QUENCHING FOR PHENOMENON FOR ONE-DIMENSIONAL P-LAPLACIAN WITH A SINGULAR BOUNDARY FLUX

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ABSTRACT
We study the finite time quenching for one-dimensional p-Laplacian with a singular boundary condition. It is shown that u quenches in finite time for all u_0, and the only quenching point is $x=1$.We also discuss the corresponding quenching rate.

Keywords: *p-Laplacian ; Singular boundary condition; Finite time quenching.*

1. INTRODUCTION

In this paper we consider the following problem

$$\begin{cases} u_t = (|u_x|^{p-2} u_x)_x + u^p, & t>0, 0<x<1, \\ u_x(0,t)=0, u_x(1,t) = -g(u(1,t)), & t>0, \\ u(x,0)=u_0(x), & 0 \le x \le 1, \end{cases} \quad (1.1)$$

Here $p>1$， and suppose(A) $g(u)>0, g'(u)<0, \lim_{u\to 0^{=}} g(u) = +\infty$ with $u>0$ or(B) $\left(\frac{g}{g'}\right)' \le 0$

with $1<p<2$. The initial value $u_0(x)$ is position and satisfies the compatibility conditions.

Equation（1.1）is known as the classical Non-Newtonian filtration equation, which is always a focus of research, and many results on stability existence and uniqueness of solutions have been obtained. In this paper our main purpose is to examine the quenching behavior of the solution of problem(1.1), that is , the solution reaches zero in a finite time, and a derivative of the solution blows up at the that moment. Mathematical models of semilinear and nonlinear parabolic equations under Dirchlet or Neumann boundary conditions have also seen considered in several papers, see[1-4].And in[5,6], the author discussed the simultaneous and non-simultaneous quenching phenomena for a system of heat equations coupled with nonlinear boundary flux separately.

In [7] , the author studied the quenching phenomena of the following problem:

$$\begin{cases} u_t = u_{xx} & t>0, 0<x<1, \\ u_x(0,t)=0, u_x(1,t) = -u^{-\beta}(1,t), & t>0, \\ u(x,0)=u_0(x)>0, & 0 \le x \le 1, \end{cases} \quad (1.2)$$

here $\beta > 0$,and it was shown the quenched infinite time for all $u_0(x)$, and the only quenching point was $x=1$.

In [8]，Keng Deng and Mingxi Xu studied the quenching phenomena of the nonlinear diffusion equation：

$$\begin{cases} (\Psi(u))_t = u_{xx}, & t>0, 0<x<1, \\ u_x(0,t)=0, u_x(1,t) = -g(u(1,t)), & t>0, \\ u(x,0)=u_0(x), & 0 \le x \le 1, \end{cases} \quad (1.3)$$

Here $\psi(u)$ is a monotone increasing function with $\psi(0)=0$； $g(u)>0, g'(u)<0$ for $u>0$, and $\lim_{u\to 0^{=}} g(u) = +\infty$.The initial value $u_0(x)$ is position and satisfies the boundary condition.

In this paper they proved the finite time quenching for the solution and established results on quenching set and rate.

In [9]， that paper was concerned with the finite time quenching phenomena for one-dimensional p-Laplacian with singular boundary flux. And it discussed the corresponding quenching rate. The equation is:

$$\begin{cases} u_t = (|u_x|^{p-2} u_x)_x, & t > 0, 0 < x < 1, \\ u_x(0,t) = 0, u_x(1,t) = -g(u(1,t)), & t > 0, \\ u(x,0) = u_0(x), & 0 \le x \le 1, \end{cases} \qquad (1.4)$$

Here $p > 0$, and the initial value $u_0(x)$ is position and satisfies the compatibility conditions. It was shown the quenched infinite time for all $u_0(x)$, and the only quenching point was $x = 1$ and it discussed the quenching rate.
In [10], that paper was concerned with the finite time quenching for a nonlinear diffusion equation with a singular boundary condition. The equation is:

$$\begin{cases} (\Psi(u))_t = u_{xx} + u^p, & t > 0, 0 < x < 1, \\ u_x(0,t) = 0, u_x(1,t) = -u^{-\beta}(1,t), & t > 0, \\ u(x,0) = u_0(x), & 0 \le x \le 1, \end{cases} \qquad (1.5)$$

