POINCARÉ-TYPE INEQUALITY FOR VARIABLE EXPONENT SPACES OF DIFFERENTIAL FORMS

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ABSTRACT
We prove both local and global Poincaré inequalities with the variable exponent for differential forms in the John domains and L^s -averaging domains, which can be considered as generalizations of the existing versions of Poincaré inequalities.

Keywords: *Poincaré inequality, Space of differential forms, John domain, Averaging domains*

1. INTRODUCTION

The Poincaré inequalities have been playing an important role in analysis and related fields during the past several decades. The study and applications of Poincaré inequalities are now ubiquitous in different areas, including PDEs and potential analysis. Some versions of the Poincaré inequality with different conditions for various families of functions or differential forms have been developed in recent years. For example, in 1989, Susan G. Staples in [1] proved the following Poincaré inequality for Sobolev functions in L^s -averaging domains. If D is an L^p -averaging domain, $p \geq n$, then there exists a constant C , such that

$$\left(\frac{1}{m(D)}\int_D |u-u_D|^p\,dx\right)^p \leq C(m(D))^{1/n}\left(\frac{1}{m(D)}\int_D |u|^p\,dx\right)^{1/p}$$

for each Sobolev function u defined in D , where the integral is the Lebesgue integral, and $m(D)$ is the Lebesgue measure of D as appeared in [1], see [2-10] for more results of the Poincaré inequality. In this paper, we will establish the Poincaré inequalities with the variable exponent for differential forms in the John domains and L^s -averaging domains, so that many existing versions of the Poincaré inequality are special cases of our new results.

In [11,12], spaces of differential forms are discussed in great details. Meanwhile in [13] $L^{p(x)}$ and $W^{k,p(x)}$ spaces are discussed and used to study the solutions of nonlinear Dirichlet boundary value problems. In this paper, we introduce $L^{p(x)}(\Omega,\Lambda^l)$ and $W^{1,p(x)}(\Omega,\Lambda^l)$ spaces. We also discuss the Poincaré inequalities with the variable exponent for differential forms in the John domains and L^s -averaging domains.

2. PRELIMINARIES

Let $e_1, e_2, ..., e_n$ denote the standard orthogonal basis of R^n. The space of all l -forms in R^n is denoted by $\Lambda^l(R^n)$. The dual basis to $e_1, e_2, ..., e_n$ is denoted by $e^1, e^2, \cdots, e^n$ and referred to as the standard basis for 1-form $\Lambda^1(R^n)$. The Grassman algebra $\Lambda(R^n) = \oplus\Lambda^l(R^n)$ is a graded algebra with respect to the exterior products. The standard ordered basis for $\Lambda(R^n)$ consists of the forms

$$1, e^1, e^2, \cdots, e^n, e^1 \wedge e^2, \cdots, e^{n-1} \wedge e^n, \cdots, e^1 \wedge e^2 \cdots \wedge e^n.$$

For $\alpha(x) = \sum_I \alpha_I(x) e^I \in \Lambda^l(R^n)$ and $\beta(x) = \sum \beta_I(x) e^I \in \Lambda^l(R^n)$, its inner product is obtained by $<\alpha,\beta> = \sum \alpha_I(x)\beta_I(x)$ with summation over all l –tuples $I = (i_1, \cdots i_l)$ and all integers $l = 0, 1, \cdots, n$. The Hodge star operator (see [14]) $\star : \Lambda(R^n) \to \Lambda(R^n)$ defined by the formulas:

$$\star 1 = e^1 \wedge e^2 \cdots \wedge e^n, \quad \alpha \wedge \star\beta = \beta \wedge \star\alpha = <\alpha,\beta> e^1 \wedge e^2 \cdots \wedge e^n.$$

Hence norm of α is given by the formula $|\alpha|^2 = <\alpha, \alpha> = \star(\alpha \wedge \star\alpha) = \sum \alpha_I(x)\alpha_I(x) \in \Lambda^0(R^n) = R^n$. Notice, the Hodge star operator is an isometric isomorphism on $\Lambda(R^n)$. Moreover

$$\star : \Lambda^l(R^n) \to \Lambda^{n-l}(R^n), \quad \star\star = (-\mathrm{I})^{l(n-l)} : \Lambda^l(R^n) \to \Lambda^l(R^n)$$

where I denotes the identity map.

