ON COMMON FIXED POINT THEOREMS FOR SEMI-COMPATIBLE AND OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS IN MENGER SPACE

Arihant Jain[1] & Basant Chaudhary[2]
[1]Department of Applied Mathematics, Shri Guru Sandipani Institute of Technology and Science,
Ujjain (M.P.) 456550, India
Email: arihant2412@gmail.com
[2]Department of Applied Mathematics, Malwa Institute of Technology, Indore (M.P.), India
Email: chaudharybasant60@gmail.com

ABSTRACT

In this paper, the concept of semi-compatibility and occasionally weak compatibility in Menger space has been applied to prove a common fixed point theorem for six self maps. Our result generalizes and extends the result of Pathak and Verma [9].

Keywords: *Probabilistic metric space, Menger space, common fixed point, compatible maps, semi-compatible maps, weak compatibility.*
AMS Subject Classification: *Primary 47H10, Secondary 54H25.*

1. INTRODUCTION

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [7]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [11] studied this concept and gave some fundamental results on this space. Sehgal and Bharucha-Reid [12] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed-point theory in Menger space.

Recently, Jungck and Rhoades [6] termed a pair of self maps to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. Sessa [13] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weak commuting maps in metric spaces. Jungck [5] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [8]. In the sequel, Pathak and Verma [9] proved a common fixed point theorem in Menger space using compatibility and weak compatibility. Using the concept of compatible mappings of type (A), Jain et. al. [2, 3] proved some interesting fixed point theorems in Menger space. Afterwards, Jain et. al. [4] proved the fixed point theorem using the concept of weak compatible maps in Menger space.

Cho, Sharma and Sahu [1] introduced the concept of semi-compatibility in a d-complete topological space. Popa [10] proved interesting fixed point results using implicit real functions and semi-compatibility in d-complete topological space. Using the concept of semi-compatible mappings in Menger space, Singh et. al. [14] proved a fixed point theorem using implicit relation.

In this paper a fixed point theorem for six self maps has been proved using the concept of semi-compatible maps and occasionally weak compatibility which turns out be a material generalization of the result of Pathak and Verma [9].

2. Preliminaries.

Definition 2.1. A mapping $F : R \to R^+$ is called a *distribution* if it is non-decreasing left continuous with

$$\inf \{ F(t) \mid t \in R \} = 0 \quad \text{and} \quad \sup \{ F(t) \mid t \in R\} = 1.$$

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by $H(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}.$

Definition 2.2. [8] A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if it satisfies the following conditions :

(t-1) $t(a, 1) = a, \quad t(0, 0) = 0$;

(t-2) $t(a, b) = t(b, a)$;

(t-3) $t(c, d) \ge t(a, b)$; for $c \ge a, d \ge b$, (t-4) $t(t(a, b), c) = t(a, t(b, c))$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3. [8] A *probabilistic metric space (PM-space)* is an ordered pair (X, F) consisting of a non empty set X and a function $F : X \times X \to L$, where L is the collection of all distribution functions and the value of F at $(u, v) \in X \times X$ is represented by $F_{u, v}$. The function $F_{u,v}$ assumed to satisfy the following conditions:

(PM-1) $F_{u,v}(x) = 1$, for all $x > 0$, if and only if $u = v$;

(PM-2) $F_{u,v}(0) = 0$;

(PM-3) $F_{u,v} = F_{v,u}$;

(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$, for all $u,v,w \in X$ and $x, y > 0$.

Definition 2.4. [8] A *Menger space* is a triplet (X, F, t) where (X, F) is a PM-space and t is a t-norm such that the inequality

(PM-5) $F_{u,w}(x + y) \geq t\{F_{u, v}(x), F_{v, w}(y)\}$, for all $u, v, w \in X$, $x, y \geq 0$.

Definition 2.5. [8] A sequence $\{x_n\}$ in a Menger space (X, F, t) is said to be *convergent* and *converges to a point* x in X if and only if for each $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$$F_{x_n, x}(\varepsilon) > 1 - \lambda \qquad \text{for all } n \geq M(\varepsilon, \lambda).$$

Further the sequence $\{x_n\}$ is said to be *Cauchy sequence* if for $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$$F_{xn, xm}(\varepsilon) > 1 - \lambda \qquad \text{for all } m, n \geq M(\varepsilon, \lambda).$$

A Menger PM-space (X, F, t) is said to be *complete* if every Cauchy sequence in X converges to a point in X.
A complete metric space can be treated as a complete Menger space in the following way:

Proposition 2.1. [8] If (X, d) is a metric space then the metric d induces mappings $F : X \times X \to L$, defined by $F_{p,q}(x) = H(x - d(p, q))$, $p, q \in X$, where

$$H(k) = 0, \quad \text{for } k \leq 0 \quad \text{and} \quad H(k) = 1, \quad \text{for } k > 0.$$

