# NECESSARY AND SUFFICIENT CONDITIONS FOR NEAR-OPTIMALITY HARVESTING CONTROL PROBLEM OF STOCHASTIC AGE-DEPENDENT SYSTEM WITH POISSON JUMPS

**Xining Li[1,*] Qimin Zhang[1,2] & Jianwei Shi[1]**
[1]School of Mathematics and Computer Science, NingXia University,
YinChuan, 750021, P. R. China
[2]Faculty of Informatics and Computer Science, North University for Nationalities, YinChuan, 750021, P. R. China
*Corresponding Author E-mail: qiminzhang89@sina.com

**ABSTRACT**
In this paper, we consider a near-optimal harvesting control problem of stochastic age-dependent system with Poisson jumps. The main aim of the paper is to establish necessary as well as sufficient conditions for near-optimality, satisfied by the harvesting control problem. The proof of the main result is based on Ekeland's variational principle and some estimates on the state and the adjoint processes. Then using It$\hat{o}$'s formula and Barkholder-Davis-Gundy's inequality, the necessary and sufficient conditions for this system are established. At last, an example is given for illustration.

**Keywords:** *Stochastic age-dependent population system; Poisson jump; Near-optimality; Necessary conditions; sufficient conditions*
**MR(2000):** *65B15;60H35;91B70;60H10* .

## 1. INTRODUCTION AND MODEL

Stochastic differential equations are very important in models of biological, chemical, physical and economical systems. Stochastic differential equations with jumps have been found many applications in areas such as economics, finance, and several areas of science and engineering. The study of stochastic age-dependent system has received a great deal of attentions. In this paper, we are going to introduce a near-optimal harvesting control problem of stochastic age-dependent system with Poisson jumps as follows:

$$\begin{cases} \dfrac{\partial p(r,t)}{\partial t}+\dfrac{\partial p(r,t)}{\partial r}=-u(r,t)p(r,t)-\mu(r,t)p(r,t)+f(r,t,p(r,t)) \\ \qquad\qquad +g(r,t,p(r,t))\dfrac{dw_t}{dt}+h(r,t,p(r,t))\dfrac{dN_t}{dt}, \\ p(0,t)=\beta(t)\displaystyle\int_{r_1}^{r_2}m(r,t)p(r,t)dr, \\ p(r,0)=p_0(r), \\ \mathsf{P}(t)=\displaystyle\int_0^A p(r,t)dr, \qquad\qquad (r,t)\in Q. \end{cases} \tag{1.1}$$

where $Q:=(0,A)\times[0,T]$. $t\in(0,T)$, $r\in(0,A)$; $A$ is the life expectancy, $0<A<+\infty$, and $p(A,t)=0$; $[r_1,r_2]$ is the fertility interval, $r_1,r_2\in(0,A)$. The other parameters mean as follows:

$p(r,t)$: the density of the population of age $r$ at time $t$; $\mu(r,t)$: the average mortality ratio of the population of age $r$ at time $t$; $\beta(t)$: the average fertility ratio of the population at time $t$; $m(r,t)$: the ratio of females in the population of age $r$ at time $t$; $p_0(r)$: the initial age distribution of the the population; $\mathsf{P}(t)$: the total population at time $t$; $u(r,t)$: the harvesting effort function, which is the control variable in the model and satisfies: $u\in\mathcal{U}_{ad}:=\{h\in L^2(Q):0\le\zeta_1\le h\le\zeta_2\ a.e.\ in\ Q\}$, where $\zeta_i\in L^2(Q)$, $i=1,2$; $f(r,t,p)+g(r,t,p)\dfrac{d\omega_t}{dt}+h(r,t,p)\dfrac{dN_t}{dt}$: the stochastic perturbation, effecting of external environment on the population system, such as tsunami, earthquakes, emigration, impacts of extra terrestrial objects and so on.

For the system (1.1), L.S. Wang and X.J. Wang [1] studied the convergence of the semi-implicit Euler method. Li, Pang, Wang [2] investigated the numerical analysis. For the system without Poisson jumps (when

$h=0$), Zhang, Han and Ma investigated the convergence of numerical analysis, the convergence of numerical solutions and the convergence of the semi-implicit Euler method, see [3,4,5]. Zhang, Liu and Nie [6] studied the existence, uniqueness and exponential stability. Zhang and Han [7] discussed the existence and uniqueness of strong solutions. For deterministic system (when $g=h=0$), there have been many works on the optimal control problem. For example, Luo [8] studied optimal harvesting control problem for an age-dependent competing system of n species. Zhao [9] and Chen [10] talked about optimal control, respectively. S. Anita [11] investigated optimal harvesting for a nonlinear age-dependent population dynamics.

As we know, there have been a few results on the topic of near-optimal control problems of stochastic age-structure. However, to the best of our knowledge, there is little work on control of stochastic age-dependent system with Poisson jumps. So the aim of this paper is to study near-optimality harvesting control problem of stochastic age-dependent system with Poisson jumps. More precisely, we establish necessary as well as sufficient conditions of near-optimality. These conditions are described in terms of an adjoint process, corresponding to the stochastic partial differential equations components and a nearly maximum condition on the Hamiltonian. Our main result is based on the Ekeland's variational principle. This paper is an extension of [4-11].

Here, the remainder of this paper will progress as follows: In section 2, we give the assumptions, notations, some basic definitions and lemmas. In section 3 and section 4, we establish necessary as well as sufficient conditions of near optimality. In section 5, we provide an example to illustrate our results.

## 2. PRELIMINARIES OF THE PROBLEM

In this paper, let

$$V = H^1([0,A]) \equiv \{\varphi \mid \in L^2([0,A]), \frac{\partial \varphi}{\partial x} \in L^2([0,A]),$$

$$where \frac{\partial \varphi}{\partial x} are\ generalized\ \ partial\ derivatives\}.$$

$V$ is a Sobolev space. $H = L^2([0,A])$ such that

$$V \rightarrowtail H \equiv H' \rightarrowtail V'.$$

$V' = H^{-1}([0,A])$ is the dual space of $V$. We denote by $|\cdot|$ and $\|\cdot\|$ the norms in $V$ and $V'$, respectively; by $\langle \cdot,\cdot \rangle$ the duality product between $V$ and $V'$, and by $(\cdot,\cdot)$ the scalar product in $H$. $K$ is a real separable Hilbert space. For an operator $B \in \mathcal{L}(K,H)$ being the space of all bounded linear operators from $K$ into $H$, we denote by $\|B\|_2$ the Hilbert-Schmidt norm, i.e.

$$\|B\|_2^2 = tr(BWB^T).$$

Throughout this paper, let $(\Omega, \mathcal{F}_t, P)$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$, satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $P$-null sets). Let $\omega_t$ be a Wiener process defined on a complete probability space $(\Omega, \mathcal{F}_t, P)$ taking its values in the separable Hilbert space $K$, with increment covariance operator $\omega$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the $\sigma$-algebras generated by $\{\omega_s, 0 \leq s \leq t\}$, then $\omega_t$ is a martingale relative to $(\mathcal{F}_t)_{t\geq 0}$. Also we assume that the Poisson process $N_t$ is independent of the Brownian motion $\omega_t$.

