2-WEAK AMENABILITY OF THE SECOND DUAL OF A BANACH ALGEBRA

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ABSTRACT

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{A}''$ be its second dual equipped with the first Arens product and let $D:\mathcal{A}\to\mathcal{A}''$ be a continuous derivation. In this paper we show that the second adjoint of D is also a derivation if we consider the fourth dual of $\mathcal{A}$ as a Banach $\mathcal{A}''$-bimodule with the natural module structures.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. A bounded linear mapping $D:\mathcal{A}\to X$ is said to be a derivation if

$$D(ab) = a\cdot D(b) + D(a)\cdot b \quad (a,b\in\mathcal{A}).$$

A derivation $D:\mathcal{A}\to X$ is said to be inner if there exists $x\in X$ such that $D(a) = a\cdot x - x\cdot a$ for each $a\in\mathcal{A}$.

A Banach algebra $\mathcal{A}$ is called weakly amenable if every derivation from $\mathcal{A}$ into the dual Banach module $\mathcal{A}'$ is inner, and 2-weakly amenable if every derivation from $\mathcal{A}$ into the second dual module $\mathcal{A}''$ is inner [4]. For example, each C^*-algebra is both weak and 2-weakly amenable and the group algebra $L^1(G)$ is always weakly amenable for locally compact group G [12].

As known, neither the weak amenability of $\mathcal{A}$ implies that of $\mathcal{A}''$, nor 2-weak amenability of $\mathcal{A}$ implies that of $\mathcal{A}''$; for example $\mathcal{A} = L^1(\mathbb{R})$ is both weak and 2-weakly amenable but $\mathcal{A}''$ neither weakly amenable nor 2-weakly amenable. This subject that whether weak amenability passes from $\mathcal{A}''$ to $\mathcal{A}$ was introduced in [6], [7] and [9]. Also for dual Banach algebras, 2-weak amenability of $\mathcal{A}''$ implies that of $\mathcal{A}$ [10].

In this parer we prove that 2-weak amenability of $\mathcal{A}''$ implies that of $\mathcal{A}$ for arbitrary Banach algebra, and under special hypotheses weak amenability of $\mathcal{A}$ is inherited from weak amenability of $\mathcal{A}''$.

The second dual space $\mathcal{A}''$ of a Banach algebra $\mathcal{A}$ admits two Banach algebra products known as first and second Arens products, each extending the product on $\mathcal{A}$. These products which we denote by $\square$ and $\blacklozenge$, respectively, can be defined as follows

$$\Phi\square\Psi = w^*-\lim_i\lim_j a_ib_j,\ \ \Phi\blacklozenge\Psi = w^*-\lim_j\lim_i a_ib_j,$$

where (a_i) and (b_j) are nets in $\mathcal{A}$ that are w^*-convergent to Φ and Ψ, respectively. The Banach algebra $\mathcal{A}$ is said to be Arens regular if $\Phi\square\Psi = \Phi\blacklozenge\Psi$ on the whole of $\mathcal{A}''$ [5]. For more information on Arens product and topological centres we refer the reader to [3] and [5].

Throughout the paper we identify an element of a Banach space X with its canonical image in X''. Also for Banach algebra $\mathcal{A}$ we have

$$a\cdot\Phi = a\square\Phi = a\blacklozenge\Phi,\ \ \Phi\cdot a = \Phi\square a = \Phi\blacklozenge a \quad (a\in\mathcal{A},\Phi\in\mathcal{A}'').$$

We also denote by $\boxdot$ and $\diamondsuit$, respectively, the first and second Arens products on $\mathcal{A}^{(4)}$ which are induced by the first Arens product $\square$ on $\mathcal{A}''$.