Here $p > 0, \beta > 0$ and $\psi(u)$ is a monotone increasing function with $\psi(0) = 0$. The initial value $u_0(x)$ is position and satisfies the boundary condition. It was shown the quenched infinite time for all $u_0(x)$, and the only quenching point was $x = 1$.
Our main purpose in this paper is to examine the quenching behavior of the solution of problem (1.1).The plan of this paper is as follows: In section 2, we will prove that quenching occurs only at $x = 1$.In section 3, we derive estimates for the quenching rate.

2. QUENCHING ON THE BOUNDARY

In this section, we prove finite time quenching for the solution of(1.1). Due to the degeneracy of the equation, the solutions considered might not be classical in general and we should discuss weak solutions. We assume that the solutions are appropriately smooth, since we may consider some approximate boundary and initial value conditions.
We need the following:

Lemma 2.1 Suppose that (A), (B) hold and the solution u of the problem (1.1) exists in $(0, T_0)$, and $u_0'(x) \le 0, (p-1)|u_0'(x)|^{p-2} u_0''(x) + u_0^p(x) \le 0$ for $0 \le x \le 1$. Then $u_x(x,t) \le 0, u_t(x,t) \le 0$ in $(0,1] \times (0, T_0]$.

Proof. Let $v(x,t) = u_x(x,t)$. Then $v(x,t)$ satisfies

$$\begin{cases} v_t = (v^{p-2} v)_{xx} + pu^{p-1} v, & t > 0, 0 < x < 1, \\ v(0,t) = 0, v(1,t) = -g(u(1,t)), & t > 0, \\ v(x,0) = u_0'(x), & 0 \le x \le 1, \end{cases} \qquad (2.1)$$

The maximum principle leads to $v \le 0$, so that $u_x(x,t) \le 0$ in $(0,1] \times (0, T_0]$. Then it is easy to see that (2.1)is non degenerate in $(0,1] \times (0, T_0]$. So u_x is a classical solution of (2.1).

Letting $w(x,t) = u_t(x,t)$, we have

$$\begin{cases} w_t = (p-1)(|u_x|^{p-2} w_x)_x + pu^{p-1} w, & t > 0, 0 < x < 1, \\ w_x(0,t) = 0, w_x(1,t) = -g'(u(1,t)) w(1,t), & t > 0, \\ w(x,0) = (p-1)|u_0'(x)|^{p-2} u_0''(x) + u_0^p(x), & 0 \le x \le 1, \end{cases} \qquad (2.2)$$

As the maximum principle, it follows that $w \le 0$, and thus $u_t(x,t) \le 0$ in $(0,1] \times (0, T_0)$.

Theorem 2.2 Suppose that(A), (B)hold, and $g^{p-1}(M) > M^p$, $u'_0(x) \le 0$ and $(p-1)$ $\left|u'_0(x)\right|^{p-2} u''_0(x) + u_0^p \le 0$ for $0 \le x \le 1$. For the little initial value, then every solution of (1.1) quenches in finite time, and the only quenching point is $x = 1$.

Proof. By the maximum principle, we know that for $0 < u(\cdot,t) \le M$ for all t in the existence interval, where $M = \max\limits_{0 \le x \le 1} u_0(x)$.

Define $F(t) = \int_0^1 u(x,t)dx$ .Then $F(t)$ satisfies

$$F'(t) = \int_0^1 u_t(x,t)dx = \int_0^1 (\left|u_x\right|^{p-2} u_x)_x + u^p)dx$$
$$= -\left|g(u(1,t))\right|^{p-2} g(u(1,t)) + \int_0^1 u^p dx \le -g^{p-1}(M) + M^P. \tag{2.3}$$

Thus $F(t) \le F(0) + (M^p - g^{P-1}(1,t))t$,which means that $F(t_0) = 0$ for some $t_0 > 0$ .From the fact that $u_x < 0$ for $0 < x \le 1$,we find that there exists $T(0 < T \le t_0)$ such that $\lim\limits_{t \to T^-} u(1,t) = 0$ .By virtue of the singular nonlinearity in the boundary condition, u must quench at $x = 1$ .In what follows, we only need to prove that quenching cannot occur in $(\frac{1}{2},1) \times (\eta, T)$ for some $\eta(0 < \eta < T)$.