Here, without loss of generality, the expected cost on the time interval $[0,T]$ is

$$\mathcal{J}(u(\cdot,\cdot)) = E\int_0^T \int_0^A p(r,t)drdt = E\int_0^T \mathcal{P}(t)dt,$$

and the value function is defined as follows:

$$V = \sup_{u(\cdot,\cdot)\in\mathcal{U}_{ad}[0,T]} \mathcal{J}(u(\cdot,\cdot)).$$

Since the objective of this paper is to study near-optimal rather than optimal controls of the system, we give the precise definition of near-optimality as given in Zhou [12].

**Definition 2.1.** ($\varepsilon$-optimal) For a given $\varepsilon > 0$, $u^{\varepsilon}(\cdot,\cdot)$ is called $\varepsilon$-optimal if

$$|\mathcal{J}(u^{\varepsilon}(\cdot,\cdot)) - V| \leq \xi(\varepsilon).$$

Holds for sufficiently small $\varepsilon$, where $\xi$ is a function of $\varepsilon$ satisfying $\xi(\varepsilon) \to 0$ as $\varepsilon \to 0$, the estimate $\xi(\varepsilon)$ is called an error bound. If $\xi(\varepsilon) = C\varepsilon^{\delta}$ for some $\delta > 0$ independent of the constant $C$, then $u^{\varepsilon}(\cdot,\cdot)$ is called near-optimal with order $\varepsilon^{\delta}$.

Throughout this paper, we will write $L_x$ for short of $\dfrac{\partial L(a,b,x)}{\partial x}$ and define

$$F(r,t,p,u) := -u(r,t)p(r,t) - \mu(r,t)p(r,t) + f(r,t,p),$$

also we assume the followings:

$(A_1)$ $\mu(r,t) \in L^l_{loc}(Q), \mu(r,t) \geq 0, \int_0^A \mu(r,t+r-A)dr = +\infty, (r,t) \in Q$, where $\mu(r,t)$ is extended by zero on $(0,A)\times(-\infty,0)$.

$(A_2)$ $f(r,t,p)$, $g(r,t,p)$, $h(r,t,p)$, $u(r,t)$ are continuous in $[0,T]$, bounded and continuously differentiable with respect to $p(r,t)$. And there exists a constant $K_1 > 0$ such that

$$| f(a,b,x) - f(a,b,x')\vee | g(a,b,x) - g(a,b,x')| \vee | h(a,b,x) - h(a,b,x')| \leq K_1 | x - x'|.$$

$(A_3)$ $0 \leq m(r,t) < 1$, $m(r,t) \in Q$, and $m(r,t) \equiv 0$, when $r < r_1$ or $r > r_2$.

$(A_4)$ $0 \leq \beta_0 < \beta(t) \leq \beta^0$, $\forall t > 0$, $\beta_0$ and $\beta^0$ are constants.

$(A_5)$ $p_0 \in L_0^{\infty}(0,A)$, $p_0(r) \geq 0$, $\forall r \in (0,A)$.

$(A_6)$ there are constants $D_0$, $D_1$, $D_2 > 0$ such that

$$| f(t,p,u) - f(t,p,u')| + | f_u(t,p,u) - f_u(t,p,u')| \leq D_0 | u - u'|,$$
$$| g(t,p,u) - g(t,p,u')| + | g_u(t,p,u) - g_u(t,p,u')| \leq D_1 | u - u'|,$$
$$| h(t,p,u) - h(t,p,u')| + | h_u(t,p,u) - h_u(t,p,u')| \leq D_2 | u - u'|.$$

The Hamiltonian function is given by

$$\begin{aligned}
& H(r,t,p,u,\phi,\psi,k,l) := \mathcal{P}(t) - \langle \phi(r,t), \frac{\partial p(r,t)}{\partial r}\rangle + (\psi(r,t), F(r,t,p,u)) \\
& + k(r,t)g(r,t,p) + l(r,t)h(r,t,p) \\
&= \mathcal{P}(t) - \langle \phi(r,t), \frac{\partial p(r,t)}{\partial r}\rangle + (-u(r,t)p(r,t) - \mu(r,t)p(r,t) \\
& + f(r,t,p), \psi(r,t)) + k(r,t)g(r,t,p) + l(r,t)h(r,t,p) \\
&= \mathcal{P}(t) - \int_0^A \phi(r,t)\frac{\partial p(r,t)}{\partial r}dr + (-u(r,t)p(r,t) - \mu(r,t)p(r,t) \\
& + f(r,t,p), \psi(r,t)) + k(r,t)g(r,t,p) + l(r,t)h(r,t,p) \\
&= \mathcal{P}(t) + p(0,t)\phi(r,t) + (-u(r,t)p(r,t) - \mu(r,t)p(r,t) \\
& + f(r,t,p), \psi(r,t)) + k(r,t)g(r,t,p) + l(r,t)h(r,t,p) \\
&= \mathcal{P}(t) + p(0,t)\phi(r,t) - (u(r,t)p(r,t) + \mu(r,t)p(r,t) \\
& - f(r,t,p))\psi(r,t) + k(r,t)g(r,t,p) + l(r,t)h(r,t,p).
\end{aligned}$$

Under the assumption $(A_2)$, there is a unique solution $p(\cdot,\cdot) \in L^2_{\mathcal{F}}([0,A]\times[0,T],H)$ which solves $(1.1)$, where $L^2_{\mathcal{F}}([0,A]\times[0,T],H)$ denotes the Hilbert space of $\mathcal{F}_1$-adapted processes $\mathcal{P}(\cdot)$ such that

$$E\int_0^t |\mathcal{P}(s)|^2 \, ds < +\infty.$$

For any $u(\cdot,\cdot) \in \mathcal{U}_{ad}$ with its corresponding state trajectory $p(\cdot,\cdot)$, We introduce the adjoint equation for our problem.

The adjoint equation can be written as:

$$\begin{cases} \frac{\partial \psi(r,t)}{\partial t} \\ \quad = -\frac{\partial \psi(r,t)}{\partial r} - \frac{\partial H}{\partial p} \\ \quad = -\frac{\partial \psi(r,t)}{\partial r} + (\mu(r,t) + u(r,t) - f_p(r,t,p))\psi(r,t) \\ \quad\quad - g_p k(r,t) - h_p l(r,t) + k(r,t) d\omega_t + l(r,t) dN_t, \\ \psi(r,T) = \psi(A,t) = 0, \qquad (r,t) \in Q. \end{cases} \tag{2.1}$$

Note that the couple $(\psi(\cdot,\cdot), k(\cdot,\cdot), l(\cdot,\cdot))$ is the adjoint process corresponding to the stochastic age-dependent system $p(r,t)$. It is a well known fact under assumption $(A_2)$. The adjoint equation admits one and only one $\mathcal{F}_t$-adapted solution $(\psi(r,t), k(r,t), l(r,t))$.

