EXPONENTIAL STABILITY OF NUMERICAL SOLUTIONS FOR A CLASS OF STOCHASTIC DELAY AGE-DEPENDENT CAPITAL SYSTEM WITH POISSON JUMPS

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ABSTRACT
Recently, numerical solutions of stochastic differential equations have received a great deal of attention. Numerical approximation schemes are invaluable tools for exploring its properties. In this paper, we introduce a class of stochastic age-dependent (vintage) capital system with Poisson jumps, and investigate the convergence of numerical approximation. It is proved that the the numerical approximation solutions converge to the analytic solutions of the equations under the given conditions. A numerical example is provided to illustrate the theoretical results.

Keywords: *Stochastic age-dependent capital system, Poisson jumps, Numerical solution, Euler approximation*
Subject classification: *(AMS) 60H10, 60H05, 60H35, 65C30*

1. INTRODUCTION

Recently, some pioneering works on the deterministic models of age-dependent (vintage) capital have been reported, see, for instance [6,7,8]. Then this deterministic models of age-dependent (vintage) capital may be described by:

$$\begin{cases} \frac{\partial K(a,t)}{\partial t}+\frac{\partial K(a,t)}{\partial a} & \\ = -\mu(a,t)K(a,t)+f(t,K(a,t)). & in\ \ Q, \\ K(0,t)=\phi(t)=\gamma(t)A(t)F(L(t),\int_0^A K(a,t)da), & in\ \ t\in[0,T], \\ K(a,0)=K_0(a), & in\ \ a\in(0,A), \\ N(t)=\int_0^A K(a,t)da, & in\ \ t\in[0.T], \end{cases} \quad (1)$$

$$\begin{cases} \frac{\partial K(a,t)}{\partial t}+\frac{\partial K(a,t)}{\partial a} & \\ = -\mu(a,t)K(a,t)+f(t,K(a,t)). & in\ \ Q, \\ K(0,t)=\phi(t)=\gamma(t)A(t)F(L(t),\int_0^A K(a,t)da), & in\ \ t\in[0,T], \\ K(a,0)=K_0(a), & in\ \ a\in(0,A), \\ N(t)=\int_0^A K(a,t)da, & in\ \ t\in[0.T], \end{cases} \quad (1)$$

where $Q=[0;A]\times[0;T]$; the stock of capital goods of age a at time t is denoted by $K(a,t)$, $N(t)$ is the total sum of the capital, a is the age of the capital, the investment $\phi(t)$ in the new capital, and the investment $f(t,K(a,t))$, in the capital of age a are the endogenous (unknown) variables. The maximum physical lifetime of capital A, the planning interval of calendar time $[0,T)$, the depreciation rate $\mu(a,t)$ of capital, and the capital density $k_0(a)$ (the initial distribution of capital over age) are given. $\gamma(t)$ denotes the accumulative rate at the moment of t; $0<\gamma(t)<1$, and $A(t)$ is the technical progress at the moment of t. Eq.(1) is a generalization of the deterministic capital equation. Eq.(1) describes the evolution of the composition of the productive capital as a function of purchasing/selling new or used capital. According to Eq.(1), machines of any age between 0 and A can be bought or sold.

The structure of $K(a,t)$ reflects different situations in economics and finance: its dependence on a illustrates the economic depreciation and physical deterioration of the capital, and represents the technological change embodied in capital. The case $K(a,t)/a > 0$ corresponds to a technical progress when new capital is more efficient. In economics such models are known as vintage capital models (VCMs). They represent a new prospective mathematical tool for modeling technological innovation. It is a fast growing area of research. Its strong impact on mathematical finance is motivated by efficient description of fundamental finance characteristics such as cost of capital, risk of investment decisions, dynamics of finance investments, market uncertainty, etc. The validity of VCMs on real data is provided, i.e., in [9,10].

Since time delay was first considered in the investment processes in [19], lots of literature such as have incorporated time lag into the dynamic economics and considered the impacts of delayed time on the whole economic system[20,21,22]. [20] analyzed an augmented IS-LM business cycle model with the capital accumulation equation that two time delays are considered in investment processes according to Kalecki's idea. Zak[21] investigated the Solow growth model with time lag, and considered that investment depended only on the capital stock at the past time and that the capital stock depreciated at the same gestation period, which it takes to produce and install capital goods.

In (1), $K(a,t)$ denotes the riskless capital. However, some important sources of uncertainty may be discontinuous, recurrent, and fluctuating. Such significant events include innovations in technique, introduction of new products, natural disasters, and changes in laws or government policies. The relationship among these events and the profitability of risky assets can be very complicated. Furthermore, there can be numerous events and economic variables that are potentially related to the profitability of risky assets. Since capital markets are incomplete, asset returns can have discontinuities of unpredictable size. This is true regardless of the number of securities available for trading. For simplicity, we consider only jump uncertainty in the market. Jump-diffusion uncertainty would then only add to the incompleteness. On the other hand, technological uncertainty is modeled as a Poisson arrival process that reduces the cost of investment, while revenue uncertainty is modeled as a diffusion process[4]. Since the financial market has one riskless and one risky asset. In order to describe this capital process situation, we suppose that the parameter $f(t,K)$ is stochastically perturbed with