Let $h(x,t) = \left|u_x\right|^{p-2} u_x + \varepsilon(x - \frac{1}{4}) g^{p-1}(M)$ in $(\frac{1}{4},1) \times (\eta, T)$, where ε is a positive constant. Then $h(x,t)$ satisfies

$$h_t = (\left|u_x\right|^{p-2} u_x)_t = (p-1)\left|u_x\right|^{p-2} u_{xt} = (p-1)\left|u_x\right|^{p-2}((\left|u_x\right|^{p-2} u_x)_{xx} + pu^{p-1}u_x)$$
$$\le (p-1)\left|u_x\right|^{p-2} h_{xx}, \quad (x,t) \in (\frac{1}{4},1) \times (\eta, T) \tag{2.4}$$

On the parabolic boundary, $h(\frac{1}{4},t) = \left|u_x(\frac{1}{4},t)\right|^{p-2} u_x(\frac{1}{4},t) < 0$ for $\eta \le t < T$;if ε is sufficiently small, $h(1,t) \le -(1 - \frac{3\varepsilon}{4})M^{-\beta} < 0$, for $\eta \le t < T$, and $h(\eta,t) \le -\left|u_x(\frac{1}{4},\eta)\right|^{p-1} + \frac{3\varepsilon}{4} g^{p-1}(M) < 0$ for $x \in [\frac{1}{4},1]$ .By the maximum principle, we find that $h(x,t) \le 0$ in $(x,t) \in (\frac{1}{4},1) \times (\eta, T)$, which leads to

$$\left|u_x\right|^{p-2} u_x + \varepsilon(x - \frac{1}{4}) g^{p-1}(M) \le 0, (x,t) \in (\frac{1}{2},1) \times (\eta, T) \tag{2.5}$$

So we have

$$-u_x \ge \left[\varepsilon(x - \frac{1}{4}) g^{p-1}(M)\right]^{\frac{1}{p-1}}, \tag{2.6}$$

Integrating (2.6) from x to 1,we obtain

$$u(x,t) \ge u(1,t) + \int_x^1 \left[\varepsilon(x - \frac{1}{4}) g^{p-1}(M)\right]^{\frac{1}{p-1}} dx \ge \int_x^1 \left[\varepsilon(x - \frac{1}{4}) g^{p-1}(M)\right]^{\frac{1}{p-1}} dx > 0.$$

It follows that $u(x,t) > 0$ if $x < 1$.

3. BOUNDS FOR THE QUENCHING

In this section, we establish on the quenching rate. We get the upper bound at first.

Theorem3.1 Suppose the hypotheses of (2.1) hold and $g''(u) > 0$ .Then there exists a positive constant C_1 such that

$$\int_0^{u(1,t)} \frac{1}{-g^{p-1}(s)g'(s)}ds \le C_1(T-t).$$

Proof We define a function $\phi(x)=|u_x(x,t)|^{p-2}u_x(x,t)+\varphi^{p-1}(x)g^{p-1}(u(x,t))$ in $(x,t)\in(0,1)\times(0,T)$ .Here $\varphi(x)$ is given as follows:

$$\varphi(x)=\begin{cases}0 & x\in[0,x_0],\\ \dfrac{(x-x_0)^l}{(1-x_0)^l} & x\in(x_0,1],\end{cases}$$

with some $x_0<1$ and $l\ge\max\left\{3,\dfrac{1}{p-1}\right\}$ is chosen so large that

$$\varphi(x)\le\frac{-u_0'(x)}{g(u_0(x))}, x_0<x\le 1.$$

It is easy to see that $\phi(0,t)=\phi(1,t)=0$,and $\phi(x,0)<0$ .And in $(0,1)\times(0,t)$, ϕ satisfies