Let $\Omega \subset R^n$ be a bounded domain. The coordinate function $x_1, x_2, \cdots, x_n$ in $\Omega \subset R^n$ are considered to be differential forms of degree 0. The 1-forms $dx_1, dx_2, ..., dx_n$ are constant function from Ω into $\Lambda^l(R^n)$. The value of dx_i is simply $e^i, i = 1, 2, \cdots, n$. Therefore every l-form $\alpha : \Omega \to \Lambda^l(R^n)$ may be written uniquely as

$$\alpha(x) = \sum \alpha_I(x)dx_I = \sum_{1 \le i_1 < \cdots < i_l \le n} \alpha_{i_1,\cdots i_l}(x)dx_{i_1} \wedge \cdots \wedge dx_{i_l}$$

where the coefficients $\alpha_{i_1,\cdots i_l}(x)$ are distributions from $D'(\Omega)$, dual to the space of smooth functions with compact support on Ω.

We use $D'(\Omega, \Lambda^l)$ to denote the space of all differential l-forms. For each form $\alpha(x) \in D'(\Omega, \Lambda^l)$ the exterior differential $d : D'(\Omega, \Lambda^l) \to D'(\Omega, \Lambda^{l+1})$ is expressed by

$$d\alpha(x) = \sum_{k=1}^{n} \sum_{1 \le i_1 < \cdots < i_l \le n} \frac{\partial \alpha_{i_1,\cdots i_l}(x)}{\partial x_k} dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_l}.$$

The formal adjoint operator, called the Hodge codifferential, is given by

$$d^\star = (-I)^{nl-1} \star d \star : D'(\Omega, \Lambda^{l+1}) \to D'(\Omega, \Lambda^l).$$

We by $C^\infty(\Omega, \Lambda^l)$ denote the space of infinitely differentiable l-forms on Ω and by $C_0^\infty(\Omega, \Lambda^l)$ denote the space $C^\infty(\Omega, \Lambda^l)$ with compact support on Ω.

The operator K_y with the case $y = 0$ was first introduced by Cartan in [15]. Then, it was extended to the following version in [12]. To each $y \in \Omega$, there corresponds a linear operator $K_y : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ defined by

$$K_y(u)(x; \xi_1, \xi_2, \cdots, \xi_{l-1}) = \int_0^1 t^{l-1}(tx + y - ty; x - y, \xi_1, \xi_2, \cdots, \xi_{l-1})\mathrm{d}t.$$

A homotopy operator $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ is defined by averaging K_y over all points $y \in \Omega$

$$Tu = \int_\Omega \phi(y) K_y(u)\mathrm{d}y$$

where $\phi \in C_0^\infty(\Omega)$ is normalized so that $\int_\Omega \phi(y)\mathrm{d}y = 1$, and we then obtain a pointwise estimate

$$|Tu(x)| \le 2^n \mu(\Omega) \int_\Omega \frac{|u(y)|}{|x - y|^{n-1}} \mathrm{d}y,$$

and the decomposition $u = d(Tu) + T(du)$. and split $\frac{\partial}{\partial x_i}(Tu) = A_i u + S_i u$, Here

$$|A_i u(x)| \le \frac{2^n \mu(\Omega)}{\mathrm{diam}(\Omega)} \int_\Omega \frac{|u(z)|}{|x - z|^{n-1}} \mathrm{d}z$$

$$S_i u(x, \xi) = \int_\Omega u(z, K_i(z, x - z), \xi)\mathrm{d}z$$

where $\xi = (\xi_1, \xi_2, \cdots, \xi_{l-1})$. Here, first notice that for each $z \in \Omega$ and $h \in R^n - \{0\}$,

(i) $K_i(z, h) \le \mu(\Omega)|h|^{-n}$,

(ii) $K_i(z, sh) = s^{-n} K_i(z, h), s > 0$,

(iii) $\int_{|h|=1} K_i(z, h) = 0$

for all $z \in \Omega$. We are now in a position to use the Calderdn-Zygmund theory of singular integrals. According to [16] we have

$$\| S_i u \|_{L^{p(x)}(\Omega)} \le C(n, p)\mu(\Omega) \| u \|_{L^{p(x)}(\Omega)}, 1 < p < \infty$$

where $C(n, p)$ does not depend on Ω.

