ON THE GROWTH OF SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS

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ABSTRACT
In this paper, we consider the differential equation $f''+A(z)f'+B(z)f=0$ where $A(z)$ and $B(z)\not\equiv 0$ are meromorphic functions. Assume that $A(z)$ has a finite deficient value, then we will give some conditions on $B(z)$ which can guarantee that every solution $f\not\equiv 0$ of the equation is of infinite order.

Keywords: *complex differential equation, deficient value, meromorphic function, growth of order*.
2010 Mathematical Subject Classification: 34M10; 30D35.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we shall consider the second order linear differential equation

$$f''+A(z)f'+B(z)f=0, \tag{1.1}$$

where $A(z)$ and $B(z)\not\equiv 0$ are meromorphic functions. We shall use the standard notations of Nevanlinna theory of meromorphic functions (see [1-3]). Especially, for a meromorphic function $f(z)$, we use the notation $\rho(f)$ and $\mu(f)$ to denote its order and lower order, respectively.

It is well known that if $A(z)$ is entire and $B(z)$ is transcendental entire, and f_1, f_2 are two linearly independent solutions of the equation (1.1), then at least one of f_1, f_2 must have infinite order. On the other hand, there are some equations of the form (1.1) that possess a solution $f\not\equiv 0$ of finite order; for example, $f(z)=e^z$ satisfies $f''+e^{-z}f'-(e^{-z}+1)f=0$.

Thus a natural question is: what conditions on $A(z)$ and $B(z)$ can guarantee that every solution $f\not\equiv 0$ of the equation (1.1) has infinite order? There are many work done on this subject. For example, from the work of Gundersen (see [4]), Hellerstein, Miles and Rossi (see [5]), we know that if $A(z)$ and $B(z)$ are entire functions with $\rho(A)<\rho(B)$; or if $A(z)$ is a polynomial, and $B(z)$ is transcendental; or if $\rho(B)<\rho(A)\le\frac{1}{2}$, then every solution $f\not\equiv 0$ of the equation (1.1) is of infinite order. More results can be found in ([6-11]).

It is well-know that deficient value plays a fundamental role in the theory of value distribution of meromorphic functions. Many important works were done on this subject (see for example [12]). There are some results on the value distribution theory of the solutions of the equation (1.1) having connections with deficient values (see [8], [10], [13]). In this paper, we shall introduce the deficient value into the studies of the equation (1.1) again, and we shall prove the following results.

Theorem 1.1 Let $A(z)$ be an meromorphic function with finite order having a finite deficient value, and let $B(z)$ be a transcendental meromorphic function with $\mu(B)<\frac{1}{2}$ having a deficient value ∞ of deficiency $\delta(\infty,B)=1$. Then every solution $f\not\equiv 0$ of the equation (1.1) is of infinite order.

The research was supported by the National Natural Science Foundation of China(Grant No.11171080) and the Foundation of Science and Technology Department of Guizhou Province (Grant No.[2010]07

Our theorem can be applied to some particular equations. Let $Q(z)$ be a non-constant polynomial, and let $B(z)$ be a transcendental entire function with $\mu(B) < \frac{1}{2}$, then Gundersen proved (see [4], theorem 7) that every solution $f \not\equiv 0$ of the equation

$$f'' + e^{Q(z)} f' + B(z) f = 0 \tag{1.2}$$

is of infinite order. Obviously, the coefficients of equation (1.2) also satisfies the conditions of Theorem 1.1.
We also note that several authors have studied the problems of whether the particular differential equation

$$f'' + e^{-z} f' + B(z) f = 0 \tag{1.3}$$

can possess a solution $f \not\equiv 0$ of finite order if $B(z) \not\equiv 0$ is entire function. If $B(z) \equiv C$ is a constant, then equation (1.3) will possess a solution $f \not\equiv 0$ of finite order if and only if $C = -n^2$ (see [7]). Therefore, when $B(z)$ is not transcendental, Theorem 1.1 is in general false. More results can be found in $[11]$.
Using the same method of the proof of Theorem 1.1, we can easily obtain the following two results.