Moreover, since $F_p$, $g_p$, $l_p$ are bounded by, there exists a constant $C > 0$ independent of $(p(\cdot,\cdot), u(\cdot,\cdot))$, such that the solution of the adjoint equation satisfy the following estimate:

$$E(\sup_{0\le t\le T} \sup_{0\le r\le A} |\psi(r,t)|^2) + E\int_Q |k(r,t)|^2 \, ds + E\int_Q |l(r,t)|^2 \, ds < C. \tag{2.2}$$

Let us recall Ekeland's variational principle which will be used in the sequel.

**Lemma 2.2.** (Ekeland[13]) Let $(V,d)$ be a complete metric space and $F : V \to R \cup \{+\infty\}$ be a lower semicontinuous function, bounded from below. If for each $\varepsilon > 0$, there exists $u^\varepsilon \in V$ such that $F(u^\varepsilon) \le \inf_{u\in V} F(u) + \varepsilon$. Then for any $\delta > 0$, there exists $u^\delta \in V$ such that

$(i)$ $F(u^\delta) \le F(u^\varepsilon)$,

$(ii)$ $d(u^\delta, u^\varepsilon) \le \delta$,

$(iii)$ $F(u^\delta) \le F(u) + \frac{\varepsilon}{\delta} d(u, u^\delta)$, for all $u \in V$.

For $u, v \in \mathcal{U}_{ad}$, we define

$$d(u,v) = dt \otimes P\{(t,\omega) \in [0,T] \times \Omega : u(r,t,\omega) \ne v(r,t,\omega)\}. \tag{2.3}$$

Where $dt \otimes q$ is the product measure of the Lebesgue measure $dt$ with the probability measure $q$. It is well known that $(\mathcal{U}_{ad}, d)$ is a complete metric space (see[12,14]).

## 3. NECESSARY CONDITIONS OF NEAR-OPTIMALITY

This is the main result of this paper. In this section, we derive necessary conditions for a control to be near-optimal.

**Lemma 3.1.** For any $0 < \alpha < 1$ and $0 < q \le 2$, there is a constant $c_1 = c_1(\alpha, q) > 0$, such that for any $u(r,t), u'(r,t) \in \mathcal{U}_{ad}$, along with the corresponding trajectories $p(r,t)$ and $p'(r,t)$, it holds that

$$E[\sup_{0\le t\le T} | p(r,t) - p'(r,t)|^q] \le c_1 d(u(r,t), u'(r,t))^{\frac{\alpha q}{2}}. \tag{3.1}$$

**Proof.** First, we assume $q = 2$. Applying It$\hat{o}$'s formula to $| p(r,t) - p'(r,t)|^2$, we obtain

$$
\begin{aligned}
&| p(r,t) - p'(r,t) |^2 \\
&= 2\int_0^t \langle -\frac{\partial p}{\partial r} + \frac{\partial p'}{\partial r} - \mu(r,s)(p-p'), p-p' \rangle ds \\
&\quad - 2\int_0^t \langle up - u'p', p-p' \rangle ds + 2\int_0^t ( f(r,s,p) - f(r,s,p'), p-p')ds \\
&\quad + 2\int_0^t (p-p', g(r,s,p) - g(r,s,p'))d\omega_s + \int_0^t \| g(r,s,p) - g(r,s,p') \|_2^2 ds \\
&\quad + 2\int_0^t (p-p', h(r,s,p) - h(r,s,p'))dN_s + \int_0^t | h(r,s,p) - h(r,s,p') |^2 \ ds \qquad (3.2) \\
&\le -2\int_0^t \langle \frac{\partial p - p'}{\partial r}, p-p' \rangle ds - 2\mu_0 \int_0^t (p-p', p-p')ds \\
&\quad - 2\int_0^t \langle up - u'p', p-p' \rangle + 2\int_0^t ( f(r,s,p) - f(r,s,p'), p-p')ds \\
&\quad + 2\int_0^t (p-p', (g(r,s,p) - g(r,s,p'))d\omega_s) + \int_0^t \| g(r,s,p) - g(r,s,p') \|_2^2 ds \\
&\quad + 2\int_0^t (p-p', (h(r,s,p) - h(r,s,p'))dN_s) + \int_0^t | h(r,s,p) - h(r,s,p') |^2 \ dN_s .
\end{aligned}
$$

where $p := p(r,t), p' = p'(r,t)$.

Considering the first term in the right hand of $(3.2)$, we have

$$
\begin{aligned}
&-\langle \frac{\partial (p-p')}{\partial r}, p-p' \rangle \\
&= -\int_0^A (p-p')dr(p-p') \\
&= -\frac{1}{2} | (p-p')^2 |_0^A \\
&= \frac{1}{2}[\beta(s)\int_0^A m(r,s)(p-p')dr]^2 ,
\end{aligned}
$$

then by ( $A_5$ ) and H$\ddot{o}$lder inequality, we can gain

$$
\begin{aligned}
&-\langle \frac{\partial (p-p')}{\partial r}, p-p' \rangle \\
&= \frac{1}{2}[\beta(s)\int_0^A m(r,s)(p-p')dr]^2 \\
&\le \frac{1}{2}\beta^2(s)\int_0^A dr \int_0^A (p-p')^2 dr \qquad (3.3) \\
&\le \frac{1}{2} A\overline{\beta}^2 \ | p-p' |^2 \ .
\end{aligned}
$$

Let us estimate the third term in the right hand of $(3.2)$

$$
\begin{aligned}
&-2\int_0^t \langle up - u'p', p-p' \rangle \\
&\le 4c_{11} | p-p' |^2 + 4(| p' |^2 + | u' - u |^2 + 4 | p-p' |^2)\chi_{u \ne u'} \quad (3.4) \\
&\le c_{12} | p-p' |^2 + c_{13} d(u,u').
\end{aligned}
$$

Now, let us turn to the second term of $(3.2)$, by Barkholder-Davis-Gundy's inequality, we have

$$\begin{aligned}
&E[\sup_{0\le s\le t}\int_0^s (p-p',(g(r,s,p)-g(r,s,p'))d\omega_r]\\
&\le 3E[\sup_{0\le s\le t}|p-p'|(\int_0^t (\|g(r,s,p)-g(r,s,p')\|_2^2 ds)^{\frac{1}{2}}]\\
&\le \frac{1}{4}E[\sup_{0\le s\le t}|p-p'|^2 + K_1\int_0^t \|g(r,s,p)-g(r,s,p')\|_2^2 ds]\\
&\le \frac{1}{4}E[\sup_{0\le s\le t}|p-p'|^2] + K_1\cdot K_2^2\int_0^t E\|p-p'\|_c^2 ds.
\end{aligned} \quad (3.5)$$

for some positive constant $K_1, K_2 > 0$.