Further if, $t : [0,1] \times [0,1] \to [0,1]$ is defined by $t(a, b) = \min\{a, b\}$. Then (X, F, t) is a Menger space. It is complete if (X, d) is complete.
The space (X, F, t) so obtained is called the *induced Menger space.*
Definition 2.6. [9] Self mappings A and S of a Menger space (X, F, t) are said to be weak compatible if they commute at their coincidence points i.e. $Ax = Sx$ for $x \in X$ implies $ASx = SAx$.
Definition 2.7. [9] Self mappings A and S of a Menger space (X, F, t) are said to be *compatible* if $F_{ASx_n,SAx_n}(x)\to 1$ for all $x>0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n,Sx_n\to u$ for some u in X, as $n \to \infty$.
Definition 2.8. [14] Self mappings A and S of a Menger space (X, F, t) are said to be semi-compatible if $F_{ASx_n,Su}(x)\to 1$ for all $x>0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$, for some u in X, as $n \to \infty$.
Definition 2.9. Self maps A and S of a N.A. Menger PM-space (X, F, t) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.
Example 2.1. Let (X, F, t) be the Menger PM-space, where X = [0, 4] Define F by

$$F_{x, y}(t) = \begin{cases} \dfrac{t}{t + |x - y|}, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \end{cases}.$$

Define $A, S : X \to X$ by

$Ax = 4x$ and $Sx = x^2$ for all $x \in X$ then $Ax = Sx$ for $x = 0$ and 4.
But $AS(0) = SA(0)$ and $AS(4) \neq SA(4)$.

Thus, S and T are occasionally weakly compatible mappings but not weakly compatible.

Remark 2.1. In view of above example, it follows that the concept of occasionally weakly compatible is more general than that of weak compatibility.

Lemma 2.1. [9] Let (X, F, *) be a Menger space with t-norm * such that the family $\{*_n(x)\}_{n\in N}$ is equicontinuous at x = 1 and let E denote the family of all functions $\phi : R^+ \to R^+$ such that ϕ is non-decreasing with $\lim_{n\to\infty} \phi^n(t)=+\infty$, $\forall$ t > 0. If $\{y_n\}_{n\in N}$ is a sequence in X satisfying the condition

$$F_{y_n, y_{n+1}}(t) \geq F_{y_{n-1}, y_n}(\phi(t)),$$

for all t > 0 and $\alpha \in [-1, 0]$, then $\{y_n\}_{n\in N}$ is a Cauchy sequence in X.

3. MAIN RESULT

Theorem 3.1. Let A, B, S, T, P and Q be self maps of a complete Menger space (X, F, *) with * = min satisfying :

(3.1.1) $P(X) \subseteq ST(X)$, $Q(X) \subseteq AB(X)$;

(3.1.2) AB = BA, ST = TS, PB = BP, QT = TQ;

(3.1.3) either P or AB is continuous;

(3.1.4) (P, AB) is semi-compatible and (Q, ST) is occasionally weak compatible;

(3.1.5) $[1 + \alpha F_{ABx, STy}(t)] * F_{Px, Qy}(t)$

$$\geq \alpha \min\{F_{Px, ABx}(t) * F_{Qy, STy}(t), F_{Px, STy}(2t) * F_{Qy, ABx}(2t)\}$$
$$+ F_{ABx, STy}(\phi(t)) * F_{Px, ABx}(\phi(t)) * F_{Qy, STy}(\phi(t)) * F_{Px, STy}(2\phi(t))$$
$$* F_{Qy, ABx}(2\phi(t))$$

for all $x, y \in X$, t > 0 and $\phi \in E$.

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. Suppose $x_0 \in X$. From condition (3.1.1) $\exists$ $x_1, x_2 \in X$ such that

$$Px_0 = STx_1 \quad \text{and} \quad Qx_1 = ABx_2.$$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Px_{2n} = STx_{2n+1} \quad \text{and} \quad y_{2n+1} = Qx_{2n+1} = ABx_{2n+2}$$

for n = 0, 1, 2, ...

Step I. Let us show that $F_{y_{n+2}, y_{n+1}}(t) \geq F_{y_{n+1}, y_n}(\phi(t))$.

For, putting x_{2n+2} for x and x_{2n+1} for y in (3.1.5) and then on simplification, we have

$$[1 + \alpha F_{ABx_{2n+2}, STx_{2n+1}}(t)] * F_{Px_{2n+2}, Qx_{2n+1}}(t)$$
$$\geq \alpha \min\{F_{Px_{2n+2}, ABx_{2n+2}}(t) * F_{Qx_{2n+1}, STx_{2n+1}}(t), F_{Px_{2n+2}, STx_{2n+1}}(2t)$$
$$F_{Qx_{2n+1}, ABx_{2n+2}}(2t)\}$$
$$+ F_{ABx_{2n+2}, STx_{2n+1}}(\phi(t)) * F_{Px_{2n+2}, ABx_{2n+2}}(\phi(t)) * F_{Qx_{2n+1}, STx_{2n+1}}(\phi(t))$$
$$* F_{Px_{2n+2}, STx_{2n+1}}(2\phi(t)) * F_{Qx_{2n+1}, ABx_{2n+2}}(2\phi(t))$$
$$[1 + \alpha F_{y_{2n+1}, y_{2n}}(t)] * F_{y_{2n+2}, y_{2n+1}}(t)$$
$$\geq \alpha \min\{F_{y_{2n+2}, y_{2n+1}}(t) * F_{y_{2n+1}, y_{2n}}(t), F_{y_{2n+2}, y_{2n}}(2t) * F_{y_{2n+1}, y_{2n+1}}(2t)\}$$
$$+ F_{y_{2n+1}, y_{2n}}(\phi(t))$$