Theorem 1.2 Let $A(z)$ be an meromorphic function having a finite deficient value, and let $B(z) \not\equiv 0$ be an meromorphic function. Suppose that there exist two constants $\alpha > 0$ and $\beta > 0$, for any given $\varepsilon > 0$, two finite set of real numbers $\{\phi_k\}$ and $\{\theta_k\}$ that satisfy $\phi_1 < \theta_1 < \phi_2 < \theta_2 < \cdots < \phi_m < \theta_m < \phi_{m+1}$ $(\phi_{m+1} = \phi_1 + 2\pi)$ and

$$\sum_{k=1}^{m} (\phi_{k+1} - \theta_k) < \varepsilon, \tag{1.4}$$

such that

$$| B(z) | \geq \exp\{(1 + o(1))\alpha | z |^{\beta}\} \tag{1.5}$$

as $z \to \infty$ in $\phi_k \leq \arg z \leq \theta_k (k = 1, 2, \cdots, m)$. Then every solution $f \not\equiv 0$ of the equation (1.1) is of infinite order.
Before stating the final result in this section, for $E \subset [0, \infty)$, we define the Lebesgue measure of E by $mes(E)$ and the logarithmic measure of $E \subset [1, \infty)$ by $m_l(E) = \int_E \frac{dt}{t}$.

Theorem 1.3 Let $A(z)$ be an meromorphic function having a finite deficient value, and let $B(z) \not\equiv 0$ be an meromorphic function such that for any $k > 0$

$$\lim_{|z| = r \to \infty} \frac{| B(z) |}{r^k} = \infty$$

holds outside a set G of r-values with $m_l(G) < \infty$. Then every solution $f \not\equiv 0$ of the equation (1.1) is of infinite order.
The paper is organized as follows. In Section 2, we shall state and prove some lemmas related with proof of our Theorems. In Section 3, we shall prove our main results.

2. LEMMAS

In this section, we state and prove some lemmas related with proof of our Theorems.

Lemma 2.1 ([14]) Let (f, Γ) denote a pair that consists of a transcendental meromorphic function $f(z)$ of finite order and a finite set

$$\Gamma = \{(k_1, j_1), (k_2, j_2), \cdots, (k_q, j_q)\}$$

of distinct pairs of integers that satisfies $k_i > j_i \geq 0$ for $i = 1, \cdots, q$. Let $\varepsilon > 0$ be a given real constant. Then the following three statements hold:

(i)There exists a set $E_1 \subset [0,2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0,2\pi) - E_1$, then there is a constant $R_0 = R_0(\psi_0) > 0$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \geq R_0$, and for all $(k, j) \in \Gamma$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho(f)-1+\varepsilon)}. \tag{2.1}$$

(ii) There exists a set $E_2 \subset (1,\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin (E_2 \bigcup [0,1])$ and for all $(k, j) \in \Gamma$, the inequality (2.1) holds.

(iii) There exiets a set $E_3 \subset [0,\infty)$ that has finite linear measure, such that for all z satisfying $|z| \notin E_3$ and for all $(k, j) \in \Gamma$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho(f)+\varepsilon)}. \tag{2.2}$$

To state the following lemma, we define the upper and lower logarithmic density of $E \subset [1,\infty)$ respectively by

$$\overline{\log dens} E = \overline{\lim_{r\to\infty}} \frac{m_l(E \bigcap [1,r])}{\log r},$$

and

$$\underline{\log dens} E = \underline{\lim}_{r\to\infty} \frac{m_l(E \bigcap [1,r])}{\log r},$$

where $m_l(E \bigcap [1,r]) = \int_{E \bigcap [1,r]} \frac{dt}{t}$.

Lemma 2.2 ([15]) Suppose that $g(z)$ is a transcendental and meromorphic with $\mu(g) < \alpha < 1$, and define $L(r,g) = \min\{|g(z)| : |z| = r\}$ and $E = \{r > 1 : \log L(r,g) > \gamma(\cos\pi\alpha + \delta(\infty,g) - 1)T(r,g)\}$, $\gamma = \frac{\pi\alpha}{\sin\pi\alpha}$. Then E has upper logarithmic density at least 1 - $\frac{\mu(g)}{\alpha}$.

Lemma 2.3 ([16]) Let $A(z)$ be a meromorphic function with $\rho(A) < +\infty$, then for any given real constants $c > 0$ and H $(\rho(A) < H)$, there exists a set $E \subset (0,\infty)$ such that $\underline{\log dens} E \geq 1 - \frac{\rho(A)}{H}$, where

$$E = \{t \mid T(te^c, A) \leq e^k T(t, A)\}, \tag{2.3}$$

and $k = cH$.