In [12], we then obtain a pointwise estimate

$$| Tu(x) |= 2^n \mu(\Omega) \int_F \frac{| u(y) |}{| x - y |^{n-1}} \mathrm{d}y$$

for $x \in F \subset \Omega$, $\operatorname{supp}\phi \subset F$, where

$$\mu(\Omega) = \frac{(\operatorname{diam}\Omega)^{n+1}}{\int_\Omega \operatorname{dist}(y, \partial\Omega)\mathrm{d}y}.$$

If Ω is a ball in R^n, then

$$\mu(\Omega) = \frac{2^{n+1} n(n+1)}{\sigma_{n-1}}$$

where σ_{n-1} denotes the surface area of the unit sphere in R^n.

We define next the following classes of differential forms with L^p-integrable coefficients.
$L^p(\Omega, \Lambda^l)$ is the space of differential l-forms with coefficients in $L^p(\Omega)$, $1 \le p \le \infty$. The norms are given by

$$\| \alpha(x) \|_{L^p(\Omega,\Lambda^l)} = (\int_\Omega | \alpha(x) |^p \, dx)^{\frac{1}{p}}, \quad 1 \le p < \infty;$$

$$\| \alpha(x) \|_{L^\infty(\Omega,\Lambda^l)} = \operatorname*{ess\,sup}_{x\in\Omega} | \alpha(x) | .$$

$W^{1,p}(\Omega, \Lambda^l)$ is the space of differential l-forms $\omega \in L^p(\Omega, \Lambda^l)$ such that $d\omega \in L^p(\Omega, \Lambda^{l+1})$, $l = 0, 1, \cdots, n-1$. For $W^{1,p}(\Omega, \Lambda^l)$ the norm is

$$\| \alpha(x) \|_{W^{1,p}(\Omega,\Lambda^l)} = \| \alpha(x) \|_{L^p(\Omega,\Lambda^l)} + \| d\alpha(x) \|_{L^p(\Omega,\Lambda^{l+1})} .$$

We define the l-form $u_\Omega \in D'(\Omega, \Lambda^l)$ by

$$u_\Omega = \begin{cases} | \Omega |^{-1} \int_\Omega u(y)\mathrm{d}y, & \text{for } l = 0 \\ d(Tu), & \text{for } l = 1, 2, \cdots, n \end{cases}$$

for $u \in L^p(\Omega, \wedge^l), (1 \le p \le \infty)$. Clearly u_Ω is a closed form and for $l > 0$, u_Ω is an exact form.
Next we recall some basic properties of variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ and variable exponent Sobolev spaces $W^{k,p(x)}(\Omega)$, where $\Omega \subset R^n$ is a bounded domain.

Let $P(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to [1, \infty]$. For $p \in P(\Omega)$ we put $\Omega_1 = \{x \in \Omega : p(x) = 1\}$, $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$, $\Omega_0 = \Omega \setminus (\Omega_1 \cup \Omega_\infty)$, $p_* = essinf_{\Omega_0} p(x)$ and $p^* = esssup_{\Omega_0} p(x)$.
Under the condition (1.3) the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions u such that $\int_{\Omega\setminus\Omega_\infty} | \lambda u(x) |^{p(x)} \, dx + esssup_{\Omega_\infty} | \lambda u(x) | < \infty$ for some $\lambda = \lambda(u) > 0$ and $L^{p(x)}(\Omega)$ is a reflexive Banach space equipped with the following norm

$$\| u \|_{L^{p(x)}(\Omega)} = \inf \{\lambda > 0 : \int_\Omega | \frac{u}{\lambda} |^{p(x)} \, dx + \operatorname{esssup}_{\Omega_\infty} | \frac{u}{\lambda} | \le 1\}.$$

The variable exponent Sobolev space $W^{k,p(x)}(\Omega)$, is the class of all functions $u \in L^{p(x)}(\Omega)$ such that $\delta_k u = \{D^{\alpha}u : |\alpha| \leq k\} \subset L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, is a reflexive Banach space equipped with the following norm

$$\| u \|_{W^{k,p(x)}(\Omega)} = \sum_{|\alpha|\leq k} \| D^{\alpha} u \|_{L^{p(x)}(\Omega)} .$$

For a differential l-form $f(x)$ on Ω we define the functional $\rho_{p(x)}$ by

$$\rho_{p(x),\Lambda^l}(f) = \int_{\Omega \setminus \Omega_\infty} | f(x) |^{p(x)} \, dx + \operatorname{esssup}_{\Omega_\infty} | f(x) | .$$