$$f(t,K) + g(t,K)\frac{dW_t}{dt} + h(t,K)\frac{dN_t}{dt}.$$

where W_t is white noise, then this environmentally perturbed system may be described by the following equation

$$\begin{cases} \frac{\partial K(a,t)}{\partial t} + \frac{\partial K(a,t)}{\partial a} \\ = -\mu(a,t)K(a,t) + f(t,K(a,t),K(a,t-\tau)) \\ + g(t,K(a,t),K(a,t-\tau))\frac{dW_t}{dt} + h(t,K(a,t),K(a,t-\tau))\frac{dN_t}{dt} & in \quad Q, \\ K(0,t) = \phi(t) = \gamma(t)A(t)F(L(t),\int_0^A K(a,t)da), & in \quad t \in [0,T], \\ K(a,t) = \varphi(a,t), & in \quad (a,t) \in \overline{R}, \\ N(t) = \int_0^A K(a,t)da, & in \quad t \in [0.T], \end{cases} \qquad (2)$$

where $\overline{R} = [0,A] \times [-\tau,0]$, $f(t,K(a,t)) + g(t,K(a,t))\frac{dW_t}{dt} + h(t,K(a,t))\frac{dN_t}{dt}$ denotes effects of external environment for capital system, such as innovations in technique, introduction of new products, natural disasters, and changes in laws and government policies, and so on. The effects of external environment has the deterministic and random parts which depend on t and $K(a,t)$. $h(t,K(a,t))$ is a jump coefficient, N_t is a scalar Poisson process with intensity λ_t.

Eq.(1) is a generalization of the deterministic age-dependent capital system. A new stochastic age-dependent (vintage) capital system is given by model (2). It is an extension of equation (1). The effects of the stochastic environmental noise and delay considerations lead to stochastic age-dependent capital system (2), which are more realistic.

However, to the best of our knowledge, there are not any numerical methods available for stochastic partial differential equations with Poisson jumps. Thus, numerical approximation schemes are invaluable tools for exploring its properties. In this paper, we use the recent mathematical technique on the stochastic capital system to estimate its numerical solutions . Some mathematical results may be found in [5,6,7]. We shall extend the idea from the papers [8,9] to the numerical solutions for stochastic age-dependent capital system with Poisson jumps. The main purpose of this paper is to investigate the convergence of numerical approximation of stochastic age-dependent capital system with Poisson jumps under the given conditions. Our work differs from these references [1,2,3] in that (a) numerical analysis is considered, and (b) Poisson jumps is involved.

In Section 2, we shall collect some basic preliminaries results which are essential for our and the Euler approximation analysis, and Euler approximation is introduced. In section 3, we give the main result that the Euler method is exponential stable in mean square sense under some conditions, and the proof of this main result is completed. In section 4, A numerical example is provided to illustrate the theoretical results. Conclusion is given in section 5.

2. PRELIMINARIES AND APPROXIMATION

Throughout this paper, let

$$V = H^1([0,A]) \equiv \{\varphi \mid \varphi \in L^2([0,A]), \frac{\partial \varphi}{\partial x} \in L^2([0,A]),$$
$$where \frac{\partial \varphi}{\partial x} are\ generalized\ partial\ derivatives\}.$$

V is a Sobolev space. $H = L^2([0,A])$ such that

$$V \rightarrowtail H \equiv H' \rightarrowtail V'.$$

Then $V' = H^{-1}([0,A])$ the dual space of V. We denote by $|\cdot|$ and $\|\cdot\|$ the norms in V and V' respectively; by $\langle \cdot,\cdot \rangle$ the duality product between V, V', and by $(\cdot,\cdot)$ the scalar product in H. K is a real separable Hilbert space. For an operator $B \in \mathcal{L}(K,H)$ be the space of all bounded linear operators from K into H, we denote by $\|B\|_2$ the Hilbert-Schmidt norm, i.e.

$$\|B\|_2^2 = tr(BWB^T).$$

$\tau > 0$ and $D : D([0,A]\times[-\tau,0];H)$ denotes the family of all right-continuous functions with left-hand limits φ from $[0,A]\times[-\tau,0]$ to H ,The space $D([0,A]\times[-\tau,0];H)$ is assumed to be equipped with the norm $|\varphi|_D = \sup_{-\tau \le x \le 0} |\varphi|$. We also use $D^b_{\mathcal{F}_0}([0,A]\times[-\tau,0];H)$ to denote the family of all almost surely bounded, $\mathcal{F}_0$ -measurable, $D([0,A]\times[-\tau,0];H)$-valued random variables. Let $L^p_V = L^p([0,A]\times[0,T];V)$ and $L^p_H = L^p([0,A]\times[0,T];H)$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtrations $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). We assume that Poisson process N_t is independent of the Brownian motion $W(t)$.