Applying $\overline{N}(t) = N(t) - \lambda t$ to the last line of $(3.2)$, we have

$$\begin{aligned}
&2\int_0^t (p-p',h(r,s,p)-h(r,s,p'))dN_s + \int_0^t |h(r,s,p)-h(r,s,p')|^2\ dN_s\\
&= 2\int_0^t\int_O (p-p')(h(r,s,p)-h(r,s,p'))d\overline{N}_s\\
&\quad + 2\lambda(\int_0^t\int_O (p-p')(h(r,s,p)-h(r,s,p'))ds)\\
&\quad + \int_0^t |h(r,s,p)-h(r,s,p')|^2\ d\overline{N}_s + \lambda\int_0^t |h(r,s,p)-h(r,s,p')|^2\ ds.
\end{aligned}$$

Then by Burkholder-Davis-Gundy's inequality, there exists positive constants $K_3$ and $K_4$, we have

$$\begin{aligned}
&E\sup_{0\le s\le t}[\int_0^s (p-p',h(r,s,p)-h(r,s,p'))d\overline{N}_s\\
&\quad + \int_0^t |h(r,s,p)-h(r,s,p')|^2\ d\overline{N}_s]\\
&\le \frac{1}{4}E\sup_{0\le s\le t}|p-p'|^2 + K_3\cdot K_4^2\int_0^t E\|p-p'\|_c^2 ds.
\end{aligned} \quad (3.6)$$

With $(3.2)-(3.6)$, for $\forall t\in[0,T]$, $(3.1)$ can be proved.

As given in K. Bahlali [15], we give the following lemma. It gives the continuity of the solutions to the adjoint equation with respect to the control variable. It plays a key role in proving the necessary conditions.

**Lemma 3.2.** For any $0<\alpha<1$ and $1<n<2$ satisfying $(1+\alpha\xi)n<2$, there is a constant $c_2 = c_2(\alpha,\xi,n) > 0$ such that for any $u(r,t)$, $u'(r,t)\in\mathcal{U}_{ad}$, along with the corresponding trajectories $p(r,t)$, $p'(r,t)$ and the solutions $(\psi(\cdot,\cdot),k(\cdot,\cdot),l(\cdot,\cdot))$, $(\psi'(\cdot,\cdot),k'(\cdot,\cdot),l'(\cdot,\cdot))$ of the corresponding adjoint equation, it holds that

$$\begin{aligned}
&E\int_0^T \{|\psi(r,t)-\psi'(r,t)|^n + \|k(r,t)-k'(r,t)\|^n\\
&\qquad + |l(r,t)-l'(r,t)|\}dt \le c_2 d(u((r,t)),u'((r,t)))^{\frac{\alpha\xi n}{2}}.
\end{aligned} \quad (3.7)$$

**Proof.** Now, let us prove $(3.7)$. Noting that $(\overline{\psi}(r,t),\bar{k}(r,t),\bar{l}(r,t)) \equiv (\psi(r,t)-\psi'(r,t),k(r,t)-k'(r,t),l(r,t)-l'(r,t))$ satisfies the following backward stochastic differential equation:

$$\begin{cases} \frac{\partial \overline{\psi}(r,t)}{\partial t} + \frac{\partial \overline{\psi}(r,t)}{\partial r} = [\mu(r,t) + u(r,t) - f_p(r,t,p)]\overline{\psi}(r,t) \\ \quad - g_p(r,t,p)\bar{k}(r,t) - h_p(r,t,p)\bar{l}(r,t) \\ \quad + [(\mu(r,t) + u(r,t) - f_p(r,t,p)) \\ \quad - (\mu'(r,t) + u'(r,t) - f_p(r,t,p'))]\psi'(r,t) \\ \quad + (g_p(r,t,p') - g_p(r,t,p))k'(r,t) + k(r,t)\frac{d\omega_t}{dt} \\ \quad + (h_p(r,t,p') - h_p(r,t,p))l'(r,t) + l(r,t)\frac{dN_t}{dt}, \\ \psi(r,T) = \psi(A,t) = 0, \qquad (r,t) \in Q. \end{cases}$$

Let $\eta$ be the solution of the following linear stochastic differential equation (SDE):

$$\begin{cases} d\eta(r,t) = \{(-u(r,t)p - \mu(r,t)p + f(r,t,p))\eta(r,t) + |\overline{\psi}(r,t)|^{n-1} sgn(\overline{\psi}(r,t))\}dt \\ \qquad + \{g_p(r,t,p)\eta(r,t) + |\bar{k}(r,t)|^{n-1} sgn(\bar{k}(r,t))\}d\omega_t \\ \qquad + \{h_p(r,t,p)\eta(r,t) + |\bar{l}(r,t)|^{n-1} sgn(\bar{l}(r,t))\}dN_t, \\ \eta(r,0) = 0. \end{cases}$$

Where $sgn(a)$ is the indicative function for any $a$. Note that the existence and uniqueness of solutions to the above equation are verified by assumption $(A_2)$ and the fact that

$$E\int_0^T \{\| \overline{\psi}(r,t)|^{q'-1} sgn(\overline{\psi}(r,t))|^2 + \| \bar{k}(r,t)|^{q'-1} sgn(\bar{k}(r,t))|^2 + \| \bar{l}(r,t)|^{q'-1} sgn(\bar{l}(r,t))|^2\}dt < +\infty.$$

Then we have

$$\begin{aligned} E\sup_{0<t\le T} |\eta(r,t)|^q &\le c_{21}E\int_0^T \{|\bar{q}(r,t)|^{qq'-q} + |\bar{k}(r,t)|^{qq'-q} + |\bar{l}(r,t)|^{qq'-q}\}dt \\ &\le c_{21}E\int_0^T \{|\bar{q}(r,t)|^{q'} + |\bar{k}(r,t)|^{q'} + |\bar{l}(r,t)|^{q'}\}dt. \end{aligned} \tag{3.8}$$

Where $q > 2$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Note that the right hand side term of $(3.8)$ is bounded due to $(2.2)$. On the other hand, applying It$\hat{o}$'s formula to $\overline{\psi}(r,t)$, $\eta(r,t)$ and taking expectations, we obtain

$$\begin{aligned} &E\int_0^T \{\overline{\psi}(r,t)\cdot[|\overline{\psi}(r,t)|^{n-1} sgn(\overline{\psi}(r,t))] + \bar{k}(r,t)\cdot[|\bar{k}(r,t)|^{n-1} sgn(\bar{k}(r,t))] \\ &+ \bar{l}(r,t)\cdot[|\bar{l}(r,t)|^{n-1} sgn(\bar{l}(r,t))]\}dt \\ = &E\int_0^T \{[-\frac{\partial \overline{\psi}(r,t)}{\partial r} + (\mu(r,t) + u(r,t) - f_p(r,t,p))\overline{\psi}]\eta(r,t) \\ &+ [(-u(r,t) - \mu(r,t) + f_p(r,t,p)) - (-u'(r,t) - \mu(r,t) + f_p(r,t,p'))]\psi'(r,t) \\ &+ (g_p(r,t,p) - g_p(r,t,p'))k'(r,t)\eta(r,t) + (h_p(r,t,p) - h_p(r,t,p'))l'(r,t)\eta(r,t)\}dt \\ &- E\int_0^t \beta(t)m(r,t)\overline{\psi}(r,t)\eta(r,t)dt. \end{aligned} \tag{3.9}$$

