NONLINEAR SEPARATION FOR CONSTRAINED MULTIOBJECTIVE OPTIMIZATION PROBLEMS AND APPLICATIONS TO PENALTY FUNCTIONS

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ABSTRACT
In this paper, we prove a nonlinear separation associated with a constrained multiobjective optimization problem in the image space, as well as the equivalence between the existence of nonlinear separation function and a saddle point condition for a generalized Lagrangian function associated with the given problem. As applications, we obtain some equivalent conditions on exact penalty methods for the given problem.

Keywords and Phrases: *Image space analysis; Nonlinear separation; Multiobjective optimization; Penalty method.*
2000 Mathematics Subject Classification: 90C29.

1. INTRODUCTION

The imgae of a constrained extremum problem was first developed in [7] by Giannessi. In recent years, there has been more and more vector variational inequalities and vector optimization problems are developed by means of the Image Space analysis (ISA)[6, 9, 10, 11, 14, 15]. The ISA is a powerful tool and a unifying scheme for studying constrained optimization problems, this approach can be applied to any kind of problems that can be expressed under the form of the impossibility of a parametric system. Many theoretical aspects results has been obtained by exploiting ISA, such as existence of optimal solutions, duality[1, 12, 17, 19], Lagrangian-type optimality condition[3, 6, 8, 16, 18], penalty methods[2, 4, 5] and so on. The optimality condition of the constrained extremum problem can be expressed under the form of the impossibility of a parametric system, which is reduced to the disjunction of two suitable subsets of the image space.

The purpose of this paper is to extend some results in [13] to constrained multiobjective optimization problem. The paper is organized as follows. In section 2, we recall the main definitions; In section 3, we recall basic properties of the ISA for constrained multiobjective optimization problem and discuss the image problem for constrained multiobjective optimization problem; In section 4, by virtue of the ISA, we characterize the (res,. regular, strongly regular) nonlinear separation and the saddle points of the generalized Lagrangian functions for constrained multiobjective optimization problem, and obtain the relation between the saddle points of the generalized Lagrangian function and the the Pareto efficient solution for the constrained multiobjective optimization problem. Also, we apply the above results to the augmented Lagrangian function; In section 5, we present some equivalent condition associated with exact penalty methods for constrained multiobjective optimization problem.

2. PRELIMINARIES

Let R^s be the s dimensional Euclidean space, where s is given positive integer. Denote by $\mathrm{R}^s_+ := \{x := (x_1, \cdots, x_s)^{\mathsf{T}} : x_i \geq 0, i = 1, \cdots, s\}$ and $\mathrm{R}^s_{++} := \{x := (x_1, \cdots, x_s)^{\mathsf{T}} : x_i > 0, i = 1, \cdots, s\}$. For $x := (x_1, \cdots, x_l)^{\mathsf{T}} \in \mathrm{R}^l$, A nonempty subset $P \subseteq \mathrm{R}^s$ is said to be a cone if $tP \subseteq P$ for all $t \geq 0$. A cone $P \subseteq \mathrm{R}^l$ is said to be a convex cone if $P + P \subseteq P$. A cone $P \subseteq \mathrm{R}^s$ is said to be a pointed cone if $P \cap (-P) = \{0\}$. The closure of P is denoted by $cl\,P$. $\langle \cdot,\cdot \rangle$ denotes the inner product,while

$$P^* = \{y \in R^s : \langle y, x \rangle \geq 0, \forall x \in P\}$$

is the positive polar cone of a convex cone $P \subseteq \mathrm{R}^s$.

This work was supported by the Natural Science Foundation of China (60804065), the Key Project of Chinese Ministry of Education (211163), Sichuan Youth Science and Technology Foundation (2012JQ0032) and Students' Science and Technology Innovation Foundation of China West Normal University (42722050)

Let $P \subseteq \mathrm{R}^s$ such that $cl P$ is a convex cone. We use the notations

$$y_1 \leq_P y_2 \Leftrightarrow y_2 - y_1 \in P, y_1, y_2 \in \mathrm{R}^s,$$

$$y_1 \not\leq_P y_2 \Leftrightarrow y_2 - y_1 \notin P, y_1, y_2 \in \mathrm{R}^s.$$

In this paper, we consider the following constrained multiobjective optimization problem:

$$\begin{aligned} &\min_{R_+^m \backslash \{0\}} f(x) \\ &s.t.\ x \in K = \{x \in X : g(x) \geq_{\mathrm{R}_+^l} 0\}. \end{aligned} \tag{1}$$

where $f : \mathrm{R}^n \to \mathrm{R}^m, g : \mathrm{R}^n \to \mathrm{R}^l$ and X is a nonempty convex subset of R^n.
If $\bar{x} \in K$ satisfying $f(\bar{x}) - f(x) \not\geq_{R_+^m \backslash \{0\}} 0, \forall x \in K$, then $\bar{x}$ is called a Pareto efficient solution of problem (1).

3. ISA FOR CONSTRAINED MULTIOBJECTIVE OPTIMIZATION PROBLEM

In this section, we develop the image space analysis for constrained multiobjective optimization problem (1).