Definition 2.1 *let* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ *be the stochastic basis and* W_t *a Wiener process. Suppose that* K_0 *is a random variable such that* $E|K_0|^2 < \infty$. *A stochastic process* $K_t \equiv K(a,t)$ *is said to be a solution on* Ω *to the stochastic age-structured capital system for* $t \in [0,T]$ *if the following conditions are satisfied:*

(1) K_t is a $\mathcal{F}_t$ -measurable random variable;

(2) $K_t \in I^p(0,T;V) \bigcap L^2(\Omega; C(0,T;V))$, $p > 1$, $T > 0$, where $I^p(0,T;V)$ denotes the space of all V-

valued processes $(K_t)_{t\in[0,T]}$ (we will write K_t for short) measurable (from $[0,T]\times\Omega$ into V), and satisfying

$$E\int_0^T \|K_t\|^p dt < \infty.$$

Here $C(0,T;V)$ denotes the space of all continuous functions from $[0,T]$ to V ;

(3) It satisfies the equation:

$$\begin{aligned}&\langle K_t, v\rangle + \int_0^t \langle \frac{\partial K_s}{\partial a}, v\rangle ds = (K_0, v) - \int_0^t \langle \mu(a,s) K_s, v\rangle ds + \int_0^t \langle f(s, K_s, K_{s-\tau}), v\rangle ds \\ &+ \int_0^t \langle g(s, K_s, K_{s-\tau})), v\rangle dW_s + \int_0^t \langle h(s, K_s, K_{s-\tau})) dN_s, v\rangle \end{aligned} \quad (3)$$

for all $v\in V$, $t\in[0,T]$, a.e. $W\in\Omega$, where the stochastic integral is understood in the It$\hat{o}$ sense.

We consider the convergence of the following stochastic age-structured capital system

$$\begin{cases} \frac{\partial K(a,t)}{\partial t} + \frac{\partial K(a,t)}{\partial a} \\ = -\mu(a,t)K(a,t) + f(t, K(a,t), K(a,t-\tau)) \\ + g(t, K(a,t), K(a,t-\tau))\frac{dW_t}{dt} + h(t, K(a,t), K(a,t-\tau))\frac{dN_t}{dt} & in \quad Q, \\ K(0,t) = \phi(t) = \gamma(t)A(t)F(L(t), \int_0^A K(a,t)da), & in \quad t\in[0,T], \\ K(a,t) = \varphi(a,t), & in \quad (a,t)\in \overline{R}, \\ N(t) = \int_0^A K(a,t)da, & in \quad t\in[0.T], \end{cases} \quad (4)$$

A is the maximal age of the capital, so

$$K(a,t) = 0, \quad \forall r \ge A.$$

Let $\Delta t = \frac{T}{N}$, for system (4) the discrete approximate solution on $t = 0, \Delta t$, $2\Delta t, \cdots, N\Delta t$ is defined by the iterative scheme

$$\begin{aligned}&Q_t^{n+1} - Q_t^n - \frac{\partial Q_t^{n+1}}{\partial a}\Delta t = -\mu(a,t)Q_t^n \Delta t \\ &+ f(t, Q_t^n, Q_t^{k-m})\Delta t + g(t, Q_t^n, Q_t^{k-m})\Delta W_n + h(t, Q_t^n, Q_t^{k-m})\Delta N_n,\end{aligned} \quad (5)$$

Here, Q_t^n is the approximation to $K(a,t_n)$, Q_t^{k-m} is the approximation to $K(a,t_k-\tau)$, for $t_n = n\Delta t$, the time increment is $\Delta t = \frac{T}{N} << 1$, with $\Delta W_n = W(t_{n+1}) - W(t_n)$ and $\Delta N_n = N(t_{n+1}) - N(t_n)$ denoting the increments of the Brownian motion and the Poisson processes,respectively.

For convenience, we shall extend the discrete numerical solution to continuous time. We first define the step function

$$\begin{aligned} z_1(t) &= \sum_{k=0}^{N-1} Q_t^k 1_{[k\Delta t, (k+1)\Delta t)}(t), \\ z_2(t) &= \sum_{k=0}^{N-1} Q_t^{k-m} 1_{[kh, (k+1)h)}(t), \end{aligned} \quad (6)$$

where 1_G is the indicator function for the set G . Then we define

$$\begin{aligned}&Q_t - K_0 + \int_0^t \frac{\partial Q_s}{\partial a} ds = -\int_0^t \mu(a,s) Z_s ds \\ &+ \int_0^t f(s, z_1(s), z_2(s)) ds + \int_0^t g(s, z_1(s), z_2(s)) dW_s + \int_0^t h(s, z_1(s), z_2(s)) dN_s,\end{aligned} \quad (7)$$

with $Q_0 = K(a,0)$, $Q_t = Q(a,t)$. First, we state the assumptions about the stochastic age-dependent capital system with Poisson jumps that will be considered:

As the standing hypotheses we always assume that the following conditions are satisfied:

(i) $\mu(a,t)$ is non-negative measurable in Q, $\gamma(t)$ and $A(t)$ are non-negative continuous in $[0;T]$ such that

$$\begin{cases} 0 \le \mu_0 \le \mu(a,t) \le \bar{\mu} < \infty, & in \;\; Q, \\ Let \gamma(t) A(t) \le \eta; \eta \;\; is\ a\ non-negative\ constant; & in \;\; [0,T], \end{cases}$$

where $\int_0^A \mu(a,t)da = +\infty$.