$* F_{y_{2n+2}, y_{2n+1}}(\phi(t)) * F_{y_{2n+1}, y_{2n}}(\phi(t)) * F_{y_{2n+2}, y_{2n}}(2\phi(t)) * F_{y_{2n+1}, y_{2n+1}}(2\phi(t))$

$F_{y_{2n+2}, y_{2n+1}}(t) + \alpha F_{y_{2n+1}, y_{2n}}(t) * F_{y_{2n+2}, y_{2n+1}}(t)$

$\geq \alpha \min\{F_{y_{2n+2}, y_{2n}}(2t), F_{y_{2n+2}, y_{2n}}(2t)\} + F_{y_{2n+1}, y_{2n}}(\phi(t)) * F_{y_{2n+2}, y_{2n+1}}(\phi(t))$

$* F_{y_{2n+2}, y_{2n}}(2\phi(t)) * 1$

$F_{y_{2n+2}, y_{2n+1}}(t) + \alpha F_{y_{2n+1}, y_{2n}}(t) * F_{y_{2n+2}, y_{2n+1}}(t)$

$\geq \alpha F_{y_{2n+2}, y_{2n}}(2t) + F_{y_{2n+1}, y_{2n}}(\phi(t)) * F_{y_{2n+2}, y_{2n+1}}(2\phi(t))$

$F_{y_{2n+2}, y_{2n+1}}(t) + \alpha F_{y_{2n+2}, y_{2n}}(2t)$

$\geq \alpha F_{y_{2n+2}, y_{2n}}(2t) + F_{y_{2n+1}, y2n}(\phi(t)) * F_{y_{2n+2}, y_{2n+1}}(\phi(t)) * F_{y_{2n+1}, y_{2n}}(\phi(t))$

$F_{y_{2n+2}, y_{2n+1}}(t) \geq F_{y_{2n+1}, y_{2n}}(\phi(t)) * F_{y_{2n+2}, y_{2n+1}}(\phi(t))$

or, $F_{y_{2n+2}, y_{2n+1}}(t) \geq F_{y_{2n+1}, y_{2n+2}}(\phi(t)) * F_{y_{2n}, y_{2n+1}}(\phi(t))$

or, $F_{y_{2n+2}, y_{2n+1}}(t) \geq \min\{F_{y_{2n+1}, y_{2n+2}}(\phi(t)), F_{y_{2n}, y_{2n+1}}(\phi(t))\}$.

If $F_{y_{2n+1}, y_{2n+2}}(\phi(t))$ is chosen 'min' then we obtain

$$F_{y_{2n+2}, y_{2n+1}}(t) \geq F_{y_{2n+2}, y_{2n+1}}(\phi(t)), \quad \forall \ t > 0$$

a contradiction as $\phi(t)$ is non-decreasing function.
Thus,

$$F_{y_{2n+2}, y_{2n+1}}(t) \geq F_{y_{2n+1}, y_{2n}}(\phi(t)), \quad \forall \ t > 0.$$

Similarly, by putting x_{2n+2} for x and x_{2n+3} for y in (3.1.5), we have

$$F_{y_{2n+3}, y_{2n+2}}(t) \geq F_{y_{2n+2}, y_{2n+1}}(\phi(t)), \quad \forall \ t > 0.$$

Using these two, we obtain

$$F_{y_{n+2}, y_{n+1}}(t) \geq F_{y_{n+1}, y_n}(\phi(t)), \quad \forall \ n = 0, 1, 2, \ldots, t > 0.$$

Therefore, by lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X, which is complete.

Hence $\{y_n\} \to z \in X$. Also its subsequences converges as follows :

$\{Px_{2n}\} \to z$ and $\{STx_{2n+1}\} \to z$,

$\{Qx_{2n+1}\} \to z$ and $\{ABx_{2n+2}\} \to z$. (3.1.6)

Case I. Suppose P is continuous.

As P is continuous and (P, AB) is semi-compatible, we get

$$PABx_{2n+2} \to Pz \ \text{ and } \ PABx_{2n+2} \to ABz. \tag{3.1.7}$$

Since the limit in Menger space is unique, we get

$$Pz = ABz. \tag{3.1.8}$$

Step II. We prove Pz = z. Put x = z, y = x_{2n+1} in (3.1.5) and let Pz ≠ z. Then

$[1 + \alpha F_{ABz, STx_{2n+1}}(t)] * F_{Pz, Qx_{2n+1}}(t)$

$\geq \alpha \min\{F_{Pz, ABz}(t) * F_{Qx_{2n+1}, STx_{2n+1}}(t), F_{Pz, STx_{2n+1}}(2t) * F_{Qx_{2n+1}, ABz}(2t)\}$

$+ F_{ABz, STx_{2n+1}}(\phi(t)) * F_{Pz, ABz}(\phi(t)) * F_{Qx_{2n+1}, STx_{2n+1}}(\phi(t)) * F_{Pz, STx_{2n+1}}(2\phi(t))$

$* F_{Qx_{2n+1}, ABz}(2\phi(t))$.

