FIXED POINT THEOREM IN FUZZY METRIC SPACE WITH THE PROPERTY (E.A.)

Arihant Jain[1], Nirmala Gupta[2] & V.K. Gupta[3]
[1]Department of Applied Mathematics, Shri Guru Sandipani Institute of Technology and Science, Ujjain (M.P.) India
[2]Department of Mathematics, Govt. Madhav Science P.G. College, Ujjain (M.P.) 456 010, India

ABSTRACT
In this paper, we prove a common fixed point theorem for semi-compatible and weakly compatible mappings in Fuzzy metric space using the property (E.A.) and implicit relation. Our result generalizes the result of Singh and Jain [14].

Keywords : *Fuzzy metric space, property (E.A.), semi-compatible and weakly compatible mappings.*
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1. INTRODUCTION
The evolution of fuzzy mathematics solely rests on the notion of fuzzy sets which was introduced by Zadeh [18] in 1965 with a view to represent the vagueness in everyday life. In mathematical programming, the problems are often expressed as optimizing some goal functions equipped with specific constraints suggested by some concrete practical situations. There exist many real-life problems that consider multiple objectives, and generally, it is very difficult to get a feasible solution that brings us to the optimum of all the objective functions. Thus, a feasible method of resolving such problems is the use of fuzzy sets [16]. In fact, the richness of applications has engineered the all round development of fuzzy mathematics. Then, the study of fuzzy metric spaces has been carried out in several ways (e.g., [2, 7]). George and Veeramani [4] modified the concept of fuzzy metric space introduced by Kramosil and Michálek [8] with a view to obtain a Hausdorff topology on fuzzy metric spaces, and this has recently found very fruitful applications in quantum particle physics, particularly in connection with both string and theory (see [3]). In recent years, many authors have proved fixed point and common fixed point theorems in fuzzy metric spaces. To mention a few, we cite [1, 5, 9, 10, 12, 13, 14, 16, 17]. As patterned in Jungck [6], a metrical common fixed point theorem generally involves conditions on commutatively, continuity, completeness together with a suitable condition on containment of ranges of involved mappings by an appropriate contraction condition. Thus, research in this domain is aimed at weakening one or more of these conditions. In this paper, we observe that the notion of common property (E.A.) relatively relaxes the required containment of the range of one mapping into the range of other and hence, we obtain a common fixed point theorem in fuzzy metric space using the concepts of semi-compatible, weakly compatible and (E.A.) property with implicit relation. Our result generalizes the result of Singh et. al. [14].
For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

2. PRELIMINARIES
Definition 2.1. [11] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$
whenever $a \leq c$ and $b \leq d$ for $a, b, c, d \in [0, 1]$.
Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2. [11] The 3-tuple $(X, M, *)$ is said to be a Fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions :
for all $x, y, z \in X$ and $s, t > 0$.
(FM-1) $M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous,
(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$.

Note that $M(x, y, t)$ can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1. [11] Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and $M(x,y,t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and all $t > 0$. Then $(X, M, *)$ is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3. [11] A sequence $\{x_n\}$ in a Fuzzy metric space $(X, M, *)$ is said to be a Cauchy sequence if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

The sequence $\{x_n\}$ is said to converge to a point x in X if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

A Fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence in it converges to a point in it.

Definition 2.4. [13] Self mappings A and S of a Fuzzy metric space $(X, M, *)$ are said to be compatible if and only if $M(ASx_n, SAx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X as $n \to \infty$.

Definition 2.5. [14] Self mappings A and S of a Fuzzy metric space $(X, M, *)$ are said to be semi-compatible if and only if $M(ASx_n, Sp, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X as $n \to \infty$.

Definition 2.6. [15] Self mappings A and S of a Fuzzy metric space $(X, M, *)$ are said to be weakly compatible if they commute at their coincidence points, i.e., $Ax = Sx$ implies $ASx = SAx$.

Definition 2.7. [1] Self mappings A and S on a fuzzy metric space $(X, M, *)$ are said to satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \text{ for some } z \in X.$$

Definition 2.8. [1] Two pairs of self mappings (A, S) and (B, T) defined on a fuzzy metric space $(X, M, *)$ are said to share the common property (E.A.) if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$ for some $z \in X$.

Implicit Function.

Following Singh and Jain [14], let Φ be the set of all real continuous functions $\phi : [0,1]^4 \to R$, non-decreasing in first argument, and satisfying the following conditions :

(i) for $u, v \geq 0$, $\phi(u, v, u, v) \geq 0$ or $\phi(u, v, v, u) \geq 0$ implies that $u \geq v$,

(ii) $\phi(u, u, 1, 1) \geq 0$ implies that $u \geq 1$.

Example 3.1. [14] Define $\phi(t_1, t_2, t_3, t_4) = 15t_1 - 13t_2 + 5t_3 - 7t_4$. Then $\phi \in \Phi$.