(ii) $f(i,0) = 0, \;\; g(i,0) = 0, \;\; i \in S$;

(iii) (local Lipschitz condition) for any bounded set $\mathcal{D} \subseteq H$, there exists a positive constant $K_1(\mathcal{D})$ such that $x, y \in \mathcal{D}, i \in S$

$$| f(i,x_1,y_1) - f(i,x_2,y_2) |^2 \vee \|g(i,x_1,y_1) - g(i,x_2,y_2)\|_2^2 \le K_1(\mathcal{D})(| x_1 - x_2 |^2 + | y_1 - y_2 |^2);$$

(iv)

$$\begin{cases} F(L,N) \ge 0 (F(L,0) = 0), \;\; \dfrac{\partial F}{\partial L} > 0, \\ 0 < \dfrac{\partial F}{\partial N} < F_1, where \;\; F_1 \;\; is\ a\ positive\ constant. \end{cases}$$

In an analogous way to the corresponding proof presented in [10], we may establish the following existence and uniqueness conclusion: under the conditions (i)-(iv), Eq.(3) has a unique continuous solution $K(a,t)$ on $(a,t) \in Q$.

Definition 2.2 *Suppose that* K_0 *is a random variable such that* $E | K_0 |^2 < \infty$. *For a given step size* $\Delta > 0$, *a numerical method is said to be exponentially stable in mean square on Eq.(2) if there is a pair of positive constants* γ *and* $\overline{N}$, *such that with initial data* K_0,

$$E | Q_t^n |^2 \le \overline{N} E | K_0 |^2 \; e^{-\gamma n \Delta}, \quad \forall n = 0,1,2,\ldots.(8)$$

3. THE MAIN RESULTS

In this section, we provide some lemmas which are necessary for the proof of our result. Because Q_t is the discrete numerical solution of Eq.(2), we first study properties of Q_t.

Lemma 3.1 *Under assumptions (i)--(iv), for any* $T > 0$,

$$\sup_{0 \le t \le T} E | Q_t |^2 \le C_{1T}, (9)$$

where C_{1T} is a positive constant independent of Δt, but it depends on Q_0 and T.

Proof. From Eq.(7), applying Itô's formula to $| Q_t |^2$ yields

$$\begin{aligned} | Q_t |^2 = & | Q_0 |^2 + 2\int_0^t \langle -\frac{\partial Q_s}{\partial a}, Q_s \rangle ds \\ & - 2\int_0^t (\mu(a,s) Z_s, Q_s) ds + 2\int_0^t (f(s,Z_s), Q_s) ds \\ & + 2\int_0^t (Q_s, g(s,Z_s)) dW_s + 2\int_0^t (Q_s, h(s,Z_s)) dN_s \\ & + \int_0^t \|g(s,Z_s)\|_2^2 ds + \lambda \int_0^t | h(s,Z_s) |^2 \; ds \\ \le & | Q_0 |^2 + 2\int_0^t \langle -\frac{\partial Q_s}{\partial a}, Q_s \rangle ds \end{aligned}$$

$$-2\int_0^t(\mu(a,s)Z_s,Q_s)ds+2\int_0^t(f(s,Z_s),Q_s)ds$$
$$+2\int_0^t(Q_s,g(s,Z_s)dW_s+2\int_0^t(Q_s,h(s,Z_s))d\overline{N}_s$$
$$+2\lambda\int_0^t(Q_s,h(s,Z_s))ds+\int_0^t\|g(s,Z_s)\|_2^2ds$$
$$+\lambda\int_0^t|h(s,Z_s)|^2\ ds.$$

where $N_t=\overline{N}_t-\lambda t$ is a compensated Poisson process.

Since

$$-\langle\frac{\partial Q_s}{\partial a},Q_s\rangle$$
$$=-\int_0^A Q_s d_a(Q_s)=\frac{1}{2}\gamma^2(s)A^2(s)[F(L(s),\int_0^A Q_s da)-F(L(s),0)]^2$$
$$\le\frac{1}{2}\eta^2(\frac{\partial F(L,N)}{\partial N}|_y)^2(\int_0^A Q_s da)\le\frac{1}{2}AF_1^2\eta^2\ |Q_s|^2$$

where $y\in(0,\int_0^A Q_s da)$.