$$\begin{aligned}\phi_t &= (p-1)|u_x|^{p-2}\phi_{xx}-(p-1)^2|u_x|^{p-2}\varphi^{p-1}(x)g^{p-3}(u)[(p-2)g'^2(u)+g(u)g''(u)]u_x^2\\ &\quad+p(p-1)|u_x|^{p-2}u^{p-1}u_x-2(p-1)^3|u_x|^{p-2}\varphi^{p-2}(x)\varphi'(x)g^{p-2}(u)g'(u)u_x\\ &\quad-(p-1)2|u_x|^{p-2}\varphi^{p-3}(x)[(p-2)\varphi'^2(x)+\varphi(x)\varphi''(x)]g^{p-2}(u)\\ &\quad+(p-1)\varphi^{p-1}(x)g^{p-2}(u)g'(u)u^p \qquad (3.1)\end{aligned}$$

Here $\left(\dfrac{g}{g'}\right)'\le 0$ for $1<p<2$,we have

$$\left(\frac{g}{g'}\right)'=\frac{g'^2-gg''}{g'^2}\le 0<p-1,$$

That it is

$$(2-p)g'^2+gg''\ge 0.$$

So we have

$$-(p-1)^2|u_x|^{p-2}\varphi^{p-1}(x)g^{p-3}(u)[(p-2)g'^2(u)+g(u)g''(u)]u_x^2<0.$$

And $u_x<0,\varphi'(x)<0,g'(u)<0,$ then

$$p(p-1)|u_x|^{p-2}u^{p-1}u_x-2(p-1)^3|u_x|^{p-2}\varphi^{p-2}(x)\varphi'(x)g^{p-2}(u)g'(u)u_x<0,$$

$$-(p-1)2|u_x|^{p-2}\varphi^{p-3}(x)[(p-2)\varphi'^2(x)+\varphi(x)\varphi''(x)]g^{p-2}(u)<0,$$

$$(p-1)\varphi^{p-1}(x)g^{p-2}(u)g'(u)u^p<0.$$

As the definition of $\varphi(x)$,and (A),(B), $g''(u)>0$,then

$$\phi_t\le(p-1)|u_x|^{p-2}\phi_{xx} \qquad (3.2)$$

By the maximum principle $\phi(x,t)\le 0$, we have

$$|u_x(x,t)|^{p-2}u_x(x,t)+\varphi^{p-1}(x)g^{p-1}(u(x,t))\le 0,\ (x,t)\in(0,1)\times(0,T).$$

That is

$$\varphi(x)g(u(x,t))\le -u_x(x,t),\quad (x,t)\in(0,1)\times(0,T).$$

By the definition of the limit, we see that $\phi_x(1,t)\ge 0$ since $\phi(x,t)\le 0$. In fact,

$$\phi_x(1,t) = \lim_{x \to 1^-} \frac{\phi(x,t) - \phi(1,t)}{x-1} \ge 0.$$

Which means

$$u_x(1,t) \ge (p-1) g^{p-1}(u(1,t))[-\varphi'(1) + g'(u(1,t))] + u^p(1,t)$$
$$\ge (p-1) g^{p-1}(u(1,t))[-\varphi'(1) + g'(u(1,t))]$$
$$\ge C_3 (p-1) g^{p-1}(u(1,t)) g'(u(1,t)) \quad (3.3)$$

If we want to (3.3) hold, then we need $(1-C_3)g'(u(1,t)) \ge \varphi'(1) = \frac{l}{1-x_0}$ ($g' < 0$, C_3 is chosen so large).

Integrating (3.3) from t to T, we get

$$\int_0^{u(1,t)} \frac{1}{-g^{p-1}(s) g'(s)} ds \le C_1 (T-t).$$

So we get the supper bound.

Then we give the lower bound, we need the following additional hypotheses.