Observe that, $\bar{x} \in K$ solves problem (1), if and only if the system (in the unknown x):

$$\begin{cases} f(\bar{x}) - f(x) \geq_{\mathrm{R}_+^m \backslash \{0\}} 0, \\ g(x) \geq_{\mathrm{R}_+^l} 0, \\ x \in X. \end{cases} \tag{1}$$

is impossible.
Let $\bar{x} \in K$. Define the mapping $A_{\bar{x}} : X \to \mathrm{R}^m \times \mathrm{R}^l$ by

$$A_{\bar{x}}(x) = (f(\bar{x}) - f(x), g(x)), \forall x \in X. \tag{2}$$

We can associate problem (1) with the following sets:

$$\mathcal{K}_{\bar{x}} = \{(u,v) \in \mathrm{R}^m \times \mathrm{R}^l : u = f(\bar{x}) - f(x), v = g(x), x \in X\} = A_{\bar{x}}(X),$$

$$\mathcal{H} = \{(u,v) \in \mathrm{R}^m \times \mathrm{R}^l : u \geq_{\mathrm{R}_+^m \backslash \{0\}} 0, v \geq_{\mathrm{R}_+^l} 0\} = \mathrm{R}_+^m \backslash \{0\} \times \mathrm{R}_+^l.$$

The set $\mathcal{K}_{\bar{x}}$ is the image associated with problem (1) at $\bar{x} \in K$, we call $\mathrm{R}^m \times \mathrm{R}^l$ image space.
It is easy to see that the system (1) is impossible if and only if

$$\mathcal{K}_{\bar{x}} \cap \mathcal{H} = \emptyset. \tag{3}$$

Consequently, $\bar{x} \in K$ solves (1) if and only if (3) is true.
In general,to prove directly whther or not (3) holds is too difficult. This is because the image set is not convex even when the functions involved enjoy some convexity properties. To overcome this difficulty, we introduce a regularization of the image $\mathcal{K}_{\bar{x}}$ with respect to the cone $cl\mathcal{H}$, denoted by,

$$\begin{aligned} \mathcal{E}_{\bar{x}} &= \mathcal{K}_{\bar{x}} - cl\mathcal{H} \\ &= \{(u,v) \in \mathrm{R}^m \times \mathrm{R}^l : u \leq_{\mathrm{R}_+^m} f(\bar{x}) - f(x), v \leq_{\mathrm{R}_+^l} g(x), x \in X\} \\ &= A_{\bar{x}}(X) - \mathrm{R}_+^m \times \mathrm{R}_+^l. \end{aligned}$$

which is called the extended image associated with problem (1) at $\bar{x} \in K$.

Proposition 3.1 $\bar{x} \in K$ is a Pareto efficient solution of problem (1) if and only if

$$\mathcal{E}_{\bar{x}} \cap \mathcal{H} = \emptyset, \tag{4}$$

or equivalently,

$$\mathcal{E}_{\bar{x}} \cap \mathcal{H}_u = \varnothing, \tag{5}$$

where

$$\mathcal{H}_u = \{(u,v) \in \mathrm{R}^m \times \mathrm{R}^l : u \geq_{\mathrm{R}_+^m \setminus \{0\}} 0, v = 0\} = \mathrm{R}_+^m \setminus \{0\} \times \{0\}.$$

Proof. It is clear that $\bar{x} \in K$ is a Pareto efficient solution of problem (1) if and only if (3) holds. We show (3) and (4) are equivalent firstly.

It suffices to show (4) implies (3), since $\mathcal{K}_{\bar{x}} \subseteq \mathcal{E}_{\bar{x}}$. To prove the reverse implication, ab absurdo, suppose that (4) does not hold. Then there exists $(u,v) \in \mathcal{E}_{\bar{x}} \cap \mathcal{H}$, i.e., $\exists \tilde{x} \in X$ such that

$$u \leq_{\mathrm{R}_+^m} f(\bar{x}) - f(\tilde{x}), \quad v \leq_{\mathrm{R}_+^l} g(\tilde{x}),$$

and

$$u \geq_{\mathrm{R}_+^m \setminus \{0\}} 0, \quad v \geq_{\mathrm{R}_+^l} 0.$$

Observe that $\mathrm{R}_+^m + \mathrm{R}_+^m \setminus \{0\} \subseteq \mathrm{R}_+^m \setminus \{0\}$. It follows that $0 \leq_{\mathrm{R}_+^m \setminus \{0\}} f(\bar{x}) - f(\tilde{x}), 0 \leq_{\mathrm{R}_+^l} g(\tilde{x})$, which leads to a contradiction, since $\mathcal{K}_{\bar{x}} \cap \mathcal{H} = \varnothing$.

Now, we prove (4) and (5) are equivalent.

Obviously, (4) implies (5), since $\mathcal{H}_u \subseteq \mathcal{H}$. Suppose to the contrary that (4) does not hold, i.e., $\exists (\hat{u}, \hat{v}) \in \mathcal{E}_{\bar{x}} \cap \mathcal{H}$. Since

$$\mathcal{E}_{\bar{x}} - cl\mathcal{H} = \mathcal{K}_{\bar{x}} - (cl\mathcal{H} + cl\mathcal{H}) = \mathcal{K}_{\bar{x}} - cl\mathcal{H} = \mathcal{E}_{\bar{x}} (0,\hat{v}) \in cl\mathcal{H},$$

it follows that $(\hat{u},\hat{v}) - (0,\hat{v}) = (\hat{u},0) \in \mathcal{E}_{\bar{x}}$. Again since $(\hat{u},0) \in \mathcal{H}_u$, this leads a contradiction with $\mathcal{E}_{\bar{x}} \cap \mathcal{H}_u = \varnothing$.