We proceed to estimate the second term in the right hand side of $(3.9)$. First

$$\begin{aligned}&\{E\int_0^T |(-u(r,t)+f_p(r,t,p))-(-u'(r,t)\\&+f_p(r,t,p'))\psi'(r,t)|^p\ dt\}^{\frac{1}{p}}\{E\int_0^T|\eta(r,t)|^q\ dt\}^{\frac{1}{q}}\\ \le\ & c_{22}\{E\int_0^T[|\bar\psi(r,t)|^q+|\bar k(r,t)|^q+|\bar l(r,t)|^q]dt\}^{\frac{1}{q}}\\&\{E\int_0^T|(-u(r,t)+f_p(r,t,p))-(-u'(r,t)\\&+f_p(r,t,p'))|^q|\psi'(r,t)|^q\ dt\}.\end{aligned}\tag{3.10}$$

From Lemma 3.1 and using the fact that $E\sup_{0\le t\le T}|\psi'(r,t)|^q<\infty$ for any $q$, it can easily check that

$$\begin{aligned}&\{E\int_0^T |(-u(r,t)+f_p(r,t,p))-(-u'(r,t)\\&+f_p(r,t,p'))\psi'(r,t)|^q\ dt\}^{\frac{1}{q}}\{E\int_0^T|\eta(r,t)|^{q'}\ dt\}^{\frac{1}{q'}}\\ \le\ & c_{23}d(u(r,t),u'(r,t))^{\frac{\alpha\beta q}{2}}.\end{aligned}\tag{3.11}$$

Using similar arguments developed above, we can easily prove that

$$\begin{aligned}&\{E\int_0^T|(g_p(r,t,p)-g_p(r,t,p'))k'(r,t)|^q\ dt\}^{\frac{1}{q}}\{E\int_0^T|\eta(r,t)|^{q'}\}^{\frac{1}{q'}}dt\\ \le\ & c_{24}d(u(r,t),u'(r,t))^{\frac{\alpha\beta q}{2}}.\end{aligned}\tag{3.12}$$

It follows from $(3.10)-(3.12)$ that

$$\begin{aligned}&E\int_0^T\{[(-u(r,t)-\mu(r,t)+f_p(r,t,p))-(-u'(r,t)-\mu(r,t)\\&+f_p(r,t,p'))]\psi'(r,t)+(g_p(r,t,p)-g_p(r,t,p'))k'(r,t)\eta(r,t)\\&+(h_p(r,t,p)-h_p(r,t,p'))l'(r,t)\eta(r,t)\}dt\\ \le\ & c_{25}d(u(r,t),u'(r,t))^{\frac{\alpha\beta q}{2}}.\end{aligned}\tag{3.13}$$

Next, we proceed to estimate the first term in the right side of $(3.9)$. By $(A_2)-(A_5)$ and using the fact that $\psi(r,t)$ is bounded, we obtain

$$\begin{aligned}&[\mu(r,t)+u(r,t)-f_p(r,t,p)]\bar\psi(r,t)\\ \le\ & c_{26}|\bar\psi(r,t)|^q\ dt\\ \le\ & c_{27}d(u(r,t),u'(r,t))^{\frac{\alpha\beta q}{2}}.\end{aligned}\tag{3.14}$$

We proceed to estimate the third term in the right side of $(3.9)$. From Lemma 3.1 and assumption

$$E\int_0^T|\beta(t)m(r,t)\bar\psi(r,t)\eta(r,t)|^q\ dt\le c_{28}d(u(r,t),u'(r,t))^{\frac{\alpha\beta q}{2}}.\tag{3.15}$$

It follows from $(3.12)-(3.14)$ that

$$E\int_0^T\{|\bar\psi(r,t)|^n+|\bar k(r,t)|^n+|\bar l(r,t)|^n\}dt\le c_{29}d(u(r,t),u'(r,t))^{\frac{\alpha\beta q}{2}}.$$

Now we have completed the proof of Lemma 3.2.

**Theorem 3.3.** Assume that $(A_6)$ hold, there is a constant $c_3$ such that for any $0\le\beta<1$, $\varepsilon>0$ and $\varepsilon$-optimal, the necessary condition

$$\max_{u\in\mathcal{U}_{ad}} E\int_0^T H(t,p^\varepsilon(r,t),u^\varepsilon(r,t),\psi^\varepsilon(r,t),k^\varepsilon(r,t),l^\varepsilon(r,t))dt$$
$$\leq \int_0^T H(t,p^\varepsilon(r,t),u(r,t),\psi^\varepsilon(r,t),k^\varepsilon(r,t),l^\varepsilon(r,t))dt + c_3\varepsilon^{\frac{\beta}{3}}. \quad (3.16)$$

hold, where $(\psi^\varepsilon(r,t),k^\varepsilon(r,t),l^\varepsilon(r,t))$ is the solution of adjoint equation $(2.1)$. $H$ is the Hamiltonian function.

**Proof.** Let us use two steps to complete it.
Step 1: From assumptions ( $A_1$ ) and ( $A_2$ ), it is easy to see that $J(u(r,t))$ is continuous on $\mathcal{U}_{ad}$ endowed with the metric defined by $(2.2)$. Applying Ekeland's variational principle with $\delta=\varepsilon^{\frac{2}{3}}$, there is an admissible control $\tilde{u}^\varepsilon(r,t)$ such that

$$d(u^\varepsilon(r,t),\tilde{u}^\varepsilon(r,t))\leq\varepsilon^{\frac{2}{3}}, \quad (3.17)$$
$$\tilde{\mathcal{J}}(\tilde{u}^\varepsilon(r,t))\leq\tilde{\mathcal{J}}(u^\varepsilon(r,t)).$$

for any $u(r,t)\in\mathcal{U}_{ad}$, where

$$\tilde{\mathcal{J}}(\tilde{u}^\varepsilon(r,t))=\mathcal{J}(u^\varepsilon(r,t))+\varepsilon^{\frac{1}{3}}d(u^\varepsilon(r,t),\tilde{u}^\varepsilon(r,t)).$$

This means that $\tilde{u}^\varepsilon(r,t)$ is optimal for the system with the new cost function $\tilde{\mathcal{J}}$. Let $t_0\in[0,T]$ and $u\in\mathcal{U}_{ad}$ be fixed. For any $\rho>0$, define the spike variation $u^0\in\mathcal{U}_{ad}[0,T]$ of

$$\tilde{u}^\rho(r,t)=\begin{cases} u(r,t), & t\in[t_0,t_0+\rho] \\ \tilde{u}^\varepsilon(r,t), & otherwise\end{cases}$$