Definition 2.1. Variable exponent Lebesgue spaces of differential l-forms $L^{p(x)}(\Omega, \Lambda^l)$ is the set of differential l-forms f such that $\rho_{p(x),\Lambda^l}(\lambda f) < \infty$ for some $\lambda = \lambda(f) > 0$ and we endow it with the following norm

$$\| f \|_{L^{p(x)}(\Omega,\Lambda^l)} = \inf\{\lambda > 0 : \rho_{p(x),\Lambda^l}(\frac{f}{\lambda}) \leq 1\}.$$

Definition 2.2. Variable exponent Sobolev spaces of differential l-forms $W^{1,p(x)}(\Omega, \Lambda^l)$ is the space of differential l-forms $f \in L^{p(x)}(\Omega, \Lambda^l)$ such that $df \in L^{p(x)}(\Omega, \Lambda^{l+1})$ with $l = 0, 1, \cdots, n-1$. For $W^{1,p(x)}(\Omega, \Lambda^l)$ the norm is defined as

$$\| f \|_{W^{1,p(x)}(\Omega,\Lambda^l)} = \| f \|_{W^{1,p(x)}(\Omega,\Lambda^l)} + \| df \|_{W^{1,p(x)}(\Omega,\Lambda^{l+1})} .$$

We first introduce the following limmas that will be used to prove the Poincaré inequalities.

Lemma 2.1. (T. Iwaniec and A. Lutoborski [12]) Let $u \in D'(Q, \Lambda^l)$ and $df \in L^{p(x)}(\Omega, \Lambda^{l+1})$. Then $u - u_Q$ is in $L^{np/(n-p)}(Q, \Lambda^l)$ and

$$(\int_Q | u - u_Q |^{np/(n-p)} \, dx)^{(n-p)/np} \leq C(n, p)(\int_Q | du |^p \, dx)^{1/p},$$

for Q a cube or a ball in R^n, $l = 0, 1, \cdots, n-1$ and $1 < p < n$.

3. THE LOCAL POINCARÉ INEQUALITIES

Lemma 3.1. Let $u \in D'(Q, \Lambda^l)$ and $du \in L^p(Q, \Lambda^{l+1})$. Let $1 < p < n$ and $p < q < p^*$ be fixed exponents. Then

$$\| u - u_Q \|_{L^q(\Omega,\Lambda^l)} \leq C(n, p) \, | Q |^{\frac{1}{n}+\frac{1}{q}-\frac{1}{p}} \| du \|_{L^p(\Omega,\Lambda^{l+1})}$$

If $p \geq n$ and $q < \infty$ then

$$\| u - u_Q \|_{L^q(\Omega,\Lambda^l)} \leq C(n, q) \, | Q |^{\frac{1}{n}+\frac{1}{q}-\frac{1}{p}} \| du \|_{L^p(\Omega,\Lambda^{l+1})}$$

for Q a cube or a ball in R^n, $l = 0, 1, \cdots, n-1$ and p^* denoted the Sobolev conjugate of $p < n, p^* = np/(n-p)$.

Proof. The case $1 < p < n$ and $q = p^*$ is by Lemma 2.1.The case $q < p^*$ follows from this by standard arguments: we choose $s \in (1, n)$ such that $s^* = q$ (or $s = p$ if $q < 1^*$). By Hölder's inequality and Lemma 2.1 we obtain

$$\begin{aligned}(\int_Q | u - u_Q |^q \, dx)^{1/q} &\leq | Q |^{\frac{1}{q}-\frac{1}{s^*}} (\int_Q | u - u_Q |^{s^*} \, dx)^{1/s^*} \\ &\leq C(n,p)|Q|^{\frac{1}{q}-\frac{1}{s^*}}(\int_Q |du|^s dx)^{1/s} \\ &\leq C(n,p)|Q|^{\frac{1}{q}-\frac{1}{s^*}}|Q|^{\frac{1}{s}-\frac{1}{p}}(\int_Q |du|^p dx)^{1/p} \\ &\leq C(n,p)|Q|^{\frac{1}{n}+\frac{1}{q}-\frac{1}{p}}(\int_Q |du|^p dx)^{1/p},\end{aligned}$$

which is clearly equivalent to the inequalities in the theorem.

Theorme 3.1. Let $u \in D'(Q, \Lambda^l)$ and $du \in L^p(Q, \Lambda^{l+1})$. If $p_Q^+ \le (p_Q^-)^*$ or $p_Q^- \ge n$ and $p_Q^+ < \infty$. Then $u - u_Q$ is in $W^{1,p(x)}(\Omega, \Lambda^l)$ and there exists a constant $C = C(n, p)$ such that

$$\| u - u_Q \|_{L^{p(x)}(\Omega,\Lambda^l)} \le C(n, p)(1+ | Q |)^2 | Q |^{\frac{1}{n}+\frac{1}{p_Q^+}-\frac{1}{p_Q^-}} \| du \|_{L^{p(x)}(\Omega,\Lambda^{l+1})}$$

for Q a cube or a ball in R^n, $l = 0, 1, \cdots, n-1$.