Consider the following image problem for problem (1):

$$\begin{gathered} max_{\mathrm{R}_+^m \setminus \{0\}} \ u \\ (u,v) \in \mathcal{K}_{\bar{x}}, v \in \mathrm{R}_+^l. \end{gathered} \tag{6}$$

If $(\bar{u},\bar{v}) \in \mathcal{K}_{\bar{x}}, \bar{v} \in \mathrm{R}_+^l$ satisfying $\bar{u} \not\leq_{\mathrm{R}_+^m \setminus \{0\}} u, \forall (u,v) \in \mathcal{K}_{\bar{x}}, v \in \mathrm{R}_+^l$, then $\bar{u}$ is called a Pareto efficient solution of (6).

Based on the extended image, we consider an equivlent image problem for problem (1):

$$\begin{gathered} max_{\mathrm{R}_+^m \setminus \{0\}} \ u \\ (u,v) \in \mathcal{E}_{\bar{x}}, v = 0. \end{gathered} \tag{7}$$

If $(\bar{u},\bar{v}) \in \mathcal{E}_{\bar{x}}, \bar{v} = 0$ satisfying $\bar{u} \not\leq_{\mathrm{R}_+^m \setminus \{0\}} u, \forall (u,v) \in \mathcal{E}_{\bar{x}}, v = 0$, then $\bar{u}$ is called a Pareto efficient solution of (7).

Proposition 3.2 Let $\bar{x} \in K$ and $\bar{v} = g(\bar{x})$. Then $\bar{x} \in K$ is a Pareto efficient solution of problem (1) if and only if

(i) $(0,\bar{v})$ is a Pareto efficient solution of (6);

or equivalently,

(ii) $(0,0)$ is a Pareto efficient solution of (7).

Proof. We first prove (1) and (i) are equivalent.

Suppose that $\bar{x} \in K$ is a Pareto efficient solution of problem (1). Then $\bar{x} \in X$ and $\bar{v} = g(\bar{x}) \in \mathrm{R}_+^l$. We have $(0,\bar{v}) = (f(\bar{x}) - f(\bar{x}), g(\bar{x})) \in \mathcal{K}_{\bar{x}}$, i.e., $(0,\bar{v})$ is a feasible solution of (6). Since $\mathcal{K}_{\bar{x}} \cap \mathcal{H} = \varnothing$, it follows that $(0,\bar{v})$ is a Pareto efficient solution of (6). Vice versa, if $(0,\bar{v})$ is a Pareto efficient solution of (6), then $\bar{x} \in X$ and $\bar{v} = g(\bar{x}) \in \mathrm{R}_+^l$, i.e., $\mathcal{K}_{\bar{x}} \cap \mathcal{H} = \varnothing$. Thus $\bar{x} \in K$ is a Pareto efficient solution of problem (1).

Now,we prove (1) and (ii) are equivalent.

Suppose that $\bar{x} \in K$ is a Pareto efficient solution of problem (1). Then we have $\mathcal{E}_{\bar{x}} \cap \mathcal{H}_u = \varnothing$ from proposition 3.1. Since $\bar{x} \in K$, one has $\bar{x} \in X, v = g(\bar{x}) \in \mathrm{R}^l_+$, it follows that $(0,\bar{v}) = (f(\bar{x}) - f(\bar{x}), g(\bar{x})) \in \mathcal{K}_{\bar{x}}$. Notice that $(0,\bar{v}) \in cl\mathcal{H}$. Then $(0,\bar{v}) - (0,\bar{v}) = (0,0) \in \mathcal{K}_{\bar{x}} - cl\mathcal{H} = \mathcal{E}_{\bar{x}}$, i.e., $(0,0)$ is a feasible solution of (7). Suppose $(0,0)$ is not a Pareto efficient of (6). Then $\exists$ $\bar{y} \in X$ such that $g(\bar{y}) \geq_{\mathrm{R}^l_+} 0$, $0 \leq_{\mathrm{R}^m_+ \setminus \{0\}} f(\bar{x}) - f(\bar{y})$. It follows that $0 \leq_{\mathrm{R}^m_+ \setminus \{0\}} f(\bar{x}) - f(\bar{y}) \leq_{\mathrm{R}^m_+} f(\bar{x}) - f(\bar{y})$. Let $\bar{u} = f(\bar{x}) - f(\bar{y})$, this leads a contradiction, since $(\bar{u},0) \notin \mathcal{E}_{\bar{x}} \cap \mathcal{H}_u$. Vice versa, if $(0,0)$ is a Pareto efficient of (8), then $\bar{x} \in X, \bar{v} = g(\bar{x}) \in \mathrm{R}^l_+$, i.e., $\mathcal{K}_{\bar{x}} \cap \mathcal{H}_u = \varnothing$. It follows that $\bar{x} \in K$ is a Pareto efficient solution of problem (1).