Letting $n \to \infty$ and using (3.1.6) and (3.1.8), we get

$[1 + \alpha F_{Pz, z}(t)] * F_{Pz, z}(t)$

$\geq \alpha \min\{F_{Pz, Pz}(t) * F_{z, z}(t), F_{Pz, z}(2t) * F_{z, Pz}(2t)\} + F_{Pz, z}(\phi(t)) * F_{Pz, Pz}(\phi(t))$
$* F_{z, z}(\phi(t)) * F_{Pz, z}(2\phi(t)) * F_{z, Pz}(2\phi(t))$

$F_{Pz, z}(t) + \alpha\{F_{Pz, z}(t) * F_{Pz, z}(t)\}$

$\geq \alpha \min\{1 * 1, F_{Pz, z}(2t) * F_{Pz, z}(2t)\} + F_{Pz, z}(\phi(t)) * 1 * 1 * F_{Pz, z}(2\phi(t))$
$* F_{z, Pz}(2\phi(t))$

$F_{Pz, z}(t) + \alpha F_{Pz, z}(t) \geq \alpha \min\{1, F_{Pz, z}(2t)\} + F_{Pz, z}(\phi(t)) * F_{Pz, z}(2\phi(t))$

$F_{Pz, z}(t) + \alpha F_{Pz, z}(t) \geq \alpha F_{Pz, z}(2t) + F_{Pz, z}(\phi(t)) * F_{Pz, z}(2\phi(t))$

$F_{Pz, z}(t) + \alpha F_{Pz, z}(t) \geq \alpha\{F_{Pz, z}(t) * F_{z, z}(t)\} + F_{Pz, z}(\phi(t)) * F_{Pz, z}(\phi(t)) * F_{z, z}(\phi(t))$

$F_{Pz, z}(t) + \alpha F_{Pz, z}(t) \geq \alpha\{F_{Pz, z}(t) * 1\} + F_{Pz, z}(\phi(t)) * 1$

$F_{Pz, z}(t) + \alpha F_{Pz, z}(t) \geq \alpha F_{Pz, z}(t) + F_{Pz, z}(\phi(t))$

$F_{Pz, z}(t) \geq F_{Pz, z}(\phi(t))$

which is a contradiction and hence, $Pz = z$
and so $z = Pz = ABz$.

Step III. Put $x = Bz$ and $y = x_{2n+1}$ in (3.1.5), we get

$[1 + \alpha F_{ABBz, STx_{2n+1}}(t)] * F_{PBz, Qx_{2n+1}}(t)$

$\geq \alpha \min\{F_{PBz, ABBz}(t) * F_{Qx_{2n+1}, STx_{2n+1}}(t), F_{PBz, STx_{2n+1}}(2t) * F_{Qx_{2n+1}, ABBz}(2t)\}$
$+ F_{ABBz, STx_{2n+1}}(\phi(t)) * F_{PBz, ABBz}(\phi(t)) * F_{Qx_{2n+1}, STx_{2n+1}}(\phi(t))$
$* F_{PBz, STx_{2n+1}}(2\phi(t)) * F_{Qx_{2n+1}, ABBz}(2\phi(t))$.

As $BP = PB$, $AB = BA$ so we have

$P(Bz) = B(Pz) = Bz$ and $AB(Bz) = B(AB)z = Bz$.

Letting $n \to \infty$ and using (3.1.6), we get

$[1 + \alpha F_{Bz, z}(t)] * F_{Bz, z}(t)$

$\geq \alpha \min\{F_{Bz, Bz}(t) * F_{z, z}(t), F_{Bz, z}(2t) * F_{z, Bz}(2t)\}$
$+ F_{Bz, z}(\phi(t)) * F_{Bz, Bz}(\phi(t)) * F_{z, z}(\phi(t)) * F_{Bz, z}(2\phi(t)) * F_{z, Bz}(2\phi(t))$

$F_{Bz, z}(t) + \alpha\{F_{Bz, z}(t) * F_{Bz, z}(t)\}$

$\geq \alpha \min\{1 * 1, F_{Bz, z}(2t)\} + F_{Bz, z}(\phi(t)) * 1 * 1 * F_{Bz, z}(2\phi(t))$

$F_{Bz, z}(t) + \alpha F_{Bz, z}(t) \geq \alpha F_{Bz, z}(2t) + F_{Bz, z}(\phi(t)) * F_{Bz, z}(2\phi(t))$

$F_{Bz, z}(t) + \alpha F_{Bz, z}(t) \geq \alpha \{F_{Bz, z}(t) * F_{z, z}(t)\} + F_{Bz, z}(\phi(t)) * F_{Bz, z}(\phi(t)) * F_{z, z}(\phi(t))$

$F_{Bz, z}(t) + \alpha F_{Bz, z}(t) \geq \alpha \{F_{Bz, z}(t) * 1\} + F_{Bz, z}(\phi(t)) * 1$

$F_{Bz, z}(t) + \alpha F_{Bz, z}(t) \geq \alpha F_{Bz, z}(t) + F_{Bz, z}(\phi(t))$

$F_{Bz, z}(t) \geq F_{Bz, z}(\phi(t))$

which is a contradiction and we get $Bz = z$ and so

$z = ABz = Az$.