Lemma 2.4 ([13]) Let $\{[t_n, 2t_n]\}$ be a sequence of closed intervals such that $t_1 \geq 1, t_{n+1} > 2t_n (n = 1,2,\cdots)$. Suppose $E \subset \cup_{n=1}^{=\infty}[t_n, 2t_n]$ such that for each n, $E \cap [t_n, 2t_n]$ consists of finitely many open intervals and such that $mes(E \bigcap [t_n, 2t_n]) \leq \frac{t_n}{L}$ ($L > 2$). Then, $\overline{\log dens} E \leq \frac{1}{L \log 2}$.

Lemma 2.5 Let $g(z)$ be a transcendental meromorphic function with $0 < \mu(g) < \frac{1}{2}$ and $\delta(\infty, g) = 1$, and let $A(z)$ be a meromorphic function with $\rho(A) < +\infty$. If $A(z)$ has a finite deficient value a with deficiency $\delta = \delta(a, A)$, then for any given constant $\varepsilon > 0$, there exists a sequence R_n with $R_n \to \infty$ ($n \to \infty$), such that the following two inequalities

$$|g(R_n e^{i\varphi})|> \exp\{R_n^{\mu(g)-\varepsilon}\},\ \varphi\in[0,2\pi),$$

and

$$mes(F_n)=:mes\{\theta\in[0,2\pi):\log|A(R_n e^{i\theta})-a|\leq -\frac{\delta}{4}T(R_n,A)\}\geq d>0$$

hold for all sufficiently large n, where d is a constant depending only on $\rho(A),\mu(g)$ and δ.

Proof. For any given constant $\varepsilon>0$, by using Lemma 2.2 to $g(z)$, there exists a set $E_1^*=E_1^*(\varepsilon,\mu(g))\subset[1,+\infty)$ with $\overline{\log}\,\overline{dens}E_1^*\geq 1-\frac{\mu(g)}{\alpha_0}$, where $\alpha_0=\frac{\mu(g)+\frac{1}{2}}{2}$, $E_1^*=\{r>1:\log L(r,g)>\gamma\cos\pi\alpha T(r,g)\}$, $\gamma=\frac{\pi\alpha}{\sin\pi\alpha}$. Hence, there exists a constant $R_0>0$ such that for all $|z|=r\in E_1=E_1^*\setminus[0,R_0]$, we have

$$|g(z)|>\exp\{r^{\mu(g)-\varepsilon}\}.$$

Set $c=\log 8$ and $H_0=\rho(A)+(1-\frac{\mu(g)}{\alpha_0})^{-1}\rho(A)$. By using Lemma 2.3, we have

$$\underline{\log}\,\underline{dens}E_2\geq 1-\frac{\rho(A)}{H_0}>\frac{\mu(g)}{\alpha_0}, \tag{2.4}$$

where $E_2=\{t\,|\,T(8t,A)\leq 8^{H_0}T(t,A)\}$. Denote $E=E_1\bigcap E_2$, then

$$\overline{\log}\,\overline{dens}E+\underline{\log}\,\underline{dens}(E_2\setminus E_1)\geq\underline{\log}\,\underline{dens}E_2\geq 1-\frac{\rho(A)}{H_0}.$$

Thus,

$$\overline{\log}\,\overline{dens}E\geq 1-\frac{\rho(A)}{H_0}-\underline{\log}\,\underline{dens}E_1^c.$$

Since $\overline{\log}\,\overline{dens}E_1+\underline{\log}\,\underline{dens}E_1^c=1$, we have

$$\overline{\log}\,\overline{dens}E\geq 1-\frac{\rho(A)}{H_0}-\frac{\mu(g)}{\alpha_0}>0. \tag{2.5}$$