Proof. Assume first that $p_Q^+ \le (p_Q^-)^*$. Since $p(x) \le p_Q^+ \le (p_Q^-)^*$ we obtain by [13, Theorem 2.8] and Lemma 2.1 that

$$\begin{aligned} \| u - u_Q \|_{L^{p(x)}(\Omega,\Lambda^l)} &\le (1+ | Q |) \| u - u_{Q_0} \|_{L^{p_Q^+}(\Omega,\Lambda^l)} \\ &\le C(n,p)(1+|Q|)|Q|^{\frac{1}{n}+\frac{1}{p_Q^+}-\frac{1}{p_Q^-}} \|du\|_{L^{p_Q^-}(\Omega,\Lambda^{l+1})} \\ &\le C(n,p)(1+|Q|)^2|Q|^{\frac{1}{n}+\frac{1}{p_Q^+}-\frac{1}{p_Q^-}} \|du\|_{L^{p(x)}(\Omega,\Lambda^{l+1})} \end{aligned}$$

The case $p_Q^- \ge n$ and $p_Q^+ < \infty$ is similar.

4. THE GLOBAL RESULTS IN δ -JOHN DOMAIN

Definition 4.1. A proper subdomain Ω is called a δ-John domain, $\delta > 0$, if there exists a point $x_0 \in \Omega$ which can be joined with any other point $x \in \Omega$ by a continuous curve $\gamma \subset \Omega$ so that

$$d(\xi, \partial\Omega) \ge \delta \, | x - \xi |$$

for each $\xi \in \gamma$. Here, $d(\xi, \partial\Omega)$ is the Euclidean distance between ξ and $\partial\Omega$.

We will need the following lemma 4.1 appeared in [17]

Lemma 4.1. Each Ω has a modified Whitney cover of cubes $F = \{Q_j\}$ which satisfy

$$\bigcup_i Q_i = \Omega, \quad \sum_{Q \in W} X_{\sqrt{\frac{5}{4}}Q} \le N X_\Omega$$

for all $x \in R^n$ and some $N > 1$ and if $Q_i \cap Q_j \ne \varnothing$, then there exists a cube $R (\notin F)$ in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover if Ω is δ-John domain, then there is a distinguished cube $Q_0 \in F$ which can be connected with every cube $Q \in F$ by a chain of cubes $Q_0, Q_1, ..., Q_k = Q$ from F and such that $Q \subset \rho Q_i, i = 0, 1, 2, ..., k$, for some $\rho = \rho(n, \delta)$.
Bounded quasiballs and bounded uniform domains are John domains. See [18] and [19]. In such domains we have the following global result.

Lemma 4.2. Let $\Omega \in R^n$ be a bounded and convex John domain. Let $u \in D'(\Omega, \Lambda^l)$ and $du \in L^p(\Omega, \Lambda^{l+1})$. Let $1 < p < n$ and $p < q < p^*$ be fixed exponents. Then

$$\| u - u_{Q_0} \|_{L^q(\Omega,\Lambda^l)} \le C(n, p) \, | \Omega |^{\frac{1}{n}+\frac{1}{q}-\frac{1}{p}} \| du \|_{L^p(\Omega,\Lambda^{l+1})}$$

If $p \ge n$ and $q < \infty$ then

$$\| u - u_{Q_0} \|_{L^q(\Omega,\Lambda^l)} \le C(n, q) \, | \Omega |^{\frac{1}{n}+\frac{1}{q}-\frac{1}{p}} \| du \|_{L^p(\Omega,\Lambda^{l+1})}$$

for $l = 0, 1, \cdots, n-1$ and p^* denoted the Sobolev conjugate of $p < n, p^* = np / (n-p)$, and the constant $N > 1$ appeared in Lemma 4.1.