Therefore, $Pz = Az = Bz = z$. (3.1.9)

Step IV. Since $P(X) \subseteq ST(X)$ there exists $u \in X$ such that

$z = Pz = STu$.

Put $x = x_{2n}$ and $y = u$ in (3.1.5), we get

$[1 + \alpha F_{ABx_{2n}, STu}(t)] * F_{Px_{2n}, Qu}(t)$

$\geq \alpha \min\{F_{Px_{2n}, ABx_{2n}}(t) * F_{Qu, STu}(t), F_{Px_{2n}, STu}(2t) * F_{Qu, ABx_{2n}}(2t)\}$

$+ F_{ABx_{2n}, STu}(\phi(t)) * F_{Px_{2n}, ABx_{2n}}(\phi(t)) * F_{Qu, STu}(\phi(t)) * F_{Px_{2n}, STu}(2\phi(t))$

$* F_{Qu, ABx_{2n}}(2\phi(t))$.

Letting $n \to \infty$ and using (3.1.6), we get

$[1 + \alpha F_{z, z}(t)] * F_{z, Qu}(t)$

$\geq \alpha \min\{F_{z, z}(t) * F_{Qu, z}(t), F_{z, z}(2t) * F_{Qu, z}(2t)\} + F_{z, z}(\phi(t)) * F_{z, z}(\phi(t))$

$* F_{Qu, z}(\phi(t)) * F_{z, z}(2\phi(t)) * F_{Qu, z}(2\phi(t))$

$F_{z, Qu}(t) + \alpha F_{z, Qu}(t) \geq \alpha \min\{F_{Qu, z}(t), F_{Qu, z}(2t)\} + F_{Qu, z}(\phi(t)) * F_{Qu, z}(2\phi(t))$

$F_{Qu, z}(t) + \alpha F_{Qu, z}(t) \geq \alpha \min\{F_{Qu, z}(t), F_{Qu, z}(t) * F_{z, z}(t)\} + F_{Qu, z}(\phi(t)) * F_{Qu, z}(\phi(t))$

$* F_{z, z}(\phi(t))$

$F_{Qu, z}(t) + \alpha F_{Qu, z}(t) \geq \alpha F_{Qu, z}(t) + F_{Qu, z}(\phi(t))$

$F_{Qu, z}(t) \geq F_{Qu, z}(\phi(t))$

which is a contradiction by lemma (2.1) and we get

Qu = z and so Qu = z = STu.

Since (Q, ST) is occasionally weak-compatible, we have

STQu = QSTu i.e. STz = Qz.

Step V. Put $x = x_{2n}$ and y = z in (3.1.5), we have

$[1 + \alpha F_{ABx_{2n}, STz}(t)] * F_{Px_{2n}, Qz}(t)$

$\geq \alpha \min\{F_{Px_{2n}, ABx_{2n}}(t) * F_{Qz, STz}(t), F_{Px_{2n}, STz}(2t) * F_{Qz, ABx_{2n}}(2t)\}$

$+ F_{ABx_{2n}, STz}(\phi(t)) * F_{Px_{2n}, ABx_{2n}}(\phi(t)) * F_{Qz, STz}(\phi(t)) * F_{Px_{2n}, STz}(2\phi(t))$

$* F_{Qz, ABx_{2n}}(2\phi(t))$.

Letting $n \to \infty$ and using (3.1.6) and step IV, we get

$[1 + \alpha F_{z, Qz}(t)] * F_{z, Qz}(t)$

$\geq \alpha \min\{F_{z, z}(t) * F_{Qz, Qz}(t), F_{z, Qz}(2t) * F_{Qz, z}(2t)\} + F_{z, Qz}(\phi(t)) * F_{z, z}(\phi(t))$

$* F_{Qz, Qz}(\phi(t)) * F_{z, Qz}(2\phi(t)) * F_{Qz, Qz}(2\phi(t))$

$F_{z, Qz}(t) + \alpha\{F_{z, Qz}(t) * F_{Qz, z}(t)\}$

$\geq \alpha \min\{1 * F_{z, Qz}(2t)\} + F_{z, Qz}(\phi(t)) * F_{z, Qz}(2\phi(t))$

$F_{Qz, z}(t) + \alpha F_{Qz, z}(t) \geq \alpha F_{Qz, z}(2t) + F_{Qz, z}(\phi(t)) * F_{Qz, z}(\phi(t)) * F_{z, z}(\phi(t))$

$F_{Qz, z}(t) + \alpha F_{Qz, z}(t) \geq \alpha \{F_{Qz, z}(t) * F_{z, z}(t)\} + F_{Qz, z}(\phi(t))$

$F_{Qz, z}(t) + \alpha F_{Qz, z}(t) \geq \alpha F_{Qz, z}(t) * F_{Qz, z}(\phi(t))$

$F_{Qz, z}(t) \geq F_{Qz, z}(\phi(t))$

which is a contradiction and we get Qz = z.