Therefore, by conditions (i) and (iii), we get that

$$|Q_t|^2\le|Q_0|^2+AF_1^2\eta^2\int_0^t|Q_s|^2\ ds$$
$$+\int_0^t|f(s,Z_s)|^2\ ds+2\overline{\mu}\int_0^t|Q_s\|Z_s|ds+\int_0^t|Q_s|^2\ ds$$
$$+\int_0^t\|g(s,Z_s)\|_2^2ds+2\int_0^t(Q_s,g(s,Z_s))dW_s$$
$$+2\int_0^t(Q_s,h(s,Z_s))d\overline{N}_s$$
$$+2\lambda\int_0^t(Q_s,h(s,Z_s))ds+\lambda\int_0^t|h(s,Z_s)|^2\ ds.$$

Now, it follows that for any $t\in[0,T]$

$$E\sup_{0\le s\le t}|Q_s|^2$$
$$\le E|Q_0|^2+(AF_1^2\eta^2+\overline{\mu}+\lambda+1)\int_0^t E\sup_{0\le s\le t}|Q_s|^2\ ds+\overline{\mu}\int_0^t E|Z_s|^2\ ds$$
$$+\int_0^t E|f(s,Z_s)|^2\ ds+\int_0^t E\|g(s,Z_s)\|_2^2ds$$
$$+2\lambda\int_0^t E|h(s,Z_s)|^2\ ds$$
$$+2E\sup_{0\le s\le t}\int_0^s(Q_r,g(\tau,Z_\tau))dW_\tau$$
$$+2E\sup_{0\le s\le t}\int_0^s(Q_r,h(\tau,Z_\tau))d\overline{N}_\tau.$$

Using condition (iii) yields

$$
\begin{aligned}
&E\sup_{0\le s\le t}|Q_s|^2\\
&\le E|Q_0|^2+(AF_1^2\eta^2+\bar{\mu}+\lambda+1)\int_0^t E\sup_{0\le s\le t}|Q_s|^2\,ds\\
&+(\bar{\mu}+2K^2+2\lambda K^2)\int_0^t|Z_s|^2 \qquad (10)\\
&+2E\sup_{0\le s\le t}\int_0^s(Q_r,g(\tau,Z_\tau))dW_\tau\\
&+2E\sup_{0\le s\le t}\int_0^s(Q_r,h(\tau,Z_\tau))d\bar{N}_\tau.
\end{aligned}
$$

By Burkholder-Davis-Gundy's inequality (see, for example, [10]), we have

$$E[\sup_{0\le s\le t}\int_0^s(Q_r,g(\tau,Z_\tau))dW_\tau]\le\frac{1}{8}E[\sup_{0\le s\le t}|Q_s|^2]+K_1\cdot K^2\int_0^t E|Z_s|^2\,ds, (11)$$

$$E[\sup_{0\le s\le t}\int_0^s(Q_r,h(\tau,Z_\tau))d\bar{N}_\tau]\le\frac{1}{8}E[\sup_{0\le s\le t}|Q_s|^2]+K_1\cdot K^2\int_0^t E|Z_s|^2\,ds, (12)$$

for some positive constant $K_1>0$. Thus, it follows from (10), (11) and (12)

$$
\begin{aligned}
&E\sup_{0\le s\le t}|Q_s|^2\\
&\le 2(AF_1^2\eta^2+2\bar{\mu}+\lambda+1+2K^2+2K_1K^2+4\lambda K^2)\int_0^t E\sup_{0\le r\le s}|Q_r|^2\,ds\\
&+2E|Q_0|^2,\quad \forall t\in[0,T].
\end{aligned}
$$

Now, Gronwall's lemma obviously implies the required result. The proof is complete.

Lemma 3.2 *Under the assumptions (i)-(iv), for any* $T>0$,

$$E\sup_{0\le t\le T}|Q_t-Z_t|^2\le C_2\Delta t\sup_{t\in[0,T]}E|Q_s|^2 .(13)$$

Proof. For $\forall t\in[0,T]$, there exists an integer k such that $t\in[k\Delta t,(k+1)\Delta t)$. We have

$$
\begin{aligned}
&Q_t-Z_t=Q_t-Q_t^k\\
&=-\int_{k\Delta t}^t\frac{\partial Q_s}{\partial a}ds-\int_{k\Delta t}^t\mu(a,s)Z_s ds\\
&\quad+\int_{k\Delta t}^t f(s,Z_s)ds+\int_{k\Delta t}^t g(s,Z_s)dW_s+\int_{k\Delta t}^t h(s,Z_s)dN_s.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|Q_t-Z_t|^2\\
&\le 5\left|\int_{k\Delta t}^t\frac{\partial Q_s}{\partial a}ds\right|^2+5\left|\int_{k\Delta t}^t\mu(a,s)Z_s ds\right|^2+5\left|\int_{k\Delta t}^t f(s,Z_s)ds\right|^2\\
&\quad+5\left|\int_{k\Delta t}^t g(s,Z_s)dW_s\right|^2+5\left|\int_{k\Delta t}^t h(s,Z_s)dN_s\right|^2 .
\end{aligned}
$$