2. THE SECOND ADJOINT OF A DERIVATION

Suppose that X is a Banach $\mathcal{A}$-bimodule. Then X'' becomes a Banach $\mathcal{A}''$-bimodule where $\mathcal{A}''$ have the first Arens product. The module actions are defined by

$$\Phi \bullet \nu = w^* - \lim_i \lim_j \widehat{a_i \cdot x_j}, \quad \nu \bullet \Phi = w^* - \lim_j \lim_i \widehat{x_j \cdot a_i}, \tag{1}$$

where (a_i) and (x_j) are nets in $\mathcal{A}$ and X that converge, in w^*-topologies, to Φ and ν, respectively. For details see [4].

From now on, we consider $\mathcal{A}''$ with the first Arens product $\square$. Then $\mathcal{A}^{(4)}$ can be made into an $\mathcal{A}''$-bimodule by two natural fashion. In the first way, $\mathcal{A}^{(4)}$ is $\mathcal{A}''$-bimodule by module actions defined as above (for $X = \mathcal{A}''$). In the second way $\mathcal{A}^{(4)}$ is $\mathcal{A}''$-bimodule by the following module structures

$$\langle \Phi \cdot \alpha, \mu \rangle = \langle \alpha, \mu \cdot \Phi \rangle, \langle \alpha \cdot \Phi, \mu \rangle = \langle \alpha, \Phi \cdot \mu \rangle, \ (\Phi \in \mathcal{A}'', \alpha \in \mathcal{A}^{(4)}), \tag{2}$$

where $\mu \cdot \Phi$ and $\Phi \cdot \mu$ are defined as follows

$$\langle \mu \cdot \Phi, \Psi \rangle = \langle \mu, \Phi \square \Psi \rangle, \langle \Phi \cdot \mu, \Psi \rangle = \langle \mu, \Psi \square \Phi \rangle.$$

Now let $D: \mathcal{A} \to \mathcal{A}''$ be a continuous derivation. If $\mathcal{A}^{(4)}$ is an $\mathcal{A}''$-bimodule by module actions as in (1), then by proposition 1.7 of [4], the second adjoint of D is also a derivation. That is for all Φ and Ψ in $\mathcal{A}''$

$$D''(\Phi \square \Psi) = D''(\Phi) \bullet \Psi + \Phi \bullet D''(\Psi).$$

The next result shows that the second adjoint of D is also a derivation if we consider $\mathcal{A}''$-bimodule structures on $\mathcal{A}^{(4)}$ as in (2).

Proposition 2.1 *Let $\mathcal{A}$ be a Banach algebra and $D: \mathcal{A} \to \mathcal{A}''$ be a continuous derivation. Then D'' is also derivation.*

Proof. Let $(a_i), (b_j)$ be bounded nets in $\mathcal{A}$ that converge, respectively, to Φ and Ψ in the w^*-topology. Then for all $\mu \in \mathcal{A}'''$ we have

$$\langle D''(\Phi \square \Psi), \mu \rangle = \lim_i \lim_j \langle D(a_i b_j), \mu \rangle$$
$$= \lim_i \lim_j \langle D(a_i) \cdot b_j, \mu \rangle + \lim_i \lim_j \langle a_i \cdot D(b_j), \mu \rangle$$
$$= \langle D''(\Phi) \cdot \Psi, \mu \rangle + \langle \Phi \cdot D''(\Psi), \mu \rangle.$$

Therefore

$$D''(\Phi \square \Psi) = D''(\Phi) \cdot \Psi + \Phi \cdot D''(\Psi).$$

Corollary 2.2 *Let $\mathcal{A}$ be a Banach algebra. Then 2-weak amenability of $\mathcal{A}''$ implies that of $\mathcal{A}$.*

Proof. Let $D: \mathcal{A} \to \mathcal{A}''$ be a continuous derivation, by above proposition $D'': \mathcal{A}'' \to \mathcal{A}^{(4)}$ is also a derivation. Since $\mathcal{A}''$ is 2-weakly amenable, there exists $\alpha \in \mathcal{A}^{(4)}$ such that $D''(\Phi) = \Phi \cdot \alpha - \alpha \cdot \Phi$ for each $\Phi \in \mathcal{A}''$. Let $\Psi = k'(\alpha)$, where $k: \mathcal{A}' \to \mathcal{A}'''$ is the canonical map. Then for all $a \in \mathcal{A}$, $D(a) = a \cdot \Psi - \Psi \cdot a$ and so D is inner. Thus $\mathcal{A}$ is 2-weakly amenable.