Step VI. Put $x = x_{2n}$ and y = Tz in (3.1.5), we have

$[1 + \alpha F_{ABx_{2n}, STTz}(t)] * F_{Px_{2n}, QTz}(t)$

$\geq \alpha \min\{F_{Px_{2n}, ABx_{2n}}(t) * F_{QTz, STTz}(t), F_{Px_{2n}, STTz}(2t) * F_{QTz, ABx_{2n}}(2t)\}$

$+ F_{ABx_{2n}, STTz}(\phi(t)) * F_{Px_{2n}, ABx_{2n}}(\phi(t)) * F_{QTz, STTz}(\phi(t)) * F_{Px_{2n}, STTz}(2\phi(t))$

$* F_{QTz, ABx_{2n}}(2\phi(t))$.

As QT = TQ and ST = TS, we have

$QTz = TQz = Tz$ and $ST(Tz) = T(STz) = Tz$.

Letting $n \to \infty$, we get

$$[1 + \alpha F_{z, Tz}(t)] * F_{z, Tz}(t)$$
$$\geq \alpha \min\{F_{z, z}(t) * F_{Tz, Tz}(t), F_{z, Tz}(2t) * F_{Tz, z}(2t)\} + F_{z, Tz}(\phi(t)) * F_{z, z}(\phi(t))$$
$$* F_{Tz, Tz}(\phi(t)) * F_{z, Tz}(2\phi(t)) * F_{Tz, z}(2\phi(t))$$
$$F_{z, Tz}(t) + \alpha\{F_{z, Tz}(t) * F_{z, Tz}(t)\} \geq \alpha \min\{1 * F_{Tz, z}(2t)\} + F_{z, Tz}(\phi(t)) * 1 * 1 * F_{Tz, z}(2\phi(t))$$
$$F_{Tz, z}(t) + \alpha F_{Tz, z}(t) \geq \alpha F_{Tz, z}(2t) + F_{Tz, z}(\phi(t)) * F_{Tz, z}(2\phi(t))$$
$$F_{Tz, z}(t) + \alpha F_{Tz, z}(t) \geq \alpha \{F_{Tz, z}(t) * F_{z, z}(t)\} + F_{Tz, z}(\phi(t)) * F_{Tz, z}(\phi(t)) * F_{z, z}(\phi(t))$$
$$F_{Tz, z}(t) + \alpha F_{Tz, z}(t) \geq \alpha F_{Tz, z}(t) + F_{Tz, z}(\phi(t))$$
$$F_{Tz, z}(t) \geq F_{Tz, z}(\phi(t))$$

which is a contradiction and we get $Tz = z$.

Now, $STz = Tz = z$ implies $Sz = z$.

Hence, $Sz = Tz = Qz = z$. (3.1.10)

Combining (3.1.9) and (3.1.10), we get

$Az = Bz = Pz = Qz = Sz = Tz = z$

i.e. z is a common fixed point of A, B, P, Q, S and T.

Case II. Suppose AB is continuous.

Since AB is continuous and (P, AB) is semi-compatible, we get

$$(AB)^2 x_{2n} \to ABz, \quad PABx_{2n} \to ABz. \tag{3.1.11}$$

Now, we prove $ABz = z$.

Step VII. Put $x = ABx_{2n}$ and $y = x_{2n+1}$ in (3.1.5) and assuming $ABz \neq z$, we get

$$[1 + \alpha F_{ABABx_{2n}, STx_{2n+1}}(t)] * F_{PABx_{2n}, Qx_{2n+1}}(t)$$
$$\geq \alpha \min\{F_{PABx_{2n}, ABABx_{2n}}(t) * F_{Qx_{2n+1}, STx_{2n+1}}(t), F_{PABx_{2n}, STx_{2n+1}}(2t)$$
$$* F_{Qx_{2n+1}, ABABx_{2n}}(2t)\} + F_{ABABx_{2n}, STx_{2n+1}}(\phi(t)) * F_{PABx_{2n}, ABABx_{2n}}(\phi(t))$$
$$* F_{Qx_{2n+1}, STx_{2n+1}}(\phi(t)) * F_{PABx_{2n}, STx_{2n+1}}(2\phi(t)) * F_{Qx_{2n+1}, ABABx_{2n}}(2\phi(t)).$$