Now, the Cauchy-Schwarz's inequality and the assumptions (i)-(iii) give

$$
\begin{aligned}
&|Q_t-Z_t|^2\\
&\le 5\Delta t\int_{k\Delta t}^t\left|\frac{\partial Q_s}{\partial a}\right|^2ds+5\bar{\mu}^2\Delta t\int_{k\Delta t}^t|Z_s|^2\,ds+5\Delta t\int_{k\Delta t}^t|f(s,Z_s)|^2\,ds\\
&\quad+5\left|\int_{k\Delta t}^t g(s,Z_s)dW_s\right|^2+10\left|\int_{k\Delta t}^t h(s,Z_s)d\bar{N}_s\right|^2
\end{aligned}
$$

$$+10|\lambda\int_{k\Delta t}^{t}h(s,Z_s)ds|^2$$
$$\le 5\Delta t\int_{k\Delta t}^{t}|\frac{\partial Q_s}{\partial a}|^2\,ds+5\bar{\mu}^2\Delta t\int_{k\Delta t}^{t}|Z_s|^2\,ds$$
$$+5(1+2\lambda^2)K^2\Delta t\int_{k\Delta t}^{t}|Z_s|^2\,ds+5|\int_{k\Delta t}^{t}g(s,Z_s)dW_s|^2$$
$$+10|\int_{k\Delta t}^{t}h(s,Z_s)d\bar{N}_s|^2,$$

whence applying the Burkholder-Davis-Gundy's inequality and conditions (ii)-(iii) leads to

$$E\sup_{t\in[0,T]}|\int_{k\Delta t}^{t}g(s,Z_s)dW_s|^2\le C_3\int_{k\Delta t}^{t}E\sup_{t\in[0,T]}|Z_s|^2\,ds,$$

and

$$E\sup_{t\in[0,T]}|\int_{k\Delta t}^{t}h(s,Z_s)d\bar{N}_s|^2\le C_3\int_{k\Delta t}^{t}E\sup_{t\in[0,T]}|Z_s|^2\,ds,$$

where C_3 is a constant. Because the differential operator $\frac{\partial}{\partial a}$ is a bounded linear operator, we obtain

$$E\sup_{t\in[0,T]}|Q_t-Z_t|^2$$
$$\le 5C_4\Delta t\sup_{t\in[0,T]}E|Q_s|^2$$
$$+5[\bar{\mu}^2\Delta t+(1+2\lambda^2)K^2\Delta t+3C_3]\Delta t\sup_{t\in[0,T]}E|Q_s|^2,$$

where C_4 is a constant. The result (13) is obtained.
We are now in a position to prove a strong convergence result.

Lemma 3.3 *Under assumptions (i)-(iv), for any* $T>0$,

$$\sup_{0\le t\le T}E|Q_t-K_t|^2\le C_T\Delta t\sup_{t\in[0,T]}E|Q_s|^2, \quad (14)$$

where C_T is independent of Δt, but it depends on T.

Proof. Combining (4) with (7) has

$$K_t-Q_t$$
$$=-\int_0^t\frac{\partial(K_s-Q_s)}{\partial a}ds-\int_0^t\mu(a,s)(K_s-Z_s)ds$$
$$+\int_0^t(f(s,K_s)-f(s,Z_s))ds+\int_0^t(g(s,K_s)-g(s,Z_s))dW_s$$
$$+\int_0^t(h(s,K_s)-h(s,Z_s))dN_s.$$

Therefore using Itô's formula, along with the Cauchy-Schwarz's inequality,(i)-(iv) yields,

$$d|K_t-Q_t|^2$$
$$=-2\langle K_t-Q_t,\frac{\partial(K_t-Q_t)}{\partial a}\rangle dt-2\langle K_t-Q_t,\mu(a,t)(K_t-Z_t)dt\rangle$$
$$+2\langle K_t-Q_t,f(t,K_t)-f(t,Z_t)\rangle dt+\|g(t,K_t)-g(t,Z_t)\|_2^2dt$$
$$+\lambda|h(t,K_t)-h(t,Z_t)|^2\,dt+2(K_t-Q_t,(g(t,K_t)-g(t,Z_t))dW_t)$$
$$+2(K_t-Q_t,(h(t,K_t)-h(t,Z_t))dN_t)$$

$$\begin{aligned}&\le AF_1^2\eta^2 |K_t-Q_t|^2\ dt+2\overline{\mu}|K_t-Q_t||K_t-Z_t|dt\\&\quad+2(\lambda+K)|K_t-Q_t||K_t-Z_t|dt+(1+\lambda)K^2|K_t-Z_t|^2\ dt\\&\quad+2(K_t-Q_t,(g(t,K_t)-g(t,Z_t))dW_t)\\&\quad+2(K_t-Q_t,(h(t,K_t)-h(t,Z_t))d\overline{N}_t),\end{aligned}$$

where dP is the differential of P relative to t. Hence, for any $t\in[0,T]$,

$$\begin{aligned}&E\sup_{s\in[0,T]}|K_s-Q_s|^2\\&\le(AF_1^2\eta^2+\overline{\mu}+\lambda+K)\int_0^T E\sup_{r\in[0,T]}|K_r-Q_r|^2\ dt\\&+[\overline{\mu}+K+(1+\lambda)K^2]E\int_0^T|K_t-Z_t|^2\ dt \qquad (15)\\&+2E\sup_{s\in[0,T]}\int_0^s(K_t-Q_t,(g(t,K_t)-g(t,Z_t))dW_t\\&+2E\sup_{s\in[0,T]}\int_0^s(K_t-Q_t,(h(t,K_t)-h(t,Z_t))d\overline{N}_t.\end{aligned}$$