Now we choose a sequence of close intervals $\{[t_n,2t_n]\}$. First we arbitrarily choose $t_1\in E(t_1\geq 1)$ and get the first interval $[t_1,2t_1]$. Let

$$t_2^{'}=\inf\{t\in E:t\geq 2t_1\},$$

and choose $t_2\in E\bigcap[2t_1,\infty)$ such that $0\leq t_2-t_2^{'}<\frac{1}{2}$. Suppose that t_{n-1} has been defined similarly as for t_2 above. Now define

$$t_n^{'}=\inf\{t\in E:t\geq 2t_{n-1}\},$$

and choose $t_n\in E\bigcap[2t_{n-1},\infty)$ such that $0\leq t_n-t_n^{'}<\frac{1}{2^{n-1}}$. In this way, we get a sequence of closed intervals. It is easy to see the sequence of closed intervals $\{[t_n,2t_n]\}$ satisfing $mes(\bigcup_{n=2}^{\infty}[t_n^{'},t_n])\leq 1$. Let

$$X = E \setminus \{(\bigcup_{n=2}^{\infty}[t_n', t_n]) \bigcup [0, t_1)\},$$

then $X \subset \bigcup_{n=1}^{\infty}[t_n, 2t_n]$ and $\overline{\log dens} X = \overline{\log dens} E$.

Let $\alpha_1, \alpha_2, \cdots, \alpha_{s_n}$ (where $s_n = n(3t_n, A-a)$) be all the zeros of $A(z)-a$, and let $\beta_1, \beta_2, \cdots, \beta_{q_n}$ (where $q_n = n(3t_n, A)$) be all the poles of $A(z)$ in $|z| \le 3t_n$. By the Boutroux-Cartan theorem (see [13]), we have

$$\prod_{j=1}^{s_n} |z-\alpha_j| > (\frac{P}{e})^{s_n}; \tag{2.6}$$

$$\prod_{j=1}^{q_n} |z-\beta_j| > (\frac{P}{e})^{q_n}, \tag{2.7}$$

except a set of points that can be enclosed in a finite number of disks $(\gamma)_m$ with the sum of total radius not exceeding $2P$. Set $L_0 = 2 + \{\log 2(1-\frac{\rho(A)}{H_0} - \frac{\mu(B)}{\alpha_0})\}^{-1}$ and $P = \frac{t_n}{2L_0}$, then $\frac{1}{L_0 \log 2} < \overline{\log dens} X$. Let G' be the set of $r \in [1, +\infty)$ such that the circle $|z| = r$ intersects the exceptional disks. Denote $G = G' \cap \bigcup_{n=1}^{\infty}[t_n, 2t_n]$. By using Lemma 2.4, we have that $\overline{\log dens} G \le \frac{1}{L_0 \log 2}$ and

$$\overline{\log dens}(X \bigcap G^c) \ge \overline{\log dens} X - \frac{1}{L_0 \log 2} > 0. \tag{2.8}$$

Hence, there exists a sequence R_n with $R_n \to \infty$ and $R_n \in E \bigcap G^c$. Without loss of generality, we can assume $R_n \in [t_n, 2t_n]$, otherwise we can use the subsequence of $\{R_{n_j}\}$ instead of $\{R_n\}$. Thus $R_n e^{i\theta} \notin \bigcup (\gamma)_m$ for all $\theta \in [0, 2\pi), n = 1, 2, \cdots$.

By (2.6), (2.7) and (2.4) and Poisson-Jensen formula ,we have

$$\begin{aligned}
\log \frac{1}{|A(R_n e^{i\theta}) - a|} &\le \frac{1}{2\pi}\int_0^{2\pi} \log + |\frac{1}{A(3t_n e^{i\varphi}) - a}| \frac{(3t_n)^2 - R_n^2}{(3t_n)^2 - 6t_n R_n \cos(\theta-\varphi) + R_n^2} d\varphi \\
&+ \sum_{j=1}^{s_n} \log |\frac{(3t_n)^2 - \overline{\alpha}_j R_n e^{i\theta}}{3t_n(R_n e^{i\theta} - \alpha_j)}| + \sum_{j=1}^{q_n} \log |\frac{(3t_n)^2 - \overline{\beta}_j R_n e^{i\theta}}{3t_n(R_n e^{i\theta} - \beta_j)}| \\
&\le \frac{3t_n + R_n}{3t_n - R_n} m(3t_n, \frac{1}{A-a}) + s_n \log \frac{e}{P} + s_n \log 5t_n + q_n \log \frac{e}{P} + q_n \log 5t_n \\
&\le 5m(3t_n, \frac{1}{A-a}) + s_n(\log 5t_n + \log \frac{2eL_0}{t_n}) + q_n(\log 5t_n + \log \frac{2eL_0}{t_n}) \\
&= 5m(3t_n, \frac{1}{A-a}) + s_n \log 10eL_0 + q_n \log 10eL_0 \\
&\le 5m(3t_n, \frac{1}{A-a}) + \frac{N(4t_n, A=a)}{\log \frac{4}{3}} \log 10eL_0 + \frac{N(4t_n, A)}{\log \frac{4}{3}} \log 10eL_0
\end{aligned}$$