Note that the two $\mathcal{A}''$-bimodule structures on $\mathcal{A}^{(4)}$ are not always equal. In the next theorem we give conditions under which two $\mathcal{A}''$-bimodule structures on $\mathcal{A}^{(4)}$ coincide.

Theorem 2.3 *Let $\mathcal{A}$ be an Arens regular Banach algebra. If $\mathcal{A}' \square \mathcal{A}'' \subseteq \mathcal{A}$, then two $\mathcal{A}''$-bimodule structures on $\mathcal{A}^{(4)}$ coincide.*

To prove this theorem, we need first the following lemma.

Lemma 2.4 *Let $\mathcal{A}$ be as in above theorem. Then*

(i) For each $\mu \in \mathcal{A}'''$ the map $\Phi \mapsto \Phi \cdot \mu$ is w^*-w^* continuous.

(ii) $\mathcal{A}''$ is Arens regular.

Proof. (i) Let $\Phi_\alpha \to \Phi$ in the w^*-topology. Since $\mathcal{A}''' = \mathcal{A}' \oplus \mathcal{A}^\perp$ $[4]$, where $\mathcal{A}^\perp$ is the annihilator of $\mathcal{A}$, for each $\mu \in \mathcal{A}'''$ there exists $f \in \mathcal{A}'$ and $\nu \in \mathcal{A}^\perp$ such that $\mu = \widehat{f} + \nu$. Then a direct verification shows that $\Phi_\alpha \cdot \mu \to \Phi \cdot \mu$ in the w^*-topology, so the map $\Phi \mapsto \Phi \cdot \mu$ is w^*-w^* continuous, as required.

(ii) Suppose that $\mathcal{A}$ is Arens regular. Then by corollary 3.16 of [5], $\mathcal{A}''$ is a dual Banach algebra with predual space $E = \mathcal{A}'$. Let $\beta \in E^\perp$, then by assumption we have

$$\langle \beta \cdot \mu, \Phi \rangle = \langle \beta, \mu \cdot \Phi \rangle = 0 \quad (\mu \in \mathcal{A}''', \Phi \in \mathcal{A}'').$$

It follows that $\beta \cdot \mu = 0$ and so $\alpha * \beta = 0 \quad (\alpha \in E^\perp)$. Similarly, $\alpha \blacklozenge \cdot \beta = 0$ for each $\alpha, \beta \in E^\perp$. Therefore $\mathcal{A}''$ is Arens regular by proposition 2.16 of [5].

Proof of Theorem 2.3. Let $\Phi \in \mathcal{A}''$, $\alpha \in \mathcal{A}^{(4)}$ and let $(a_i), (F_j)$ be bounded nets in $\mathcal{A}$ and $\mathcal{A}''$ that are w^*-convergent to Φ and α, respectively. By lemma 2.4, the map $\Phi \mapsto \Phi \cdot \mu$ is w^*-w^* continuous and so

$$\begin{aligned}
\langle \alpha \cdot \Phi, \mu \rangle &= \langle \alpha, \Phi \cdot \mu \rangle = \lim_j \langle \Phi \cdot \mu, F_j \rangle \\
&= \lim_j \lim_i \langle a_i \cdot \mu, F_j \rangle \\
&= \lim_j \lim_i \langle \mu, F_j \cdot a_i \rangle \\
&= \lim_j \lim_i \langle \widehat{F_j \cdot a_i}, \mu \rangle \\
&= \langle \alpha \bullet \Phi, \mu \rangle.
\end{aligned}$$