Proposition 3.3 Let $\bar{x} \in K$. If there exists $\tilde{x} \in K$ such that $f(\tilde{x}) \leq_{\mathrm{R}^m_+} f(x), \forall x \in K$, then $\mathcal{E}_{\bar{x}} \cap cl\mathcal{H}_u = \{(u,v) : 0 \leq_{\mathrm{R}^m_+} u \leq_{\mathrm{R}^m_+} f(\bar{x}) - f(\tilde{x}), v = 0\}$.

Proof. If there exists $\tilde{x} \in K$ such that $f(\tilde{x}) \leq_{\mathrm{R}^m_+} f(x), \forall x \in K$, then

$$f(\bar{x}) - f(x) \leq_{\mathrm{R}^m_+} f(\bar{x}) - f(\tilde{x}), g(x) \geq_{\mathrm{R}^l_+} 0, x \in X. \tag{8}$$

Consequently, $u \leq_{\mathrm{R}^m_+} f(\bar{x}) - f(\tilde{x}), \forall (u,v) \in \mathcal{K}_{\bar{x}}, v \in \mathrm{R}^l_+$. We declare that

$$u \leq_{\mathrm{R}^m_+} f(\bar{x}) - f(\tilde{x}), \forall (u,v) \in \mathcal{E}_{\bar{x}}, v \in \mathrm{R}^l_+. \tag{9}$$

In fact, if $(u,v) \in \mathcal{E}_{\bar{x}}, v \in \mathrm{R}^l_+$, then there exists $\hat{y} \in X$ such that $u \leq_{\mathrm{R}^m_+} f(\bar{x}) - f(\hat{y}), 0 \leq_{\mathrm{R}^l_+} v \leq_{\mathrm{R}^l_+} g(\hat{y})$. It follows that $\hat{y} \in K$, and from (8), we have

$$u \leq_{\mathrm{R}^m_+} f(\bar{x}) - f(\hat{y}) \leq_{\mathrm{R}^m_+} f(\bar{x}) - f(\tilde{x}),$$

which implies that (9) holds. Since $cl\mathcal{H}_u = \{(u,v) \in \mathrm{R}^m \times \mathrm{R}^l : u \geq_{\mathrm{R}^m_+} 0, v = 0\}$, from (9) we can obtain $\mathcal{E}_{\bar{x}} \cap cl\mathcal{H}_u = \{(u,v) : 0 \leq_{\mathrm{R}^m_+} u \leq_{\mathrm{R}^m_+} f(\bar{x}) - f(\tilde{x}), v = 0\}$.

4. NONLINEAR SEPARATION AND SADDLE POINTS FOR CONSTRAINED MULTIOBJECTIVE OPTIMIZATION PROBLEM

In order to prove (3), we will show $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ lie in two disjoint level sets of a suitable nonlinear separation function by ISA. Let us introduce the class of functions $\omega : \mathrm{R}^m \times \mathrm{R}^l \times (\mathrm{R}^m \times \Gamma) \to \mathrm{R}^1$, defined by:

$$\omega(u,v;\theta,\gamma) = \langle \theta, u \rangle + \varpi(v;\gamma), \quad \theta \in \mathrm{R}^m, \gamma \in \Gamma, \tag{1}$$

where $\varpi : \mathrm{R}^l \times \Gamma \to \mathrm{R}^1$, Γ is a given parameter set.

Definition 4.1 Given a parameter set Γ. We say that $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ are nonlinearly separable if and only if there exist $\bar{\theta} \in \mathrm{R}^m, \bar{\gamma} \in \Gamma$ and $\omega(u,v;\theta,\gamma) = \langle \theta, u \rangle + \varpi(v;\gamma) \not\equiv 0$, such that:

$$\omega(u,v;\bar{\theta},\bar{\gamma}) = \langle \bar{\theta}, u \rangle + \varpi(v;\bar{\gamma}) \geq 0, \ \forall (u,v) \in \mathcal{H}, \tag{2}$$

$$\omega(u,v;\bar{\theta},\bar{\gamma}) = \langle \bar{\theta}, u \rangle + \varpi(v;\bar{\gamma}) \leq 0, \ \forall (u,v) \in \mathcal{K}_{\bar{x}}. \tag{3}$$

Moreover, if $\bar{\theta} \in \mathrm{R}^m_+ \setminus \{0\}$, then the separation is regular; if $\bar{\theta} \in \mathrm{R}^m_{++}$, the separation is strongly regular.

Lemma 4.1 Suppose there exist $\bar{\theta} \in \mathrm{R}^m, \bar{\gamma} \in \Gamma$ such that (2). Then $\bar{\theta} \in \mathrm{R}^m_+$ and $\varpi(v;\bar{\gamma}) \geq 0, \forall v \in \mathrm{R}^l_+$.