Proof. First, we use Lemma 4.1 for the bounded and convex John domain Ω. There is a modified Whitney cover of cubes $F = \{Q_j\}$ for Ω such that

$$\bigcup_i Q_i = \Omega \ , \quad \sum_{Q\in W} X_{\sqrt{\frac{5}{4}}Q} \le NX_\Omega$$

for all $x \in R^n$ and some $N > 1$ and if $Q_i \cap Q_j \ne \varnothing$, then there exists a cube $R(\notin F)$ in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, there is a distinguished cube $Q_0 \in F$ which can be connected with every cube $Q \in F$ by a chain of cubes $Q_0, Q_1, ..., Q_k = Q$ from F and such that $Q \subset \rho Q_i, i = 0,1,2,...,k$, for some $\rho = \rho(n,\delta)$. Then by the elementary inequality $(a+b)^s \le 2^s(|a|^s + |b|^s), s \ge 0$, we have

$$\begin{aligned}
&(\textstyle\int_\Omega | u - u_{Q_0} |^q \,dx)^{1/q} \\
&=(\textstyle\int_{\cup Q_i} |u-u_{Q_0}|^q dx)^{1/q} \\
&\le(\sum_{Q_i\in F} 2^q \textstyle\int_{Q_i} |u-u_{Q_i}|^q dx + 2^q \int_{Q_i} |u_{Q_0}-u_{Q_i}|^q dx)^{1/q} \qquad (4.1)\\
&\le C(n,p)((\sum_{Q_i\in F} \textstyle\int_{Q_i} |u-u_{Q_i}|^q dx)^{1/q} + (\sum_{Q_i\in F} \int_{Q_i} |u_{Q_0}-u_{Q_i}|^q dx)^{1/q})
\end{aligned}$$

The first sum in above formula can be estimated by using Lemma 2.1

$$\begin{aligned}
\sum_{Q_i\in F} \textstyle\int_{Q_i} | u - u_{Q_i} |^q \,dx &\le C(p,q,n) \sum_{Q_i\in F} | Q_i |^{(\frac{1}{n}+\frac{1}{q}-\frac{1}{p})q} (\textstyle\int_{Q_i} | du |^p \,dx)^{\frac{q}{p}} \\
&\le C(p,q,n)|\Omega|^{(\frac{1}{n}+\frac{1}{q}-\frac{1}{p})q} \sum_{Q_i\in F} (\textstyle\int_\Omega |du|^p \chi_{Q_i} dx)^{\frac{q}{p}} \\
&\le C(p,q,n,N)|\Omega|^{(\frac{1}{n}+\frac{1}{q}-\frac{1}{p})q} (\textstyle\int_\Omega |du|^p dx)^{\frac{q}{p}}
\end{aligned}$$

So that

$$(\sum_{Q_i\in F} \textstyle\int_{Q_i} | u - u_{Q_i} |^q \,dx)^{1/q} \le C(p,q,n,N) \,|\, \Omega \,|^{\frac{1}{n}+\frac{1}{q}-\frac{1}{p}} (\textstyle\int_\Omega | du |^p \,dx)^{\frac{1}{p}} \qquad (4.2)$$

To estimate the second sum in above formula, we need the property of John domain. Fix a cube $Q_i \in F$ and let $Q_0 = Q_{i_0}, Q_{i_1}, ..., Q_{i_{k-1}} = Q$ be the chain in Lamma 2.2. Then we have

$$| u_{Q_0} - u_{Q_i} | \le \sum_{j=0}^{k-1} | u_{Q_{i_j}} - u_{Q_{i_{j+1}}} | \qquad (4.3)$$

The chain Q_{i_j} also has property that, for each j, $j = 0,1,\cdots,k-1$, $Q_{i_j} \cap Q_{i_{j+1}} \ne \Phi$. Thus, there exists a cube Ω_j such that $\Omega_j \subset Q_{i_j} \cap Q_{i_{j+1}}$ and $Q_{i_j} \cup Q_{i_{j+1}} \subset N\Omega_j, N > 1$. so,

$$\frac{\max\{| Q_{i_j} |, | Q_{i_{j+1}} |\}}{Q_{i_j} \cap Q_{i_{j+1}}} \le \frac{\max\{| Q_{i_j} |, | Q_{i_{j+1}} |\}}{| \Omega_j |} \le C(N) \qquad (4.4)$$