Therefore we conclude that $\alpha \cdot \Phi = \alpha \bullet \Phi$. Since $\mathcal{A}''$ is Arens regular we have

$$\lim_j \lim_i \widehat{a_i \cdot F_j} = \lim_i \lim_j \widehat{a_i \cdot F_j}$$

hence for each $\mu \in \mathcal{A}'''$

$$\begin{aligned}
\langle \Phi \cdot \alpha, \mu \rangle &= \langle \alpha, \mu \cdot \Phi \rangle = \lim_j \langle \mu \cdot \Phi, F_j \rangle \\
&= \lim_j \lim_i \langle \mu \cdot a_i, F_j \rangle \\
&= \lim_j \lim_i \langle \mu, a_i \cdot F_j \rangle \\
&= \lim_j \lim_i \langle \widehat{a_i \cdot F_j}, \mu \rangle \\
&= \lim_i \lim_j \langle \widehat{a_i \cdot F_j}, \mu \rangle \\
&= \langle \Phi \bullet \alpha, \mu \rangle.
\end{aligned}$$

This shows that $\Phi \cdot \alpha = \Phi \bullet \alpha$, as required.

Let $\mathcal{A} = l^1$, with pointwise product. Then $\mathcal{A}$ is an Arens regular Banach algebra which is not reflexive but satisfies $\mathcal{A}''W\mathcal{A}'' \subseteq \mathcal{A}$ [5]. Therefore by theorem 2.3 two $\mathcal{A}''$-bimodule structures on $\mathcal{A}^{(4)}$ coincide.

3. WEAK AND 2-WEAK AMENABILITY OF $\mathcal{A}''$

Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity(=BAI). We say that $\mathcal{A}'$ factors on the left (right)

if $A' = A' \cdot A$ ($A' = A \cdot A'$), and factors if both equalities $A \cdot A' = A' = A' \cdot A$ hold [11].
In this section we show that if A'' is 2-weakly amenable, then A' factors and then we give conditions under which weak amenability of A'' implies that of A.

Proposition 3.1 *Let A be a 2-weakly amenable Banach algebra. Then A has a BAI.*
Proof. Assume that X is a Banach A-bimodule whose underlying space is A, equipped with the following module actions

$$a \cdot x := ax \quad and \quad x \cdot a := 0 \quad (a \in A, x \in X).$$

Let $D : A \to X''$ be the canonical embedding of A into its second dual. Then D is a continuous derivation. Since A is 2-weakly amenable there exists $\Phi \in X''$ such that $a = D(a) = a \cdot \Phi$ for each $a \in A$. Let (a_i) be a bounded net in A such that $a_i \to \Phi$ in the w^*-topology. Then for each $f \in A'$ we have

$$\langle f, aa_i \rangle = \langle a_i, f \cdot a \rangle \to \langle \Phi, f \cdot a \rangle = \langle a \cdot \Phi, f \rangle = \langle f, a \rangle.$$

Therefore $aa_i \to a$ in the weak topology and so A has a bounded RAI by proposition 4, § 11 of [2]. In a similar way, A a bounded LAI. Thus, A has a BAI.
Let A be a Banach algebra with a BAI. Then it is well-known that A' factors if and only if A'' is unital with each of Arens products [11]. Therefore by lemma 1 of [6], and above proposition we have the following.

Corollary 3.2 *Let A be a Banach algebra with a BAI. Then*

(i) If $(A'', \square)$ is 2-weakly amenable, then A' factors on the left.

(ii) If $(A'', \Diamond)$ is 2-weakly amenable, then A' factors on the right.

Since every dual Banach algebra with a BAI is unital [13], the next result is immediate.