Proof. Ab absurdo, if $\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2, \cdots, \bar{\theta}_m)^{\mathsf{T}} \notin \mathrm{R}^m_+$, then there exists $i \in \{1, 2, \cdots, m\}$ such that $\bar{\theta}_i < 0$. Let $I_0 = \{i : \bar{\theta}_i < 0\}$ and $v \in \mathrm{R}^l_+$ be fixed, and $\{u_k\} \subseteq \mathrm{R}^m$ be a sequence of points. Denote by $(u_k)_i$ the i th component of u_k,

$$(u_k)_i = \begin{cases} k, & \text{if} \, i \in I_0 \\ 0, & \text{if} \, i \notin I_0 \end{cases}$$

then $u_k \geq_{\mathrm{R}^m_+ \setminus \{0\}} 0$. $\|u_k\| \to \infty$, if $k \to +\infty$. It follows that $\{(u_k, v)\} \subseteq \mathcal{H}$ and $\lim_{k\to+\infty} \omega(u_k, v; \bar{\theta}, \bar{\gamma}) = \lim_{k\to+\infty} [\langle \bar{\theta}, u_k \rangle + \varpi(v;\bar{\gamma})] = -\infty$, a contradiction from (2). Therefore $\bar{\theta} \in \mathrm{R}^m_+$. Now, we prove $\varpi(v;\bar{\gamma}) \geq 0, \forall v \in \mathrm{R}^l_+$. Let $v \in \mathrm{R}^l_+$ be fixed, and $\{u_k\} \subseteq \mathrm{R}^m$ be a sequence of points such that

$$u_k \geq_{\mathrm{R}^m_+ \setminus \{0\}} 0, \|u_k\| \to 0, \mu \pm k \to +\infty.$$

Then, $\{(u_k, v)\} \subseteq \mathcal{H}$ and $\varpi(v;\bar{\gamma}) = \lim_{k\to+\infty} \omega(u_k, v; \bar{\theta}, \bar{\gamma}) \geq 0$.

In view of Lemma 4.1, we can restrict our analysis to the class of functions $\omega(u, v; \theta, \gamma)$ such that $\theta \in \mathrm{R}^m_+, \gamma \in \Gamma$ and $\varpi(v;\bar{\gamma}) \geq 0, \forall v \in \mathrm{R}^l_+$. This class of functions will be called separation functions. It is clear that when $\bar{\theta} = 0$, the existence of a nonlinear separation can not guarantee the $\mathcal{K}_{\bar{x}} \cap \mathcal{H} = \varnothing$. However, if $\bar{\theta} \in \mathrm{R}^m_{++}$, i.e., the separation is strongly regular, strict inequality holds in (2), then (3) holds. Consequently, the following proposition is clear.

Proposition 4.1 Suppose $\bar{x} \in K$. If $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ are strongly regular nonlinearly separable, then $\bar{x} \in K$ is a Pareto efficient solution of problem (1).

Proof. Suppose $\bar{x} \in K$. If $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ are nonlinearly separable, then there exist $\bar{\theta} \in \mathrm{R}^m, \bar{\gamma} \in \Gamma$ such that (2) and (3) hold. Since this separation is strongly regular, then $\bar{\theta} \in \mathrm{R}^m_{++}$, i.e., the strict inequality holds in (2). Therefore, $\mathcal{K}_{\bar{x}} \cap \mathcal{H} = \varnothing$, i.e., $\bar{x} \in K$ is a Pareto efficient solution of (1).

In the following, without loss of generality, we assume $\bar{\theta} \in \mathrm{R}^m_{++}$, denoted the strongly regular separation functions by:

$$\omega(u, v; \bar{\theta}, \gamma) = \langle \bar{\theta}, u \rangle + \varpi(v;\gamma). \tag{4}$$

Theorem 4.1 Let $\bar{x} \in K$. Consider the class of strongly regular separation functions (4) that fulfill the following assumptions:

$$\inf_{\gamma\in\Gamma} \varpi(v;\gamma) = -\infty, \forall v \notin \mathrm{R}^l_+, \tag{5}$$

$$\inf_{\gamma\in\Gamma} \varpi(v;\gamma) = 0, \forall v \in \mathrm{R}^l_+. \tag{6}$$

Then,

$$\sup_{(u,v)\in\mathcal{K}_{\bar{x}}} \inf_{\gamma\in\Gamma} [\langle \bar{\theta}, u \rangle + \varpi(v;\gamma)] = \sup_{\substack{(u,v)\in\mathcal{K}_{\bar{x}} \\ v\in\mathrm{R}^l_+}} \langle \bar{\theta}, u \rangle. \tag{7}$$

Proof. Since (5) and (6) hold, for every fixed $u \in \mathrm{R}^m$, we have

$$\inf_{\gamma\in\Gamma} [\langle \bar{\theta}, u \rangle + \varpi(v;\gamma)] = \begin{cases} -\infty, & \text{if} \, v \notin \mathrm{R}^l_+, \\ \langle \bar{\theta}, u \rangle, & \text{if} \, v \in \mathrm{R}^l_+. \end{cases} \tag{8}$$

Notice that $(\bar{u},\bar{v}) = (f(\bar{x}) - f(\bar{x}), g(\bar{x})) = (0, g(\bar{x})) \in \mathsf{K}_{\bar{x}}$ and $\bar{v} \in \mathbb{R}^l_+$, since $\bar{x} \in K$. Taking the supremum in (8) leads to (7).

Remark 4.1 When the separation function $\omega(\cdot,\cdot;\theta,\gamma)$ is linear, i.e., $\varpi(v;\gamma) = \langle \gamma, v\rangle, \gamma \in (\mathbb{R}^l_+)^* = \mathbb{R}^l_+$, (2) and (3) are obvious.