By (4.4) and Lemma 2.2, we have

$$\begin{aligned}
| u_{Q_{i_j}} - u_{Q_{i_{j+1}}} |^q &= \frac{1}{| Q_{i_j} \cap Q_{i_{j+1}} |} \textstyle\int_{Q_{i_j} \cap Q_{i_{j+1}}} | u_{Q_{i_j}} - u_{Q_{i_{j+1}}} |^q \,dx \\
&\le \frac{C(n)}{\max\{| Q_{i_j} |, | Q_{i_{j+1}} |\}} \textstyle\int_{Q_{i_j} \cap Q_{i_{j+1}}} | u_{Q_{i_j}} - u_{Q_{i_{j+1}}} |^q \,dx \\
&\le C(n) \sum_{k=j}^{j+1} \frac{1}{| Q_{i_k} |} \textstyle\int_{Q_{i_k}} | u - u_{Q_{i_k}} |^q \,dx \\
&\le C(n) \sum_{k=j}^{j+1} \frac{| Q_{i_k} |^{(\frac{1}{n}+\frac{1}{q}-\frac{1}{p})q}}{| Q_{i_k} |} (\textstyle\int_{Q_{i_k}} | du |^p \,dx)^{q/p}
\end{aligned}$$

Then by (4.3) and the elementary inequality $|\sum_{i=1}^{M} t_i|^s \le M^s \sum_{i=1}^{M} |t_i|$, we finally obatain

$$\sum_{Q_i \in F} \int_{Q_i} |u_{Q_0} - u_{Q_i}|^q \, dx \le C(p,q,n) \sum_{Q_i \in F} \int_{Q_i} \sum_{j=0}^{k-1} \sum_{k=j}^{j+1} |Q_{i_k}|^{(\frac{1}{n}-\frac{1}{p})q} (\int_\Omega |du|^p \, dx)^{q/p}$$

$$\le C(p,q,n) \sum_{Q_i \in F} \int_{Q_i} |Q_i|^{(\frac{1}{n}-\frac{1}{p})q} (\int_\Omega |du|^p \, dx)^{q/p}$$

$$\le C(p,q,n) \sum_{Q_i \in F} |Q_i|^{1+(\frac{1}{n}-\frac{1}{p})q} (\int_\Omega |du|^p \, dx)^{q/p}$$

$$\le C(p,q,n) |\Omega|^{(\frac{1}{n}+\frac{1}{q}-\frac{1}{p})q} (\int_\Omega |du|^p \, dx)^{q/p}$$

We have completed the proof of Lemma 4.2.

Theorme 4.1. Let $\Omega \in R^n$ be a bounded John domain. Let $u \in D'(\Omega, \Lambda^l)$ and $du \in L^{p(x)}(\Omega, \Lambda^{l+1})$. If $p_\Omega^+ \le (p_\Omega^-)^*$ or $p_\Omega^- \ge n$ and $p_\Omega^+ < \infty$. Then $u - u_Q$ is in $W^{1,p(x)}(Q, \Lambda^l)$ and there exists a constant $C = C(n,p)$ such that

$$\| u - u_{Q_0} \|_{L^{p(x)}(\Omega, \Lambda^l)} \le C(n,p)(1+|\Omega|)^2 |\Omega|^{\frac{1}{n}+\frac{1}{p_\Omega^+}-\frac{1}{p_\Omega^-}} \| du \|_{L^{p(x)}(\Omega), \Lambda^{l+1}}$$

for $l = 0, 1, \cdots, n-1$.

Proof. Assume first that $p_\Omega^+ \le (p_\Omega^-)^*$. Since $p(x) \le p_\Omega^+ \le (p_\Omega^-)^*$ we obtain by [13, Theorem 2.8] and Lemma 2.1 that

$$\| u - u_{Q_0} \|_{L^{p(x)}(\Omega, \Lambda^l)} \le (1+|\Omega|) \| u - u_{Q_0} \|_{L^{p_\Omega^+}(\Omega, \Lambda^l)}$$

$$\le C(n,p)(1+|\Omega|)|\Omega|^{\frac{1}{n}+\frac{1}{p_\Omega^+}-\frac{1}{p_\Omega^-}} \|du\|_{L^{p_\Omega^-}(\Omega, \Lambda^{l+1})}$$

$$\le C(n,p)(1+|\Omega|)^2 |\Omega|^{\frac{1}{n}+\frac{1}{p_\Omega^+}-\frac{1}{p_\Omega^-}} \|du\|_{L^{p(x)}(\Omega, \Lambda^{l+1})}$$

The case $p_\Omega^- \ge n$ and $p_\Omega^+ < \infty$ is similar.