Corollary 3.3 *Let A be a dual Banach algebra which is an ideal of A''. If A is 2-weakly amenable, then A is reflexive.*

Examples 3.4 (i) *Let $A = L^1(G)$ for a non-discrete locally compact group G. Since A' dose not factors [11], A'' is not 2-weakly amenable with each of Arens products by corollary 3.2.*

(ii) Let $A = K(c_0)$, the operator algebra of all compact linear operators on the sequence space c_0. Then A has a BAI, A' factors on the left but $A' \neq A \cdot A'$ [5]. Hence $(A'', \Diamond)$ is not 2-weakly amenable by corollary 3.2.

Let A be a Banach algebra and $D : A \to A'$ be a continuous derivation. Then $D'' : A'' \to A'''$ is not derivation, in general. In [6], Dales, *et al*. characterized this argument and proved that D'' is a derivation if and only if $D''(A'') \cdot A'' \subseteq A'$, see also [7].

Now let A be an Arens regular Banach algebra. If A' is WSC, then every derivation $D : A \to A'$ is weakly compact [1], and so D'' is a derivation by corollary 7.2 of [6]. Now assume that A'' is WSC. Then by applying lemma 3.11 of [14] and theorem 3.1 of [8], D'' is also a derivation. We summarize these observations in the next result.

Proposition 3.5 *Suppose that A is an Arens regular Banach algebra with continuous derivation $D : A \to A'$. If either A' or A'' is WSC, then D'' is a derivation.*

Corollary 3.6 *Let A be an Arens regular Banach algebra such that A'' is WSC. If A'' is weakly amenable, then so is A.*
Proof. Let $D : A \to A'$ be a continuous derivation, then by proposition 5.1 $D'' : A'' \to A'''$ is also a derivation. Since A'' is weakly amenable, there exists $\mu \in A'''$ such that $D''(\Phi) = \Phi \cdot \mu - \mu \cdot \Phi$ for each $\Phi \in A''$.

Setting $f = k'(\mu)$, where $k : \mathsf{A} \to \mathsf{A}''$ is the canonical map, then $D(a) = a \cdot f - f \cdot a$ for all $a \in \mathsf{A}$ and so D is inner. Thus A is weakly amenable.

Proposition 3.7 *Let* A *be a dual Banach algebra with predual space* E*. Suppose that* $\mathsf{A}' \cdot \mathsf{A} \subseteq E$ *and let* $D : \mathsf{A} \to E$ *be an inner derivation. Then* $D'' : \mathsf{A}'' \to E''$ *is also inner derivation.*

Proof. Let (a_i) and (b_j) be bounded nets in A that are w^*-convergent to Φ and $\Psi \in \mathsf{A}''$, respectively. Let $D : \mathsf{A} \to E$ be a derivation, then for each $\Lambda \in E'$ we have

$$\lim_i \lim_j \langle D(a_i) \cdot b_j, \Lambda \rangle = \langle D''(\Phi) \cdot \Psi, \Lambda \rangle.$$

Since $D''(\mathsf{A}'') \cdot E' \subseteq \mathsf{A}' \cdot \mathsf{A} \subseteq E$, it follows that $D''(\Psi) \cdot \Lambda$ is w^*-continuous linear functional on A'' and so we conclude that

$$\lim_i \lim_j \langle a_i \cdot D(b_j), \Lambda \rangle = \langle \Phi \cdot D''(\Psi), \Lambda \rangle.$$

So, $D''(\Phi \mathsf{W} \Psi) = D''(\Phi) \cdot \Psi + \Phi \cdot D''(\Psi)$. Since D is inner, there exists $f \in E$ such that $D(a) = a \cdot f - f \cdot a$ for all $a \in \mathsf{A}$. Therefore we have

$$D''(\widehat{a_i}) = \widehat{D(a_i)} = \widehat{a_i \cdot f - f \cdot a_i} \to \Phi \cdot f - f \cdot \Phi.$$

This shows that $D''(\Phi) = \Phi \cdot f - f \cdot \Phi$, thus D'' is inner.

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