The fact that

$$\tilde{\mathcal{J}}(\tilde{u}^\varepsilon(r,t))\leq\tilde{\mathcal{J}}(u^\rho(r,t)),$$

and

$$d(u^\rho(r,t),\tilde{u}^\varepsilon(r,t))\leq\rho.(3.18)$$

From $(3.18)$, Lemma 2.2 and Taylor's expansion, we derive

$$\begin{aligned}
-\rho\varepsilon^{\frac{1}{3}} &\leq \mathcal{J}(u^\rho(r,t))-\mathcal{J}(\tilde{u}^\varepsilon(r,t))\\
&= E\int_0^T[\mathcal{P}(t,p^\rho,u^\rho)-\mathcal{P}(t,p^\varepsilon,\tilde{u}^\varepsilon)]dt\\
&= E\int_0^T\{[\mathcal{P}(t,p^\rho,u^\rho)-\mathcal{P}(t,\tilde{p}^\varepsilon,u^\rho)]+[\mathcal{P}(t,\tilde{p}^\varepsilon,u^\rho)-\mathcal{P}(t,\tilde{p}^\varepsilon,\tilde{u}^\varepsilon)]\}dt\\
&\leq E\{\int_0^T\mathcal{P}_p(t,\tilde{p}^\varepsilon,u^\rho)(p^\rho-\tilde{p}^\varepsilon)dt+\int_{\tilde{t}}^{\tilde{t}+\rho}[\mathcal{P}(t,\tilde{p}^\varepsilon,u)-\mathcal{P}(t,\tilde{p}^\varepsilon,\tilde{u})]dt+0(\rho)\}.
\end{aligned} \quad (3.19)$$

Let we notice assumption $(A_2)$ and Lemma 3.1,

$$-\rho\varepsilon^{\frac{1}{3}}\leq E\int_{\tilde{t}}^{\tilde{t}+\rho}[\mathcal{P}(t,\tilde{p}^\varepsilon,u)-\mathcal{P}(t,\tilde{p}^\varepsilon,\tilde{u}^\varepsilon)]dt+E\int_{\tilde{t}}^{\tilde{t}+\rho}\mathcal{P}(u-\tilde{u}^\varepsilon)\tilde{\psi}^\varepsilon(r,t)dt.(3.20)$$

Let $\rho\to0$, we get

$$-\varepsilon^{\frac{1}{3}}\leq E[\mathcal{P}(\tilde{t},\tilde{p}^\varepsilon,u)-\mathcal{P}(\tilde{t},\tilde{p}^\varepsilon,\tilde{u}^\varepsilon)]+E[p(u-\tilde{u}^\varepsilon)\tilde{\psi}^\varepsilon(r,t)].(3.21)$$

i.e.

$$-\varepsilon^{\frac{1}{3}}\leq E[H(r,\tilde{t},\tilde{p}^\varepsilon,u,\tilde{\psi}^\varepsilon,\tilde{k}^\varepsilon,\tilde{l}^\varepsilon)-H(r,\tilde{t},\tilde{p}^\varepsilon,\tilde{u},\tilde{\psi}^\varepsilon,\tilde{k}^\varepsilon,\tilde{l}^\varepsilon)]$$

Step 2: Necessary condition for $(\tilde{p}^{\varepsilon},\tilde{u}^{\varepsilon})$.

Let us replace all the $(\tilde{p}^{\varepsilon},\tilde{u}^{\varepsilon})$ by $(p^{\varepsilon},u^{\varepsilon})$ in $(3.21)$. Then we estimate the following difference

$$E\int_0^T[\mathcal{P}(t,\tilde{p}^{\varepsilon},u^{\rho})-\mathcal{P}(t,\tilde{p}^{\varepsilon},\tilde{u}^{\varepsilon})]dt-E\int_0^T[\mathcal{P}(t,p^{\varepsilon},u^{\rho})-\mathcal{P}(t,p^{\varepsilon},u^{\varepsilon})]dt$$
$$+E\int_0^T p(u^{\rho}-\tilde{u}^{\varepsilon})\tilde{\psi}^{\varepsilon}dt-E\int_0^T p(u^{\rho}-u^{\varepsilon})\psi^{\varepsilon}dt$$
$$\le c_{31}\varepsilon^{\frac{\beta}{3}}.$$

Then, we have

$$-c_{32}\varepsilon^{\frac{\beta}{3}}\le E\int_0^T[\mathcal{P}(t,p^{\varepsilon},u^{\rho})-\mathcal{P}(t,p^{\varepsilon},u^{\varepsilon})]dt+E\int_0^T p(u^{\rho}-u^{\varepsilon})\psi^{\varepsilon}dt$$
$$=E\int_0^T H(t,p^{\varepsilon},u^{\rho},\psi^{\varepsilon},k^{\varepsilon},l^{\varepsilon})dt-E\int_0^T H(t,p^{\varepsilon},u^{\varepsilon},\psi^{\varepsilon},k^{\varepsilon},l^{\varepsilon})dt.$$

So the equation $(3.17)$ is proved.

## 4. SUFFICIENT CONDITIONS OF NEAR-OPTIMALITY

In this section, we will prove that the near-maximum condition on the Hamiltonian function is a sufficient condition for near-optimality, under additional assumption.

**Theorem 4.1.** Let $(p^{\varepsilon}(r,t),u^{\varepsilon}(r,t))$ be the near-optimal solution of the state equation, and $(\psi^{\varepsilon}(r,t),k^{\varepsilon}(r,t),l^{\varepsilon}(r,t))$ is the solution of the adjoint equation, corresponding to $(\lambda^{\varepsilon}(r,t),u^{\varepsilon}(r,t))$. Assume that $H(t,\cdots,\psi^{\varepsilon}(r,t),k^{\varepsilon}(r,t),l^{\varepsilon}(r,t))$ is concave for $a.e.t\in[0,T]$, if for some $\varepsilon>0$,

$$E\int_0^T H(t,\lambda^{\varepsilon}(r,t),u^{\varepsilon}(r,t),\psi^{\varepsilon}(r,t),k^{\varepsilon}(r,t),l^{\varepsilon}(r,t))dt$$
$$\ge\sup_{u(r,t)\in\mathcal{U}_{ad}[0,T]}E\int_0^T H(t,\lambda^{\varepsilon}(r,t),u(r,t),\psi^{\varepsilon}(r,t),k^{\varepsilon}(r,t),l^{\varepsilon}(r,t))dt-\varepsilon^{\gamma}. \quad (4.1)$$

Then

$$\mathcal{J}(u^{\varepsilon}(r,t))\le\inf_{u(r,t)\in\mathcal{U}_{ad}[0,T]}\mathcal{J}(u(r,t))+D\varepsilon^{\frac{1}{2}}. \quad (4.2)$$

Where $D>0$ is a constant, which is independent from $\varepsilon$.

**Proof.** Let us fix $\varepsilon>0$ and define a new metric $\tilde{d}$ on $\mathcal{U}_{ad}$ as follows

$$\tilde{d}(u(r,t),u^{\varepsilon}(r,t))=E\int_0^T y^{\varepsilon}(r,t)\,|\,u(r,t)-u^{\varepsilon}(r,t)\,|\,dt. \quad (4.3)$$

Where

$$y^{\varepsilon}(r,t)=1+|\,\psi^{\varepsilon}(r,t)\,|\ge 1. \quad (4.4)$$

Define a function $I$ on $\mathcal{U}_{ad}[0,T]$ by:

$$I(u(r,t))=E\int_0^T H(t,\lambda^{\varepsilon}(r,t),u(r,t),(\psi^{\varepsilon}(r,t),k^{\varepsilon}(r,t),l^{\varepsilon}(r,t)))dt$$

A simple computation shows that

$$|\,I(u(r,t))-I(u¡®(r,t))\,|\le D_1E\int_0^T y^{\varepsilon}(r,t)\,|\,u(r,t)-u^{\varepsilon}(r,t)\,|\,dt.$$

which implies that $I$ is continuous on $\mathcal{U}_{ad}$ with respect to $\tilde{d}$.