5. THE GLOBAL RESULTS IN L^s-AVERAGING DOMAINS

Definition 5.1. (S.G. Staples [1]) A proper subdomain Ω is called an L^s-averaging domain, $s \ge 1$, if there exists a constant C such that

$$\left(\frac{1}{|\Omega|} \int_\Omega |u - u_\Omega|^s \, dx\right)^{1/s} \le C \sup_{B \subset \Omega} \left(\frac{1}{|B|} \int_B |u - u_B|^s \, dx\right)^{1/s}$$

for all $u \in L^s_{loc}(\Omega)$. Here, $|\Omega|$ is the n-dimensional Lebesgue measure of Ω and supremum is over all ball $B \subset \Omega$. In such domains we have the following global result.

Lemma 5.1. Let $\Omega \in R^n$ be an L^q-averaging domain, $q > 1$. Let $u \in D'(\Omega, \Lambda^l)$ and $du \in L^p(\Omega, \Lambda^{l+1})$. Then

$$\| u - u_\Omega \|_{L^q(\Omega, \Lambda^l)} \le C(n,q) |\Omega|^{\frac{1}{n}+\frac{1}{q}-\frac{1}{p}} \| du \|_{L^p(\Omega, \Lambda^{l+1})}$$

for $l = 0, 1, \cdots, n-1$ and $p \ge n$.

Proof.

$$\frac{1}{|\Omega|} \int_\Omega |u - u_\Omega|^q \, dx \le \sup_{B \subset \Omega} \frac{1}{|B|} \int_B |u - u_B|^q \, dx$$

$$\le \sup_{B \subset \Omega} \frac{1}{|B|} |B|^{(\frac{1}{n}+\frac{1}{q}-\frac{1}{p})q} (\int_\Omega |du|^p \, dx)^{q/p}$$

$$\le \sup_{B \subset \Omega} |B|^{(\frac{1}{n}-\frac{1}{p})q} (\int_\Omega |du|^p \, dx)^{q/p}$$

So that

$$\| u - u_{\Omega} \|_{L^q(\Omega,\Lambda^l)} \leq C(n,p)\,|\,\Omega\,|^{\frac{1}{n}+\frac{1}{q}-\frac{1}{p}} \| du \|_{L^p(\Omega,\Lambda^{l+1})} .$$

Theorem 5.1. Let $\Omega \in R^n$ be an L^s-averaging domain. Let $u \in D'(\Omega, \Lambda^l)$ and $du \in L^{p(x)}(\Omega, \Lambda^{l+1})$. If $n \leq p(x) \leq s$, then $u - u_Q$ is in $W^{1,p(x)}(Q, \Lambda^l)$ and there exists a constant $C = C(n, p)$ such that

$$\| u - u_{\Omega} \|_{L^{p(x)}(\Omega,\Lambda^l)} \leq C(n,p)(1+|\,\Omega\,|)^2\,|\,\Omega\,|^{\frac{1}{n}+\frac{1}{p_\Omega^+}-\frac{1}{p_\Omega^-}} \| du \|_{L^{p(x)}(\Omega,\Lambda^{l+1})}$$

for $l = 0, 1, \cdots, n-1$.

Proof. Since $n \leq p_\Omega^- \leq p(x) \leq p_\Omega^+ \leq s$ we obtain by [13, Theorem 2.8], Lemma 2.1 that

$$\begin{aligned}
\| u - u_{\Omega} \|_{L^{p(x)}(\Omega,\Lambda^l)} &\leq (1+|\,\Omega\,|)\, \| u - u_{\Omega} \|_{L^{p_\Omega^+}(\Omega,\Lambda^l)} \\
&\leq C(n,p)(1+|\Omega|)|\Omega|^{\frac{1}{p_\Omega^+}-\frac{1}{s}} \|u-u_\Omega\|_{L^s(\Omega,\Lambda^l)} \\
&\leq C(n,p)(1+|\Omega|)|\Omega|^{\frac{1}{n}+\frac{1}{p_\Omega^+}-\frac{1}{p_\Omega^-}} \|du\|_{L^{p_\Omega^-}(\Omega,\Lambda^{l+1})} \\
&\leq C(n,p)(1+|\Omega|)^2|\Omega|^{\frac{1}{n}+\frac{1}{p_\Omega^+}-\frac{1}{p_\Omega^-}} \|du\|_{L^{p(x)}(\Omega,\Lambda^{l+1})}
\end{aligned}$$

We know the any δ-John domain is an L^s-domain (See [1]), $s > 1$. Hence, Lemma 5.1 and Theorem 5.1 holds if Ω is a δ-John domain.

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