Now, we consider the generalized Lagrangian function associated with problem (1), defined by

$$\mathsf{L}^{\omega}(x;\theta,\gamma) = \langle \theta, f(x)\rangle - \varpi(g(x);\gamma). \tag{9}$$

For a given separation function $\omega(u,v;\theta,\gamma) = \langle \theta, u\rangle + \varpi(v;\gamma)$, we shall present the equivalence between a nonlinear separation for problem (1) and the existence of saddle points of the generalized Lagrangian function (9). Obviously, when $\varpi(v;\gamma) = \langle \gamma, v\rangle$ and $\theta = m = 1$, (9) collapses to the standard Lagrangian function.

Theorem 4.2 Suppose that (5) and (6) hold. Then the sets $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ are nonlinearly separable and $\bar{x} \in K$, if and only if there exists $(\bar{\theta},\bar{\gamma}) \in \mathbb{R}^m_+ \times \Gamma$ such that $(\bar{x},\bar{\gamma})$ is a saddle point for $\mathsf{L}^{\omega}(x;\bar{\theta},\gamma)$ on $X \times \Gamma$. i.e.,

$$\mathcal{L}^{\omega}(\bar{x};\bar{\theta},\gamma) \le \mathcal{L}^{\omega}(\bar{x};\bar{\theta},\bar{\gamma}) \le \mathcal{L}^{\omega}(x;\bar{\theta},\bar{\gamma}), \quad \forall (x,\gamma) \in X \times \Gamma.$$

Proof. Sufficiency. Suppose there exists $(\bar{\theta},\bar{\gamma}) \in \mathbb{R}^m_+ \times \Gamma$ such that $(\bar{x},\bar{\gamma})$ is a saddle point for $\mathsf{L}^{\omega}(x;\bar{\theta},\gamma)$ on $X \times \Gamma$, i.e.,

$$\langle \bar{\theta}, f(\bar{x})\rangle - \varpi(g(\bar{x});\gamma) \le \langle \bar{\theta}, f(\bar{x})\rangle - \varpi(g(\bar{x});\bar{\gamma}) \le \langle \bar{\theta}, f(x)\rangle - \varpi(g(x);\bar{\gamma}), \forall (x,\gamma) \in X \times \Gamma. \tag{10}$$

Let $\bar{v} = g(\bar{x})$. From above we have

$$\varpi(\bar{v};\gamma) \ge \varpi(\bar{v},\bar{\gamma}) \ge \langle \bar{\theta}, u\rangle + \varpi(v;\bar{\gamma}), \forall (u,v) \in \mathcal{K}_{\bar{x}}, \forall \gamma \in \Gamma. \tag{11}$$

We now prove $\bar{x} \in K$. It suffices to show $\bar{v} \in \mathbb{R}^l_+$. Ab absudo, suppose $\bar{v} \notin \mathbb{R}^l_+$. Then by (5), it follows that $\inf_{\gamma\in\Gamma} \varpi(\bar{v},\gamma) = -\infty$. This contradicts the first inequality in (11). Thus $\bar{v} \in \mathbb{R}^l_+$. Again from the first inequality in (11) and (6), it follows that

$$0 = \inf_{\gamma\in\Gamma} \varpi(\bar{v},\gamma) \ge \varpi(\bar{v},\bar{\gamma}). \tag{12}$$

Since $\bar{v} \in \mathbb{R}^l_+$, one has $\varpi(\bar{v},\bar{\gamma}) \ge 0$. This associated with (12) implies:

$$\varpi(\bar{v},\bar{\gamma}) = 0. \tag{13}$$

Therefore, from above and the second inequality in (11) it follows that

$$0 \ge \langle \bar{\theta}, u\rangle + \varpi(v;\bar{\gamma}) = \omega(u,v;\bar{\theta},\bar{\gamma}), \forall (u,v) \in \mathcal{K}_{\bar{x}}. \tag{14}$$

It is clear that $\omega(u,v;\bar{\theta},\bar{\gamma}) \ge 0, \forall (u,v) \in \mathcal{H}$, which yields $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ are nonlinearly separable.

Necessity. Suppose $\bar{x} \in K$ and $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ are nonlinearly separable. Let $(\bar{u},\bar{v}) = (0, g(\bar{x}))$ be the image of the points $\bar{x}$ through the mapping $A_{\bar{x}}$, associated with (14). We can obtain $\varpi(\bar{v},\bar{\gamma}) \le 0$. Since $\bar{v} \in \mathbb{R}^l_+$ and from (6), it follows that $\varpi(\bar{v},\bar{\gamma}) \ge 0$, i.e., (13) holds. Again from (6), one has $\varpi(\bar{v},\gamma) \ge 0, \forall \gamma \in \Gamma$, i.e., (11) holds. Consequently, (10) is true.

Remark 4.2 Similarly, we can prove the following results. Suppose that (5) and (6) hold. Then the sets $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ are regular (res., strongly regular) nonlinearly separable and $\bar{x} \in K$, if and only if there exists $(\bar{\theta},\bar{\gamma}) \in \mathbb{R}^m_+ \setminus \{0\} \times \Gamma$ (res., $(\bar{\theta},\bar{\gamma}) \in \mathbb{R}^m_{++} \times \Gamma$) such that $(\bar{x},\bar{\gamma})$ is a saddle point for $\mathcal{L}^{\omega}(x;\bar{\theta},\gamma)$ on $X \times \Gamma$. Specially, when the separation function is strongly regular, from proposition 4.1, it follows that the existence of Pareto efficient solution of problem (1) can be obtained by the saddle point condition of the generalized Lagrangian function.