$$\leq 5T(3t_n,\frac{1}{A-a})+\{T(4t_n,\frac{1}{A-a})+T(4t_n,\frac{1}{A})\}\frac{\log 10eL_0}{\log\frac{4}{3}}$$

$$\leq \{5+\frac{2\log 10eL_0}{\log\frac{4}{3}}\}T(8t_n,A)$$

$$\leq 8^{H_0}\{5+\frac{2\log 10eL_0}{\log\frac{4}{3}}\}T(t_n,A),\quad (n\geq N_0).$$

In addition, there exists a constant $n_0 > N_0$ such that for all $n > n_0$, we have

$$\frac{\delta}{2}T(R_n,A)<\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{|A(R_ne^{i\theta})-a|}d\theta$$

$$\leq \frac{1}{2\pi}\int_{F_n}\log\frac{1}{|A(R_ne^{i\theta})-a|}d\theta+\frac{\delta}{4}T(R_n,A).$$

Hence,

$$\frac{\delta}{4}T(R_n,A)\leq\frac{1}{2\pi}\int_{F_n}\log\frac{1}{|A(R_ne^{i\theta})-a|}d\theta$$

$$\leq \frac{1}{2\pi}8^{H_0}\{5+\frac{2\log 10eL_0}{\log\frac{4}{3}}\}T(R_n,A)mes(F_n).$$

So

$$d=\frac{\frac{\delta}{4}}{\frac{1}{2\pi}8^{H_0}\{5+\frac{2\log 10eL_0}{\log\frac{4}{3}}\}}\leq mes(F_n). \tag{2.9}$$

The proof of Lemma 2.5 is completed.

Remark If g is a transcendental meromorphic function with $\mu(g)=0$ and $\delta(\infty,g)=1$, according to Lemma 2.2, we only need to give an appropriate modification and Lemma 2.5 still holds.

3. PROOF OF THEOREM 1.1-1.3

In this section,we prove Theorem 1.1-1.3.

Proof of Theorem 1.1 Suppose that $f\not\equiv 0$ is a solution of the equation (1.1) with $\rho(f)<+\infty$. We shall seek a contradiction. Let a be a finite deficient value of $A(z)$ with deficiency $\delta=\delta(a,A)$. From equation (1.1), we have the following inequalities

$$|B(z)|\leq|\frac{f''(z)}{f(z)}|+|\frac{f'(z)}{f(z)}|(|A(z)-a|+|a|). \tag{3.1}$$

By Lemma 2.1, there exists a set $E_1\subset[1,+\infty)$ with $m_l(E_1)<+\infty$ such that the following inequality

$$|\frac{f^{(k)}(z)}{f(z)}|\leq|z|^{2\rho(f)},\quad k=1,2 \tag{3.2}$$

holds for all z with $|z|=r\notin(E_1\bigcup[0,r_0]),r_0>1$.

In the following, we treat two cases.

Case 1: $0 < \mu(B) < \frac{1}{2}$. By using Lemma 2.5 to $A(z)$, there exists a sequence $\{R_n\}$ with $R_n < R_{n+1}$ and $R_n \to \infty$ such that for every n, we have

$$mes(F_n) =: mes\{\varphi \in [0,2\pi) : \log|A(R_n e^{i\varphi}) - a| \le -\frac{\delta}{4} T(R_n, A)\} \ge d > 0, \tag{3.3}$$

and

$$|B(R_n e^{i\varphi})| > \exp\{R_n^{\frac{1}{2}\mu(B)}\},\ \varphi \in [0,2\pi), \tag{3.4}$$

where d is a constant depending only on $\rho(A)$, $\mu(B)$ and δ.