By Burkholder-Davis-Gundy's inequality, we have

$$\begin{aligned}&E\sup_{s\in[0,T]}\int_0^s(K_t-Q_t,(g(t,K_t)-g(t,Z_t))dW_t)\\&\le\frac{1}{8}E[\sup_{0\le s\le T}|K_t-Q_t|^2+K_1\int_0^T E|K_t-Z_t|^2\ dt,\end{aligned}\qquad(16)$$

and

$$\begin{aligned}&E\sup_{s\in[0,T]}\int_0^s(K_t-Q_t,(h(t,K_t)-h(t,Z_t))d\overline{N}_t)\\&\le\frac{1}{8}E[\sup_{0\le s\le T}|K_t-Q_t|^2+K_1\int_0^T E|K_t-Z_t|^2\ dt,\end{aligned}\qquad(17)$$

where K_1 is a positive constant. Therefore, inserting (16) and (17) into (15) gives

$$\begin{aligned}&E\sup_{s\in[0,T]}|K_s-Q_s|^2\\&\le(AF_1^2\eta^2+\overline{\mu}+\lambda+K)\int_0^T E\sup_{r\in[0,s]}|K_r-Q_r|^2\ ds\\&+[\overline{\mu}+K+(1+\lambda)K^2+4K_1]\int_0^T|K_t-Z_t|^2\ dt\\&+\frac{1}{2}E\sup_{s\in[0,T]}|K_s-Q_s|^2\ .\end{aligned}$$

Hence,

$$\begin{aligned}&E\sup_{s\in[0,T]}|K_s-Q_s|^2\\&\le 2(AF_1^2\eta^2+\overline{\mu}+\lambda+K)\int_0^T E\sup_{r\in[0,s]}|K_r-Q_r|^2\ ds\\&\quad+2[\overline{\mu}+K+(1+\lambda)K^2+4K_1]\int_0^T|K_t-Z_t|^2\ dt\\&\le 2(\overline{\beta}^2A+\overline{\mu}+\lambda+K)\int_0^T E\sup_{r\in[0,s]}|K_r-Q_r|^2\ ds\\&\quad+4[\overline{\mu}+K+(1+\lambda)K^2+4K_1]\int_0^T(|Q_t-Z_t|^2+|K_t-Q_t|^2)dt\end{aligned}$$

$$\leq 2[AF_1^2\eta^2 + \lambda + 3\overline{\mu} + 3K + 2(1+\lambda)K^2 + 8K_1]\int_0^T E\sup_{r\in[0,s]} |K_r - Q_r|^2\, ds$$

$$+4[\overline{\mu} + K + (1+\lambda)K^2 + 8K_1]\int_0^T |Q_t - Z_t|^2\, dt.$$

Applying Lemma 3.2, we obtain a bound of the form

$$E\sup_{s\in[0,T]} |K_s - Q_s|^2 \quad\leq\quad D_1\Delta t + D_2\int_0^T E\sup_{r\in[0,s]} |K_r - Q_r|^2\, ds,$$

where $D_1 = 4(AF_1^2\eta^2 + K + (1+\lambda)K^2 + 8K_1)TC_2 \sup_{t\in[0,T]} E|Q_s|^2$, and $D_2 = 2[AF_1^2\eta^2 + 3\overline{\mu} + \lambda + 3K + 2(1+\lambda)K^2 + 8K_1]$. By applying the Gronwall's inequality, we have the following inequality

$$E\left(\sup_{s\in[0,T]} |K_s - Q_s|^2\right) \leq D_1\Delta t exp(D_2 T).$$

By Lemma 3.1, (14) is obtained. The proof is proved.

Lemma 3.4 *Under assumptions (i)--(iii), the trivial solution of Eq.(2) is exponentially stable in mean square. That is, there is a pair of positive constants* γ *and* M *such that, for any* K_0

$$E|K_t|^2 \leq ME|K_0|^2\, e^{-\gamma t}, \quad \forall t \geq 0.(18)$$

The proof of this lemma is an analogous to that of Theorem in [11].
Now we are in a position to give the main result.

Theorem 3.5 *Under assumptions (i)--(iv), the Euler method applied to Eq.(2) is exponentially stable in mean square.*

The proof of this Theorem is an analogous to that of Theorem 2.2 in [12].