It is well known the generalized Lagrangian function contains many special Lagrange-type function as its special case, such as the classic Lagrangian function, the exponential-type Lagrangian function and the augmented Lagrangian function and so on. Next, we will introduce an example associated with generalized Lagrangian function, which is called the augmented Lagrangian function $\hat{\mathcal{L}}: X \times \mathbb{R}_+^m \times \mathbb{R}^l \to \mathbb{R}$, defined by:

$$\hat{\mathcal{L}}(x;\theta,\lambda) = \langle \theta, f(x)\rangle + \inf_{g(x)-z \geq_{\mathbb{R}_+^l} 0} [-\langle \lambda, z\rangle + c\sigma(z)], \tag{15}$$

where $c > 0$ is a real number and $\sigma : \mathbb{R}^l \to \mathbb{R}$ is a function satisfying

$$arg\ min_{z\in\mathbb{R}^l} \sigma(z) = \{0\}, \sigma(0) = 0.$$

Actually, to show the augmented Lagrangian function (15) is a special case of the generalized Lagrangian function (9), we only should assume $\Gamma = \mathbb{R}^l$ and

$$\varpi(v;\gamma) = \sup_{z \leq_{\mathbb{R}_+^l} v} [\langle \gamma, z\rangle - c\sigma(z)], \tag{16}$$

where c and σ are the same as in (15). Then (9) reduces to

$$\mathcal{L}^{\varpi}(x;\theta,\gamma) = \langle \theta, f(x)\rangle - \varpi(g(x);\gamma) = \langle \theta, f(x)\rangle - \sup_{z \leq_{\mathbb{R}_+^l} g(x)} [\langle \gamma, z\rangle - c\sigma(z)].$$

Letting $\gamma = \lambda$, allows (15).

If we consider the augmented Lagrangian function (15), then (5) and (6) always hold.

Lemma 4.2 Suppose $\Gamma = \mathbb{R}^l, \varpi(v;\gamma)$ is defined by (16). Then (5) and (6) are true.

Proof. Since $\langle \gamma, z\rangle - c\sigma(z) \leq \langle \gamma, z\rangle, \forall (z,\gamma) \in \mathbb{R}^l \times \Gamma.$, it follows that

$$\varpi(v;\gamma) = \sup_{z \leq_{\mathbb{R}_+^l} v} [\langle \gamma, z\rangle - c\sigma(z)] \leq \sup_{z \leq_{\mathbb{R}_+^l} v} \langle \gamma, z\rangle, \forall (v,\gamma) \in \mathbb{R}^l \times \Gamma.$$

If $\gamma \in \mathbb{R}_+^l$, then

$$\sup_{z \leq_{\mathbb{R}_+^l} v} \langle \gamma, z\rangle = \langle \gamma, v\rangle, \forall v \in \mathbb{R}^l$$

and it follows that

$$\inf_{\gamma\in\Gamma} \varpi(v;\gamma) \leq \inf_{\gamma\in\mathbb{R}_+^l} \langle \gamma, v\rangle = -\infty, \forall v \notin \mathbb{R}_+^l,$$

which proves (5). If $v \in \mathbb{R}_+^l$, then $\varpi(v;\gamma) \geq 0, \forall \gamma \in \Gamma$, and so

$$\inf_{\gamma\in\Gamma} \varpi(v;\gamma) \geq 0.$$

Again since $\varpi(v;\gamma) \leq \langle \gamma, v\rangle, \forall (v,\gamma) \in \mathbb{R}^l \times \mathbb{R}_+^l$, we have $\inf_{\gamma\in\mathbb{R}_+^l} \langle \gamma, v\rangle = 0$, $\forall v \in \mathbb{R}_+^l$. As a consequence,

$$\inf_{\gamma\in\Gamma} \varpi(v;\gamma) \leq \inf_{\gamma\in\mathbb{R}_+^l} \varpi(v;\gamma) \leq 0,$$

which implies (6) is true.

The following is an immediate consequence of Theorem 4.2 and Lemma 4.2.

Corollary 4.1 Suppose $\Gamma = \mathbb{R}^l$ and the separation function is $\omega(u,v;\theta,\lambda) = \langle \theta, u\rangle + \varpi(v,\lambda)$, where $\varpi(v,\lambda)$ is defined by (16). Then, the sets $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}$ are nonlinearly separable and $\bar{x} \in K$ if and only if there exists $(\bar{\theta},\bar{\gamma}) \in \mathbb{R}_+^m \times \Gamma$ such that $(\bar{x},\bar{\gamma})$ is a saddle point for the augmented Lagrangian $\hat{\mathcal{L}}(x;\bar{\theta},\lambda)$ on $X \times \Gamma$.