For every $n \ge n_0$, we choose $\theta_n \in F_n$. From (3.1), (3.2) and (3.3), we get

$$|B(R_n e^{i\theta_n})| \le |z|^{2\rho(f)} (1 + \exp\{-\frac{\delta}{4} T(R_n, A)\} + |a|),\ n \ge n_0. \tag{3.5}$$

Thus,

$$\exp\{R_n^{\frac{1}{2}\mu(B)}\} < |z|^{2\rho(f)} (1 + \exp\{-\frac{\delta}{4} T(R_n, A)\} + |a|).$$

Obviously, when n is sufficiently large, this is a contradiction.

Case 2: $\mu(B) = 0$. By using Lemma 2.2, there exists a set $E_2 \subset [1,+\infty)$ with $\overline{\log dens} E_2 = 1$ such that for all z satisfying $|z| = r \in E_2$, we have

$$\log|B(z)| > \frac{\pi}{4} T(r, B). \tag{3.6}$$

It follows from the proof of Lemma 2.5, there exists a sequence $\{R_n\}$ such that (3.3) and (3.6) hold. From (3.1), (3.2) and (3.3), we can obtain (3.5). Hence, from (3.5) and (3.6), we get

$$\exp^{\frac{\pi}{4} T(R_n, B)} \le R_n^{2\rho(f)} (1 + \exp\{-\frac{\delta}{4} T(R_n, A)\} + |a|)(n \ge n_0). \tag{3.7}$$

But $B(z)$ is a transcendental meromorphic function, so we have

$$\underline{\lim}_{r\to\infty} \frac{T(r, B)}{\log r} = +\infty.$$

Therefore, we can easily obtain a contradiction from (3.7). The proof of Theorem 1.1 is completed.

Proof of Theorem 1.2 Suppose that $f \not\equiv 0$ is a solution of the equation (1.1) with $\rho(f) < +\infty$. Let a be a finite deficient value of $A(z)$ with deficiency $\delta = \delta(a, A)$. By Lemma 2.1, there exists a set $E_1 \subset [0,+\infty)$ with $m_l(E_1) < +\infty$ such that the inequality (3.2) holds for all z with $|z| = r \notin (E_1 \bigcup [0, r_1])(r_1 > 1)$. From the proof of Lemma 2.5, there still exists a sequence $\{R_n\}$ that satisfies (3.3). Let $0 < \varepsilon < \frac{d}{2}$. Then for every integer n, we choose $\varphi_n \in F_n \bigcap (\bigcup_{k=1}^{m} [\phi_k, \theta_k])$. So there exists an integer $k \in \{1,2,\cdots,m\}$ such that for every integer n satisfies $\varphi_n \in F_n \bigcap [\phi_k, \theta_k]$ (otherwise we use the subsequence φ_{n_j} instead of φ_n). Thus, from our hypothesis, we have

$$|B(R_n e^{i\varphi_n})| \ge \exp\{(1 + o(1))\alpha R_n^{\beta}\}, \tag{3.8}$$

as $n \to \infty$. From (3.1), (3.2) and (3.3), we get

$$|B(R_ne^{i\varphi_n})|\le|z|^{2\rho(f)}(1+\exp\{-\frac{\delta}{4}T(R_n,A)\}+|a|)\quad n\ge n_0. \tag{3.9}$$

Obviously, from (3.8) and (3.9), we can lead a contradiction. Therefore, every solution $f\not\equiv 0$ of the equation (1.1) is of infinite order.

Proof of Theorem 1.3 Suppose that $f\not\equiv 0$ is a solution of the equation (1.1) with $\rho(f)<+\infty$. Let a be a finite deficient value of $A(z)$ with deficiency $\delta=\delta(a,A)$. From Lemma 2.1, there exists a set $E_2\subset[0,+\infty)$ with $m_l(E_2)<+\infty$ such that the inequality (3.2) holds for all z with $|z|=r\notin(E_2\bigcup[0,r_2])(r_2>1)$. We only need to use $E^*=E\setminus(E_2\bigcup[0,r_2]\bigcup G)$ to instead of E of Lemma 2.5. There still exists a sequence $\{R_n\}$ satisfying (3.3) and for all $k>0$,

$$\lim_{n\to\infty}\frac{|B(R_ne^{i\theta})|}{R_n^k}=\infty. \tag{3.10}$$

From (3.5) and (3.10) we can obtain a contradiction. Therefore, every solution $f\not\equiv 0$ of the equation (1.1) is of infinite order.

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