By $(4.1)$ and Ekeland's lemma, there exists $\tilde{u}^{\varepsilon}(r,t)\in\mathcal{U}_{ad}[0,T]$ such that

$$\tilde{d}(u^{\varepsilon}(r,t),\tilde{u}^{\varepsilon}(r,t))\le\varepsilon,$$

and

$$E\int_0^T \tilde{H}(t, p^\varepsilon(r,t), \tilde{u}(r,t))dt = \max_{u(r,t)\in\mathcal{U}_{ad}[0,T]} E\int_0^T \tilde{H}(t, p^\varepsilon(r,t), u(r,t))dt, \quad (4.5)$$

where

$$\begin{aligned}\tilde{H}(t, p^\varepsilon(r,t), u(r,t)) &= H(t, p^\varepsilon(r,t), u(r,t), (\psi^\varepsilon(r,t), k^\varepsilon(r,t), l^\varepsilon(r,t))) \\ &\quad - \varepsilon^{\frac{1}{2}} y^\varepsilon(r,t) \,|\, u(r,t) - \tilde{u}(r,t) \,|.\end{aligned} \quad (4.6)$$

By standard arguments, it can be proved that the integral maximum condition $(4.6)$ implies a pointwise maximum condition, namely for $a.e.t \in [0,T]$ and $p-a.s.$.

$$\tilde{H}(t, p^\varepsilon(r,t), \tilde{u}^\varepsilon(r,t)) = \max_{u(r,t)\in\mathcal{U}_{ad}} \tilde{H}(t, p^\varepsilon(r,t), u(r,t)). \quad (4.7)$$

Using assumption $(A_2)$, we can prove that

$$\begin{aligned}& H_u(t, p^\varepsilon(r,t), \tilde{u}^\varepsilon(r,t), (\psi^\varepsilon(r,t), k^\varepsilon(r,t), l^\varepsilon(r,t))) \\ &\le D_2 y^\varepsilon(r,t) \,|\, u^\varepsilon(r,t) - \tilde{u}^\varepsilon(r,t) \,| + \varepsilon^{\frac{1}{2}} y^\varepsilon(r,t).\end{aligned} \quad (4.8)$$

By the concavity of $H(t, p, u, \psi^\varepsilon(r,t), k^\varepsilon(r,t), l^\varepsilon(r,t))$, we have

$$\begin{aligned}& H(t, p, u, (\psi^\varepsilon(r,t), k^\varepsilon(r,t), l^\varepsilon(r,t))) \\ & - H(t, p^\varepsilon, u^\varepsilon, (\psi^\varepsilon(r,t), k^\varepsilon(r,t), l^\varepsilon(r,t))) \\ \le\ & H_p(t, p^\varepsilon, u^\varepsilon, (\psi^\varepsilon(r,t), k^\varepsilon(r,t), l^\varepsilon(r,t))) \\ & + H_u(t, p^\varepsilon, u^\varepsilon, (\psi^\varepsilon(r,t), k^\varepsilon(r,t), l^\varepsilon(r,t))).\end{aligned} \quad (4.9)$$

By integrating both sides and noting and $(4.8)$, then we can obtain

$$\begin{aligned}& E\int_0^T \{H(t, p, u, (\psi^\varepsilon(r,t), k^\varepsilon(r,t), l^\varepsilon(r,t))) \\ & - H(t, p^\varepsilon, u^\varepsilon, (\psi^\varepsilon(r,t), k^\varepsilon(r,t), l^\varepsilon(r,t)))\}dt \\ \le\ & E\int_0^T H_p(t, p^\varepsilon, u^\varepsilon, (\psi^\varepsilon(r,t), k^\varepsilon(r,t), l^\varepsilon(r,t)))(p - p^\varepsilon)dt + D_3 \varepsilon^{\frac{1}{2}}.\end{aligned} \quad (4.10)$$

On the other hand, by applying It$\hat{o}$'s formula respectively to $\psi^\varepsilon(p - p^\varepsilon)$ and by taking expectations, we can obtain

$$\begin{aligned}& E[\psi^\varepsilon(r,T)(p(r,T) - p^\varepsilon(r,T))] + E\int_0^T \{k^\varepsilon(r,t)(g(r,t,p) - g(r,t,p^\varepsilon)) \\ & + l^\varepsilon(\cdot)[h(r,t,p) - h(r,t,p^\varepsilon)]\}dt \\ =\ & E\int_0^T H_p(t, p^\varepsilon, u^\varepsilon, (\psi^\varepsilon(r,t), k^\varepsilon(r,t), l^\varepsilon(r,t)))(p(r,t) - p^\varepsilon(r,t))dt \\ & + E\int_0^T \psi^\varepsilon(r,t)[(-u(r,t) - \mu(r,t) + f_p(r,t,p)) \\ & - (-u^\varepsilon(r,t) - \mu(r,t) + f_p^\varepsilon(r,t,p))]dt \\ =\ & 0.\end{aligned} \quad (4.11)$$

Combining $(4.10)$ and $(4.11)$, using the fact ahead, then we can obtain

$$\mathcal{J}(u^\varepsilon(r,t)) \le \mathcal{J}(u(r,t)) + D_4 \varepsilon^{\frac{1}{2}}.$$

Since $u(r,t)$ is arbitrary, so the desired result follows.

**5. AN EXAMPLE**

Consider the following stochastic age-dependent system:

$$\begin{cases} \frac{\partial p}{\partial t}+\frac{\partial p}{\partial r}=up-\frac{1}{(1-r)^2}p+2pt-pt\frac{d\omega_t}{dt}+p\frac{dN_t}{dt}, & (r,t)\in[0,1]\times[0,1], \\ p(0,t)=t^2\int_0^1 p(r,t)dr, & (r,t)\in[0,1]\times[0,1], \\ p(r,0)=\exp(-\frac{1}{1-r}), & t\in[0,1], \\ \mathcal{P}(t)=\int_0^A p(r,t)dr. \end{cases} \quad (5.1)$$

Here,without loss of generality, our expected cost of the near-optimal control problem is

$$\mathcal{J}(u(\cdot,\cdot))=E\int_0^1\int_0^1 p(r,t)drdt=E\int_0^1\mathcal{P}(t)dt.$$

In the system, $\omega_t$ is a real standard Brownian motion. We can get this problem in our formulation by taking $H=L^2([0,1]\times[0,1])$, $V=W_0^1([0,1])$ (a sobolev space with elements satisfying the boundary conditions above), $u(r,t)=-u$, $\mu(r,t)=\frac{1}{(1-r)^2}$, $f(r,t,p)=2pt$, $g(r,t,p)=-pt$, $h(r,t,p)=p$, $\beta(t)=t^2$ and $m(r,t)=1$. Clearly, the operators $f(r,t,p)$, $g(r,t,p)$, $\mu(r,t)$ and $h(r,t,p)$ satisfy the assumptions in our system.