Letting $n \to \infty$ and using (3.1.11), we get

$$[1 + \alpha F_{ABz, z}(t)] * F_{ABz, z}(t)$$
$$\geq \alpha \min\{F_{ABz, ABz}(t) * F_{z, z}(t), F_{ABz, z}(2t) * F_{z, ABz}(2t)\} + F_{ABz, z}(\phi(t))$$
$$* F_{ABz, ABz}(\phi(t)) * F_{z, z}(\phi(t)) * F_{ABz, z}(2\phi(t)) * F_{z, ABz}(2\phi(t))$$
$$F_{ABz, z}(t) + \alpha \{F_{ABz, z}(t) * F_{ABz, z}(t)\}$$
$$\geq \alpha \min\{1 * 1, F_{ABz, z}(2t)\} + F_{ABz, z}(\phi(t)) * 1 * 1 * F_{ABz, z}(2\phi(t))$$
$$F_{ABz, z}(t) + \alpha F_{ABz, z}(t) \geq \alpha \min\{1, F_{ABz, z}(2t)\} + F_{ABz, z}(\phi(t)) * F_{ABz, z}(2\phi(t))$$
$$F_{ABz, z}(t) + \alpha F_{ABz, z}(t) \geq \alpha \{F_{ABz, z}(t) * F_{z, z}(t)\} + F_{ABz, z}(\phi(t)) * F_{ABz, z}(\phi(t)) * F_{z, z}(\phi(t))$$
$$F_{ABz, z}(t) + \alpha F_{ABz, z}(t) \geq \alpha F_{ABz, z}(t) + F_{ABz, z}(\phi(t))$$
$$F_{ABz, z}(t) \geq F_{ABz, z}(\phi(t))$$

which is a contradiction and we get $ABz = z$.

Step VIII. Put $x = z$ and $y = x_{2n+1}$ in (3.1.5), we get

$$[1 + \alpha F_{ABz, STx_{2n+1}}(t)] * F_{Pz, Qx_{2n+1}}(t)$$
$$\geq \alpha \min\{F_{Pz, ABz}(t) * F_{Qx_{2n+1}, STx_{2n+1}}(t), F_{Pz, STx_{2n+1}}(2t) * F_{Qx_{2n+1}, ABz}(2t)\}$$
$$+ F_{ABz, STx_{2n+1}}(\phi(t)) * F_{Pz, ABz}(\phi(t)) * F_{Qx_{2n+1}, STx_{2n+1}}(\phi(t)) * F_{Pz, STx_{2n+1}}(2\phi(t))$$

$* F_{Qx_{2n+1}, ABz}(2\phi(t)).$

Letting $n \to \infty$ and using (3.1.6), we get

$$[1 + \alpha F_{z, z}(t)] * F_{Pz, z}(t) \geq \alpha \min\{F_{Pz, z}(t) * F_{z, z}(t), F_{Pz, z}(2t) * F_{z, z}(2t)\} + F_{z, z}(\phi(t)) * F_{Pz, z}(\phi(t)) * F_{z, z}(\phi(t)) * F_{Pz, z}(2\phi(t)) * F_{z, z}(2\phi(t))$$

$$F_{Pz, z}(t) + \alpha F_{Pz, z}(t) \geq \alpha \min\{F_{Pz, z}(t), F_{Pz, z}(2t)\} + F_{Pz, z}(\phi(t)) * F_{Pz, z}(2\phi(t))$$

$$F_{Pz, z}(t) + \alpha F_{Pz, z}(t) \geq \alpha \min\{F_{Pz, z}(t), F_{Pz, z}(t) * F_{z, z}(t)\} + F_{Pz, z}(\phi(t)) * F_{z, z}(\phi(t))$$

$$F_{Pz, z}(t) + \alpha F_{Pz, z}(t) \geq \alpha \min\{F_{Pz, z}(t), F_{Pz, z}(t)\} + F_{Pz, z}(\phi(t))$$

$$F_{Pz, z}(t) + \alpha F_{Pz, z}(t) \geq \alpha F_{Pz, z}(t)\} + F_{Pz, z}(\phi(t))$$

$$F_{Pz, z}(t) \geq F_{Pz, z}(\phi(t))$$

which is a contradiction and hence, we get Pz = z.
Hence, Pz = z = ABz.
Further using step III, we get Bz = z.
Thus ABz = z gives Az = z and so Az = Bz = Pz = z.
Also, it follows from steps IV, V and VI that

Sz = Tz = Qz = z.

Hence, we get

Az = Bz = Pz = Sz = Tz = Qz = z

i.e. z is a common fixed point of A, B, P, Q, S and T in this case also.

Uniqueness :

Let z_1 be another common fixed point of A, B, P, Q, S and T. Then

$Az_1 = Bz_1 = Pz_1 = Sz_1 = Tz_1 = Qz_1 = z_1$, assuming $z \neq z_1$.

Put $x = z$ and $y = z_1$ in (3.1.5), we get

$$[1 + \alpha F_{ABz, STz_1}(t)] * F_{Pz, Qz_1}(t) \geq \alpha \min\{F_{Pz, ABz}(t) * F_{Qz_1, STz_1}(t), F_{Pz, STz_1}(2t) * F_{Qz_1, ABz}(2t)\} + F_{ABz, STz_1}(\phi(t)) * F_{Pz, ABz}(\phi(t)) * F_{Qz_1, STz_1}(\phi(t)) * F_{Pz, STz_1}(2\phi(t)) * F_{Qz_1, ABz}(2\phi(t))$$

$$[1 + \alpha F_{z, z_1}(t)] * F_{z, z_1}(t) \geq \alpha \min\{F_{z, z}(t) * F_{z_1, z_1}(t), F_{z, z_1}(2t) * F_{z_1, z}(2t)\} + F_{z, z_1}(\phi(t)) * F_{z, z}(\phi(t)) * F_{z_1, z_1}(\phi(t)) * F_{z, z_1}(2\phi(t)) * F_{z_1, z}(2\phi(t))$$