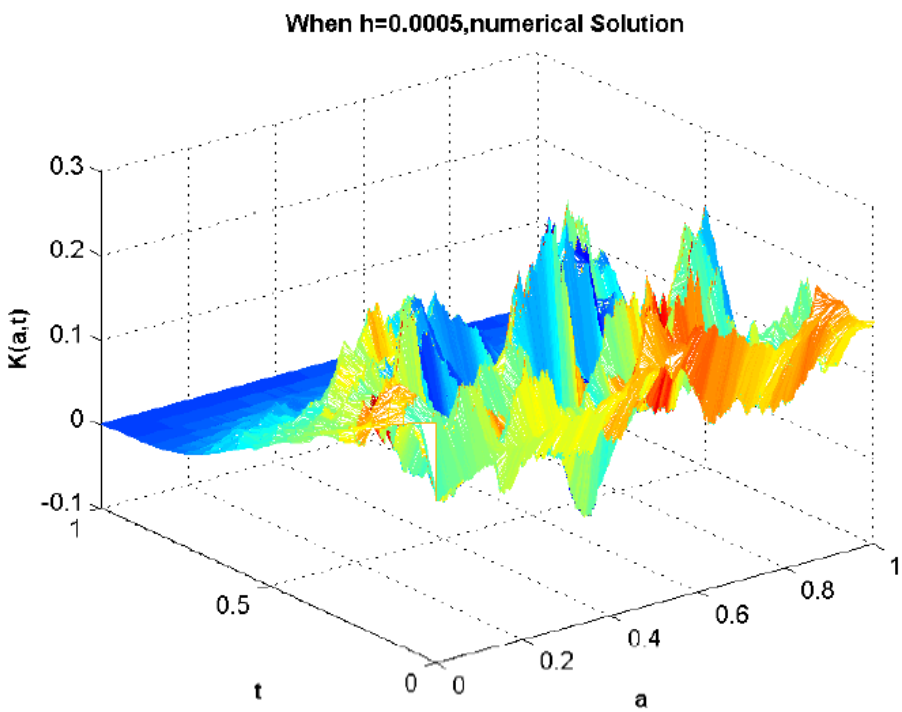

Figure 1: Numerical Simulations of stochastic age-dependent capital system

4. AN EXAMPLE

Consider the following stochastic age-dependent capital system with Poisson jumps

$$\begin{cases} \frac{\partial K}{\partial t}+\frac{\partial K}{\partial a}=-\frac{1}{(1-a)^2}K+2Kt-tKdW_t+KdN_t, & \\ & in\ (0,A)\times(0,T), \\ K(0,t)=\frac{t^2}{(1-t)^2}\int_0^1 K(a,t)da, & in\ (0,T), \\ K(a,0)=exp(-\frac{1}{1-a}), & in\ (0,A), \\ N(t)=\int_0^1 K(a,t)da, & in\ (0,T), \end{cases} \tag{19}$$

Here W_t is a real standard Brownian motion , N_t is a scalar Poisson process with intensity 1. Take $T=1$, $A=1$ in Eq.(19). We can set this problem in our formulation by taking $H=L^2([0,1]\times[0,1])$, $V=W_0^1([0,1])$ (a Sobolev space with elements satisfying the boundary conditions above), $\mu(a,t)=\frac{1}{(1-a)^2}$, $\gamma(t)A(t)=t^2$, $F(L(t),N(t))=\frac{1}{(1-t)^2}\int_0^1 K(a,t)da$, $L(t)=\frac{1}{(1-t)^2}$, $f(t,K)=2Kt$, $h(t,K)=K$ and $g(t,K)=-tK$, $K(a,0)=exp(-\frac{1}{1-a})$.

Clearly, the operators f, g and h satisfy conditions (ii) and (iii), $F(L,N)$ and $\mu(a,t)$ satisfy conditions (i) and (iv). Consequently, the approximate solution will converge to the true solution of (19) for any $(a,t)\in(0,1)\times(0,T)$ in the sense of Theorem 3.5.

Obviously, $K(a,t)$ in (19) cannot be solved explicitly. It is necessary to know the numerical approximation $Q(a,t)$ of $K(a,t)$. Take $\Delta t=0.005$, $\Delta a=0.05$.Fig. 1 is numerical simulations of the stochastic age-dependent capital system with Poisson jumps with 1000 experiments, where $K(a,t)=EQ(a,t)=\frac{1}{1000}\sum_{k=1}^{1000}Q_k(a,t)$. It clearly reveals the age-dependent capital system tendency.

5. CONCLUSION

Some important sources of uncertainty may be discontinuous, recurrent, and fluctuating. Such significant events include innovations in technique, introduction of new products, natural disasters, and changes in laws or government policies. The relationship among these events and the profitability of risky assets can be very complicated. Furthermore, there can be numerous events and economic variables that are potentially related to the profitability of risky assets. In order to describe this situation, this paper introduces a class of stochastic age-dependent capital dynamic system. To the best of our knowledge, there are not any numerical methods available for stochastic partial differential equations with Poisson jumps. Thus, numerical approximation schemes are invaluable tools for exploring its properties. In this paper, we extend the idea from the papers [8,9] to the numerical solutions for stochastic age-dependent capital system with Poisson jumps. The main purpose of this paper is to investigate the convergence of numerical approximation of stochastic age-dependent capital system with Poisson jumps under the given conditions. Using the recent mathematical technique for the stochastic differential equations. We obtain the condition which can ensure the approximate solution that converges to the true solution for stochastic age-dependent capital system. At the same time, we propose the numerical solution for stochastic age-dependent capital system with Poisson jumps. The approach is based on constructing a discrete-time approximation to exact solution by considering the jump time. An example has demonstrated our theory.

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