To solve this problem, we write down the Hamiltonian function:

$$\begin{aligned} H(r,t,p,u,\psi,k,l) = \mathcal{P}(t)+p(0,t)\phi(r,t)-(-u(r,t)p+\frac{p}{(1-r)^2} \\ -2pt)\psi(r,t)-k(r,t)pt+l(r,t)p \end{aligned}$$

So the adjoint equation is

$$\begin{cases} \frac{\partial\psi(r,t)}{\partial t}=-\frac{\partial\psi(r,t)}{\partial r}+(\frac{1}{(1-r)^2}-u(r,t)-2t)\psi(r,t) \\ \qquad +tk(r,t)-l(r,t)+k(r,t)d\omega_t+l(r,t)dN_t, \\ \psi(r,T)=0 \\ \psi(A,t)=0 \end{cases} \quad (5.2)$$

Solving equation $(5.1)$ and $(5.2)$, for any admissible control $u(\cdot,\cdot)$, if we have Lemma 2.2 and Lemma 3.2, then $(3.16)$ is valid, which is the necessary and sufficient conditions for near-optimality in this control problem.
The corresponding Hamiltonian are

$$H(r,t,p^\varepsilon,u^\varepsilon,\psi^\varepsilon,k^\varepsilon,l^\varepsilon)=\mathcal{P}^\varepsilon+p^\varepsilon(0,t)\phi^\varepsilon-(\frac{p^\varepsilon}{(1-r)^2}-u^\varepsilon p^\varepsilon-2p^\varepsilon t)\psi^\varepsilon-k^\varepsilon p^\varepsilon t+p^\varepsilon l^\varepsilon$$

and

$$H(r,t,p^\varepsilon,u^\varepsilon,\psi^\varepsilon,k^\varepsilon,l^\varepsilon)=\mathcal{P}^\varepsilon+p^\varepsilon(0,t)\phi^\varepsilon-(\frac{p^\varepsilon}{(1-r)^2}-up^\varepsilon-2p^\varepsilon t)\psi^\varepsilon-k^\varepsilon p^\varepsilon t+p^\varepsilon l^\varepsilon$$

If $u(\cdot,\cdot)$ is $\varepsilon$-optimal, the necessary condition for $(5.1)$ is

$$\begin{aligned} & E\int_0^1[\mathcal{P}^\varepsilon+p^\varepsilon(0,t)\phi^\varepsilon-(\frac{p^\varepsilon}{(1-r)^2}-u^\varepsilon p^\varepsilon-2p^\varepsilon t)\psi^\varepsilon-k^\varepsilon p^\varepsilon t+p^\varepsilon l^\varepsilon]dt \\ \le & E\int_0^1[\mathcal{P}^\varepsilon+p^\varepsilon(0,t)\phi^\varepsilon-(\frac{p^\varepsilon}{(1-r)^2}-up^\varepsilon-2p^\varepsilon t)\psi^\varepsilon-k^\varepsilon p^\varepsilon t+p^\varepsilon l^\varepsilon]dt \quad (5.3) \\ & -c\varepsilon^\gamma. \end{aligned}$$

where $c$ is a constant.
For example, if we let

$$u^{\varepsilon}(r,t)=1-\varepsilon^{\frac{1}{2}},$$

where $\varepsilon > 0$ is a sufficiently small parameter, then $u^{\varepsilon}(r,t)$ is a candidate $\varepsilon$-optimal control.

Moreover, because of $p(r,t)$, the solution of $u(r,t)$, $\mu(r,t)$, $f(r,t,p)$, $g(r,t,p)$ and $h(r,t,p)$, satisfying the assumption $(A_6)$, we can conclude that

$$\mathcal{J}(u^{\varepsilon}(\cdot,\cdot)) \le \sup_{u(\cdot,\cdot)\in\mathcal{U}_{ad}[0,1]} \mathcal{J}(u(\cdot,\cdot)) + \varepsilon^{\frac{1}{2}}.$$

So we gain that $(5.3)$ is also the sufficient condition for $(5.1)$.

**6. ACKNOWLEDGEMENTS**
The authors would like to thank the referees for their very helpful suggestions which greatly improved this paper. The authors would like to thank the referees for their very helpful comments which greatly improved this paper. The research was supported by The National Natural Science Foundation (No. 11061024; 11261043) (China)

**REFERENCES**

L.S. Wang, X.J. Wang, Convergence of the semi-implicit Euler method for stochastic age-dependent population equations with Poisson jumps, Appl. Math. Model. 34 (2010) 2034-2044.
R.H. Li, W.K. Pang, Q.H. Wang, Numerical analysis for stochastic age-dependent population equations with Poisson jumps, J. Math. Anal. Appl. 327 (2007) 1214-1224.
Q.M. Zhang, C. Han, Convergence of numerical solutions to stochastic age-structured population system, Applied Mathematics and Computation. 118 (1) (2005) 134-146.
Q.M. Zhang, Convergence of numerical solutions for a class of stochastic age-dependent capital system with Markovian switching, Economic Modelling. 28 (2011) 1195-1125.
W.J. Ma, Q.M. Zhang, Convergence of the Semi-implicit Euler Method for Stochastic Age-Dependent Population Equations with Markovian Switching, Lecture Notes in Computer Science. 6377 (2010) 407-414.
Q.M. Zhang, W.A. Liu, Z.K. Nie, Existence, uniqueness and exponential stability for stochastic age-dependent population, Appl. Math. Comput. 154 (2004) 183-201.
Q.M. Zhang, C.Z. Han, Existence and Uniqueness of Strong Solutions for Stochastic Age-Dependent Population, Lecture Notes in Computer Science. 3521 (2005) 151-161.
Z.X. Luo, Optimal harvesting control problem for an age-dependent competing system of n species, Appl. Math. Comput. 183 (2006) 119-127.
C. Zhao, M.S. Wang, P. Zhao, Optimal Control of Harvesting for Age-Dependent Predator-Prey System, Math. Comput. Model. 42 (2005) 573-584.
J.J. Chen, Z.R. He, Optimal control for a class nonlinear age-distributed population systems, Appl. Math. Comput. 214 (2009) 574-580.
S. Anita, Optimal harvesting for a nonlinear age-dependent population dynamics, Math. Anal. Appl. 226 (1998) 6-22.
X.Y. Zhou, Stochastic near-optimal controls: Necessary and sufficient conditions for near-optimality, SIAM. J. Control Optim. 36 (3) (1998) 929-947.
I. Ekeland, Non convex minimisation problems, Bull. Amer. Math. Soc. NS. 1 (1979) 324-353.
B. Mezerdi, Necessary conditions for optimality for a diffusion with nonsmooth drift, Stochastic. 24 (1998) 305-326.
K. Bahlali, N. Khelfallah, B. Mezerdi, Necessary and sufficient conditions for near-optimality in Stochastic control of FBSDEs, System Control Letters. 58 (2009) 857-864.