z_t)\Delta t,$$

which reduces to

$$y^{\Delta}(t) = e_p(\sigma(t),0)X^{\Delta}(t) + p(t)y(t) + f(t,z_t).$$

Applying conditional expectation E_t to each side, we obtain

$$E_t(y^{\Delta}(t)) = e_p(\sigma(t),0)E_t(X^{\Delta}(t)) + p(t)y(t) + f(t,z_t).$$

Since the arbitrary stochastic process X satisfies that $E_t(X^{\Delta}(t)) = 0$, we have

$$E_t(y^{\Delta}(t)) = p(t)y(t) + f(t,z_t).$$

Remark 2.1 For a time scale which has all its points as isolated, the condition $E_t(X^{\Delta}(t)) = 0$ on the arbitrary stochastic process X can be recognized as the martingale property in discrete time, i.e. $E_t(X(\sigma(t))) = X(t)$.

Example 2.1 For $\mathbb{T} = \mathbb{Z}$ and $p(t) = \dfrac{1-a}{a}$, the equation (2) has the form

$$y_t = aE_t[y_{t+1}] + g(t,z_t). \tag{3}$$

It follows from Theorem 2.1 that the solution of the equation (3) is

$$y_t = a^{-t}X(t) - a^{-t}\sum a^t g(t,z_t).$$

3. NON-UNIQUENESS OF SOLUTIONS

In deterministic setting, a linear dynamic equation with an initial condition produces a unique solution (see [1]). Here in stochastic setting, it is important to note that the dynamic equation (2) with an initial condition might not have a unique solution. Let's consider the following linear dynamic equation on $\mathbb{T} = \mathbb{Z}$

$$E_t[\Delta y_t] = ay_t \tag{4}$$

with initial condition $y_0 = 0,$ where a is constant.

Using Theorem 2.1 we obtain the solution as

$$y_t = (1+a)^t X(t),$$

where $X(t)$ is an arbitrary martingale with $E_t(X^{\Delta}(t)) = 0$.

By the initial condition we have $X(0) = 0.$ A zero stochastic process obviously satisfies $X(0) = 0$ and $E_t(X^{\Delta}(t)) = 0$. On the other hand, the discrete Poisson process martingale $N(t) - t$ is a nonconstant martingale with $N(0) = 0.$ As a conclusion, for any arbitrary stochastic process X, we may have more than one solution to the initial value problem (4).

4. CONCLUSION

Over the years, the rational expectation assumption has been used in many fields of economics, including finance, labor economics, and industrial organization. In dynamic modeling use of such an assumption produces stochastic difference equations. There are so many papers devoted to study these equations by many researchers. In this paper, for the first time, continuous time rational expectation models

$$E_t[y'_t] = p(t)y_t + f(t, z_t)$$

are introduced and solved. The stochastic dynamic equations on time scales will bring a new dimension to dynamic modeling techniques in economics.

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