5. SEPARATION AND PENALTY METHODS

In order to prove (3), we introduce the strongly regular separation function to obtain saddle point condition. We always assume $\bar{\theta} \in \mathbb{R}_{++}^m$. Consider the following extremum problem:

$$min[\langle \bar{\theta}, f(x)\rangle + \psi(x;\omega)] \quad s.t. \quad x \in K, \tag{1}$$

where $\psi : \mathrm{R}^n \times \Omega \to \mathrm{R}$, Ω is a parameter set. $\langle \overline{\theta}, f(x)\rangle + \psi(x;\omega)$ is called a penalty function for Problem (1).
If there exists $\overline{\omega} \in \Omega$ such that a solution of (1), say $\overline{x}$, is a Pareto efficient solution of problem (1), then we say that $\langle \overline{\theta}, f(x)\rangle + \psi(x;\overline{\omega})$ is an exact penalty function for problem (1) at $\overline{x}$.

Proposition 5.1 Suppose that $\overline{x} \in K$ and 5 and (6) hold. If

$$\inf_{\gamma\in\Gamma} \sup_{(u,v)\in \mathrm{K}_{\overline{x}}} \omega(u,v;\overline{\theta},\gamma) \le 0, \tag{2}$$

then $\overline{x}$ is a Pareto efficient solution of problem (1).

Proof. Notice that $\overline{x}$ is a Pareto efficient solution of problem (1) is equivalent to $\mathcal{K}_{\overline{x}} \cap \mathcal{H} = \emptyset$. Ab absudo, suppose there exists $(\tilde{u},\tilde{v}) \in \mathcal{K}_{\overline{x}} \cap \mathcal{H}$. Since $\overline{x} \in K$ and (5) and (6) hold, we have

$$\sup_{(u,v)\in \mathrm{K}_{\overline{x}}} \omega(u,v;\overline{\theta},\gamma) \ge \langle \overline{\theta},\tilde{u}\rangle + \varpi(\tilde{v};\gamma) \ge \langle \overline{\theta},\tilde{u}\rangle > 0, \forall \gamma \in \Gamma,$$

Thus it follows that

$$\inf_{\gamma\in\Gamma} \sup_{(u,v)\in \mathrm{K}_{\overline{x}}} \omega(u,v;\overline{\theta},\gamma) \ge \langle \overline{\theta},\tilde{u}\rangle > 0,$$

a contradiction with (2).
The following statements are related with the existence of an exact penalty function at $\overline{x} \in K$.
Denote by $\rho(x,B) = \inf\{\|x-b\|_2 : b\in B\}$ the distance from a point $x \in \mathrm{R}^l$ to a set $B \subseteq \mathrm{R}^l$. Consider the function $\mathcal{L}^{\omega} : X \times \mathrm{R}_+ \setminus \{0\} \to \mathrm{R}$, defined by:

$$\mathcal{L}^{\omega}(x;\gamma) = \langle \overline{\theta}, f(x)\rangle + \gamma\rho(g(x),\mathrm{R}^l_+). \tag{3}$$

Theorem 5.1 Let $\overline{x} \in K$. The following statements are equivalent:

(i)there exists $\overline{\gamma} \in \Gamma = \mathrm{R}_+ \setminus \{0\}$ such that,

$$\omega(u,v;\overline{\theta},\overline{\gamma}) \le 0, \forall (u,v) \in \mathrm{K}_{\overline{x}},$$

where $\omega(u,v;\overline{\theta},\gamma) = \langle \overline{\theta},u\rangle + \varpi(v;\gamma) = \langle \overline{\theta},u\rangle - \gamma\rho(v,\mathrm{R}^l_+), \gamma \in \Gamma$,

(ii) $\mathrm{L}^{\omega}(x;\gamma)$ defined by (3) is an exact penalty function for problem (1) at $\overline{x}$,

(iii)there exists $\overline{\gamma} \in \Gamma = \mathrm{R}_+ \setminus \{0\}$ such that

$$\langle \overline{\theta}, f(\overline{x}) - f(x)\rangle \le \overline{\gamma}\rho(g(x),\mathrm{R}^l_+), \forall x \in X.$$

Proof. We first show (i) implies (ii). From (i), it follows that $\varpi(v;\gamma) = -\gamma\rho(v,\mathrm{R}^l_+)$. Since (5), (6) and (2) is true, we can obtain $\overline{x}$ is a Pareto efficient solution of problem (1) from the Proposition 5.1. Since $\omega(u,v;\overline{\theta},\overline{\gamma}) \le 0, \forall (u,v) \in \mathrm{K}_{\overline{x}}$, it is easy to know $\overline{x}$ is a solution of (1). Again from the difinition of exact penalty function at $\overline{x}$, we obtain (ii) holds.

Now, we show (ii) implies (iii). Since $\mathcal{L}^{\omega}(x;\gamma)$ is an exact penalty function for problem(1) at $\overline{x}$, there exists $\overline{\gamma} \in \mathrm{R}_+ \setminus \{0\}$ such that

$$\langle \overline{\theta}, f(\overline{x})\rangle + \overline{\gamma}\rho(g(\overline{x}),\mathrm{R}^l_+) \le \langle \overline{\theta}, f(x)\rangle + \overline{\gamma}\rho(g(x),\mathrm{R}^l_+), \forall x \in X.$$

Again since $\overline{x}$ is a Pareto efficient solution of problem (1), we have $g(\overline{x}) \in \mathrm{R}^l_+$, it follows that

$$\langle \overline{\theta}, f(\overline{x}) - f(x)\rangle \le \overline{\gamma}\rho(g(x),\mathrm{R}^l_+), \forall x \in X.$$

(iii) $\Longrightarrow$ (i) is obvious.

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