$$F_{z, z_1}(t) + \alpha\{F_{z, z_1}(t) * F_{z, z_1}(t)\} \geq \alpha \min\{1, F_{z, z_1}(2t)\} + F_{z, z_1}(\phi(t)) * F_{z, z_1}(2\phi(t))$$

$$F_{z, z_1}(t) + \alpha F_{z, z_1}(t) \geq \alpha F_{z, z_1}(2t)\} + F_{z, z_1}(\phi(t)) * F_{z, z_1}(\phi(t)) * F_{z, z}(\phi(t))$$

$$F_{z_1, z}(t) + \alpha F_{z_1, z}(t) \geq \alpha\{F_{z_1, z}(t) * F_{z, z}(t)\} + F_{z_1, z}(\phi(t)) * 1$$

$$F_{z_1, z}(t) + \alpha F_{z_1, z}(t) \geq \alpha F_{z_1, z}(t) + F_{z_1, z}(\phi(t))$$

$$F_{z_1, z}(t) \geq F_{z_1, z}(\phi(t))$$

which is a contradiction.
Hence $z = z_1$ and so z is the unique common fixed point of A, B, S, T, P and Q.

This completes the proof.

Remark 3.1. If we take B = T = I, the identity map on X in theorem 3.1, then condition (3.1.2) is satisfied trivially and we get

Corollary 3.1. Let A, S, P and Q be self maps of a complete Menger space (X, F, *) with * = min satisfying :

(a) $P(X) \subseteq S(X)$, $Q(X) \subseteq A(X)$;

(b) either P or A is continuous;

(c) (P, A) is semi-compatible and (Q, S) is occasionally weak compatible;

(d) $[1 + \alpha F_{Ax,\, Sy}(t)] * F_{Px,\, Qy}(t) \geq \alpha \min\{F_{Px,\, Ax}(t) * F_{Qy,\, Sy}(t),\, F_{Px,\, Sy}(2t) * F_{Qy,\, Ax}(2t)\}$
$+ F_{Ax,\, Sy}(\phi(t)) * F_{Px,\, Ax}(\phi(t)) * F_{Qy,\, Sy}(\phi(t)) * F_{Px,\, Sy}(2\phi(t))$
$* F_{Qy,\, Ax}(2\phi(t))$

for all $x, y \in X$, $t > 0$ and $\phi \in E$.
Then A, S, P and Q have a unique common fixed point in X.

Remark 3.2. In view of remark 3.1, corollary 3.1 is a generalization of the result of Pathak and Verma [9] in the sense that condition of compatibility of the first pair of self maps has been restricted to semi-compatibility.

REFERENCES

[1]. Cho, Y.J., Sharma, B.K. and Sahu, R.D., Semi-compatibility and fixed points, Math. Japon. 42 (1), (1995), 91-98.
[2]. Jain, Arihant and Singh, Bijendra, Common fixed point theorem in Menger space through compatible maps of type (A), Chh. J. Sci. Tech. 2 (2005), 1-12.
[3]. Jain, Arihant and Singh, Bijendra, A fixed point theorem in Menger space through compatible maps of type (A), V.J.M.S. 5(2), (2005), 555-568.
[4]. Jain, Arihant and Singh, Bijendra, Common fixed point theorem in Menger Spaces, The Aligarh Bull. of Math. 25 (1), (2006), 23-31.
[5]. Jungck, G., Compatible mappings and common fixed points, Internat. J. Math. and Math. Sci. 9(4), (1986), 771-779.
[6]. Jungck, G. and Rhoades, B.E., Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. 29(1998), 227-238.
[7]. Menger, K., Statistical metrics, Proc. Nat. Acad. Sci. USA. 28(1942), 535 -537.
[8]. Mishra, S.N., Common fixed points of compatible mappings in PM-spaces, Math. Japon. 36(2), (1991), 283-289.
[9]. Pathak, H.K. and Verma, R.K., Common fixed point theorems for weakly compatible mappings in Menger space and application, Int. Journal of Math. Analysis, Vol. 3, 2009, No. 24, 1199-1206.
[10]. Popa, V., Fixed points for non-surjective expansion mappings satisfying an implicit relation, Bul. Stiint. Univ. Baia Mare Ser. B Fasc. Mat.-Inform. 18 (1), (2002), 105–108.
[11]. Schweizer, B. and Sklar, A., Statistical metric spaces, Pacific J. Math. 10 (1960), 313-334.
[12]. Sehgal, V.M. and Bharucha-Reid, A.T., Fixed points of contraction maps on probabilistic metric spaces, Math. System Theory 6(1972), 97- 102.
[13]. Sessa, S., On a weak commutativity condition of mappings in fixed point consideration, Publ. Inst. Math. Beograd 32(46), (1982), 146-153.
[14]. Singh, B. and Jain, S., Semi-compatibility and fixed point theorem in Menger space using implicit relation, East Asian Math. J. 21 (1), (2005), 65-76.