There exists a constant $\sigma(-\infty < \sigma \le \sigma_0 = \min\left\{1, 2 - \frac{1}{\varphi - 1}\right\})$, such that

(a) $(g^{(p-1)(\sigma-1)}(u) g'(u))'' < 0,$

(b) $(g^{(p-1)(\sigma-1)}(u) g'(u))' \ge 0,$

(c) $(g^{(p-1)(\sigma-1)-1}(u) g'(u))' < 0,$

Theorem3.2 Suppose that the hypotheses of Theorem2.1 hold. Assume that the hypotheses (a),(b),(c) hold. Then there exists a positive constant C_2 such that

$$\int_0^{u(1,t)} \frac{1}{-g^{p-1}(s) g'(s)} ds \ge C_2 (T-t).$$

Proof Let $d(u) = g^{(p-1)(\sigma-1)}(u) g'(u).$ That the hypotheses (a),(b),(c)are equivalent to

(a1) $d''(u) < 0,$

(b1) $d'(u) \ge 0,$

(c1) $d(u) g'(u) > d'(u) g(u).$

let $t \to T,$ we consider

$$\psi(x,t) = u_t - \varepsilon d(u)(-u_x)^{(p-1)(2-\sigma)} ((x,t) \in (1-T+\tau, 1) \times (\tau, T)),$$

where $\varepsilon > 0$. Though a fairly complicated calculation, we find that

$$\psi_t = (p-1)|u_x|^{p-2} \psi_{xx} - (p-1)(p-2)(-u_x)^{p-} 3(-u_{xx}) \psi_x + C(x,t)\psi + \varepsilon R(x,t)(-u_x)^{(p-1)(3-\sigma)+1},$$
(3.4)

where

$$C(x,t) = \varepsilon(2-\sigma)[(p-1)(2-\sigma)-1](-u_x)^{(p-1)(1-\sigma)-1} d(u) u_t + \varepsilon[(p-1)(5-2\sigma)-1] d'(u)$$
$$(-u_x)^{(p-1)(2-\sigma)} + \varepsilon^2 (2-\sigma)[(p-1)(2-\sigma)-1] d^2(u)(-u_x)^{(p-1)(3-2\sigma)-1} + pu^{p-1}.$$
$$R(x,t) = (p-1) d''(u) + \varepsilon[(p-1)-(5-2\sigma)-1] d'(u) d(u)(-u_x)^{(p-1)(1-\sigma)-1} + \varepsilon^2 (2-\sigma)$$
$$[(p-1)(2-\sigma)-1] d^3(u)(-u_x)^{2(p-1)(1-\sigma)-2} + p d(u) u^{p-1} (-u_x)^p.$$

since (a1),(b1)hold, and $d(u) < 0,$ we can see that $C(x,t) > 0, R(x,t) < 0.$ So we have

$$\psi_t < (p-1)|u_x|^{p-2} \psi_{xx} - (p-1)(p-2)(-u_x)^{p-} 3(-u_{xx}) \psi_x + C(x,t)\psi, (x,t) \in (1-T+\tau, 1) \times (\tau, T).$$ O

n the parabolic boundary, since $x=1$ is the only quenching point, if ε is small enough, then both $\psi(1-T+\tau,t)$,and $\psi(x,\tau)<0.$ At $x=1$,in view of（C1), we have

$$\psi_x(1,t)=-[1-\varepsilon(2-\sigma)]g'(u(1,t))\psi(1,t)-\varepsilon[(1-\varepsilon(2-\sigma))g'(u(1,t))d(u(1,t))-d'(u(1,t))g(u(1,t))]$$
$$g^{(p-1)(2-\sigma})(u(1,t))-\varepsilon(2-\sigma)u^p(1,t)g'(u(1,t))$$
$$\leq -[1-\varepsilon(2-\sigma)]g'(u(1,t))\psi(1,t)$$

Provided ε is sufficiently small. Making the maximum principle, we have $\psi(x,t)\leq 0$ in $(x,t)\in(1-T+\tau,1)\times(\tau,T).$ In particular, $\psi(1,t)\leq 0,$ that is

$$u_t(1,t)\leq \varepsilon d(u(1,t))(-u_x(1,t))^{(p-1)(2-\sigma)}=\varepsilon g^{p-1}(u(1,t))g'(u(1,t)). \qquad (3.5)$$

Integration of（3.5）over (t,T) then lends to

$$\int_0^{u(1,t)}\frac{1}{-g^{p-1}(s)g'(s)}ds\geq C_2(T-t).$$

So we give the lower boundary.

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