MAIN RESULT:

Theorem 3.1. Let A, B, S and T be self mappings of a fuzzy metric space $(X, M, *)$. Assume that there exists $\phi \in \Phi$ such that

(3.1) $\phi(M(Ax, By, kt), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, kt)) \geq 0,$

$\phi(M(Ax, By, kt), M(Sx, Ty, t), M(Ax, Sx, kt), M(By, Ty, t)) \geq 0,$

for all $x, y \in X$, $k \in (0, 1)$ and $t > 0$. Suppose that the pairs (A, S) and (B, T) share the common property (E.A.) and S(X) and T(X) are closed subsets of X. Then, the pair (A, S) as well as (B, T) have a point of

coincidence each. Further, A, B, S and T have a unique common fixed point provided the pair (A, S) is semi-compatible and (B, T) is weakly compatible.
Proof. Since the pairs (A, S) and (B, T) share the common property (E.A.), then there exist two sequences {xn} and {yn} in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

for some $z \in X$.
Since S(X) is closed subset of X, then

$$\lim_{n\to\infty} Sx_n = z \in S(X).$$

Therefore, there exists a point $u \in X$ such that $Su = z$.
Then by (3.1), we have
$\phi(M(Au, By_n, kt), M(Su, Ty_n, t), M(Au, Su, t), M(By_n, Ty_n, kt)) \geq 0,$
$\phi(M(Au, By_n, kt), M(Su, Ty_n, t), M(Au, Su, kt), M(By_n, Ty_n, t)) \geq 0,$
which on making $n \to \infty$ reduces to

$$\phi(M(Au, z, kt), M(Su, z, t), M(Au, Su, t), M(z, z, kt)) \geq 0,$$
$$\phi(M(Au, z, kt), M(Su, z, t), M(Au, Su, kt), M(z, z, t)) \geq 0,$$

or, equivalently,

$$\phi(M(Au, z, kt), 1, M(Au, z, t), 1) \geq 0,$$
$$\phi(M(Au, z, kt), 1, M(Au, z, kt), 1) \geq 0,$$

which gives $M(Au, z, t) = 1$ for all $t > 0$,
that is, Au = z. Hence, Au = Su. Therefore, u is a coincidence point of the pair (A, S).
Since T(X) is closed subset of X, then

$$\lim_{n\to\infty} Ty_n = z \in T(X).$$

Therefore, there exists a point $w \in X$ such that Tw = z.
Now, we assert that Bw = z.
Indeed, again using (3.1), we have
$\phi(M(Ax_n, Bw, kt), M(Sx_n, Tw, t), M(Ax_n, Sx_n, t), M(Bw, Tw, kt)) \geq 0,$
$\phi(M(Ax_n, Bw, kt), M(Sx_n, Tw, t), M(Ax_n, Sx_n, kt), M(Bw, Tw, t)) \geq 0,$
which on making $n \to \infty$ reduces to

$$\phi(M(z, Bw, kt), M(z, z, t), M(z, z, t), M(Bw, z, kt)) \geq 0,$$
$$\phi(M(z, Bw, kt), M(z, z, t), M(z, z, kt), M(Bw, z, t)) \geq 0,$$

or, equivalently,

$$\phi(M(z, Bw, kt), 1, 1, M(Bw, z, kt)) \geq 0,$$
$$\phi(M(z, Bw, kt), 1, 1, M(Bw, z, t)) \geq 0,$$

which gives $M(z, Bw, t) = 1$ for all $t > 0$,
that is, Bw = z. Hence, Tw = Bw = z, which shows that w is a coincidence point of the pair (B, T).
Since (A, S) is semi-compatible, so $\lim_{n\to\infty} ASx_n = Sz$.
Also, $\lim_{n\to\infty} ASx_n = Az$.
Since the limit in fuzzy metric space is unique, so Sz = Az.
Now we assert that z is a common fixed point of the pair (A, S).
Using (3.1), we have
$\phi(M(Az, Bw, kt), M(Sz, Tw, t), M(Az, Sz, t), M(Bw, Tw, kt)) \geq 0,$
$\phi(M(Az, Bw, kt), M(Sz, Tw, t), M(Az, Sz, kt), M(Bw, Tw, t)) \geq 0,$
or, $\phi(M(Az, z, kt), M(Az, z, t), M(Az, Az, t), M(z, z, kt)) \geq 0,$
$\phi(M(Az, z, kt), M(Az, z, t), M(Az, Az, kt), M(z, z, t)) \geq 0,$
or, $\phi(M(Az, z, kt), M(Az, z, t), 1, 1) \geq 0,$
$\phi(M(Az, z, kt), M(Az, z, t), 1, 1) \geq 0,$
which gives $M(Az, z, t) = 1$ for all $t > 0$.
Hence, Az = z = Sz.
Now, since w is a coincidence point of B and T and the pair (B, T) is weakly compatible, so we have

$$BTw = TBw \text{ implies } Bz = Tz = z.$$

Hence, z is a common fixed point of both the pairs (A, S) and (B, T).

For uniqueness, let v ($v \neq z$) be another common fixed point of A, B, S and T. Taking x = z and y = v in (3.1), we have

$$\phi(M(Az, Bv, kt), M(Sz, Tv, t), M(Az, Sz, t), M(Bv, Tv, kt)) \geq 0,$$

$$\phi(M(Az, Bv, kt), M(Sz, Tv, t), M(Az, Sv, kt), M(Bv, Tv, t)) \geq 0,$$

which gives z = v.

Hence, z is the unique common fixed point of the mappings A, B, S and T.

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