

## ON THE LOCAL PROPERTY OF GENERAL INDEXED SUMMABILITY OF A FACTORED FOURIER SERIES

Mahendra Misra<sup>1</sup>, B.P. Padhy<sup>2</sup>, Dattaram Bisoyi<sup>3</sup> & U.K. Misra<sup>4</sup>

<sup>1</sup>P.G.Department of Mathematics N.C. College (Autonomous) Jajpur, Odisha

<sup>2</sup>Roland Institute of Technology Golanthara-760008, Odisha, India.

<sup>3</sup>Department of Mathematics L.N.Mahavidyalaya Kodala, Ganjam, Odisha, India

<sup>4</sup>Department of Mathematics National Institute of Science and Technology, Pallur Hills, Brahmapur, Odisha, India

**Email:** mahendramisra@2007.gmail.com<sup>1</sup>, iraady@gmail.com<sup>2</sup>,  
dbisoyi2@gmail.com<sup>3</sup> & umakanta\_misra@yahoo.com<sup>4</sup>

### ABSTRACT

In this paper we have established a theorem on the local property of  $\left| \overline{N}, p_n, \alpha_n; f \right|_k$  summability of a factored Fourier series.

**Keywords:**  $\left| \overline{N}, p_n \right|_k$  - summability,  $\left| \overline{N}, p_n, \alpha_n \right|_k$  - summability,  $\left| \overline{N}, p_n, \alpha_n; \delta \right|_k$  - summability  $\left| \overline{N}, p_n, \alpha_n; f(\delta) \right|_k$  - summability and Fourier series.

**AMS Classification No:** 40D25

### 1. INTRODUCTION

Let  $\sum a_n$  be a given infinite series with sequence of partial sums  $\{s_n\}$ . Let  $\{p_n\}$  be a sequence of positive real constants such that

$$(1.1) \quad P_n = \sum_{v=0}^n p_v \rightarrow \infty \text{ as } n \rightarrow \infty \quad (P_{-i} = p_{-i} = 0, i \geq 1)$$

The sequence -to-sequence transformation

$$(1.2) \quad t_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v$$

defines the sequence  $\{t_n\}$  of the  $\left| \overline{N}, p_n \right|_k$ -means of the sequence  $\{s_n\}$  generated by the sequence of coefficients  $\{p_n\}$ . The series  $\sum a_n$  is said to be summable  $\left| \overline{N}, p_n \right|_k$ ,  $k \geq 1$ , if

$$(1.3) \quad \sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |t_n - t_{n-1}|^k < \infty.$$

For  $k=1$ ,  $\left| \overline{N}, p_n \right|_k$  -summability is same as  $\left| N, p_n \right|_k$  -summability.

When  $p_n = 1$  for all  $n$  and  $k = 1$ ,  $\left| \overline{N}, p_n \right|_k$  -summability is same as  $\left| C, 1 \right|_k$  -summability.

Also if we take  $k = 1$  and  $p_n = \frac{1}{(n+1)}$ , summability is equivalent to the summability  $\left| R, \log n, 1 \right|_k$ .

For any sequence  $\{c_n\}$  we use the following notation

$$\Delta c_n = c_n - c_{n-1}, \quad \Delta^2 c_n = \Delta(\Delta c_n).$$

A sequence  $\{\lambda_n\}$  is said to be convex if  $\Delta^2 \lambda_n \geq 0$  for every positive integer 'n'.

Let  $\{\alpha_n\}$  be any sequence of positive numbers. The series  $\sum a_n$  is said to be summable  $\left| \overline{N}, p_n, \alpha_n \right|_k$ ,  $k \geq 1$ , if

$$(1.4) \quad \sum_{n=1}^{\infty} \alpha_n^{k-1} |t_n - t_{n-1}| < \infty.$$

and the series  $\sum a_n$  is said to be  $|\overline{N}, p_n, \alpha_n; \delta|_k, k \geq 1, \delta \geq 0$ , summable if

$$(1.5) \quad \sum_{n=1}^{\infty} \alpha_n^{\delta k + k - 1} |T_n - T_{n-1}|^k < \infty.$$

Let  $f$  be a function of  $\delta$  and if

$$(1.6) \quad \sum_{n=1}^{\infty} \{f(\delta)\}^k (\alpha_n)^{k-1} |T_n - T_{n-1}|^k < \infty,$$

then the series  $\sum a_n$  is said to be  $|\overline{N}, p_n, \alpha_n; f(\delta)|_k, k \geq 1$  summable.

Putting  $f(\delta) = \alpha_n^\delta, |\overline{N}, p_n, \alpha_n; f(\delta)|_k$  Summability reduces to  $|\overline{N}, p_n, \alpha_n; \delta|_k$  Summability.

Let  $f(t)$  be a periodic function with period  $2\pi$ , integrable in the sense of Lebesgue over  $(-\pi, \pi)$ . Without loss of generality we may assume that the constant term in the Fourier series of  $f(t)$  is zero, so that

$$(1.7) \quad f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$

**2. KNOWN THEOREMS**

Dealing with the  $|\overline{N}, p_n|_k$ -summability of an infinite series Bor[1] proved the following theorem:

**2.1. Theorem:**

Let  $k \geq 1$  and let the sequences  $\{p_n\}$  and  $\{\lambda_n\}$  be such that

$$(2.1.1) \quad \Delta X_n = O\left(\frac{1}{n}\right),$$

$$(2.1.2) \quad \sum_{n=1}^{\infty} X_n^{k-1} \frac{|\lambda_n|^k + |\lambda_{n+1}|^k}{n} < \infty,$$

$$(2.1.3) \quad \sum_{n=1}^{\infty} (X_n^k + 1) |\Delta \lambda_n| < \infty,$$

Where  $X_n = \frac{P_n}{np_n}$ . Then the summability  $|\overline{N}, p_n|_k$  of the series  $\sum_{n=1}^{\infty} A_n(t) \lambda_n X_n$  at a point can be ensured by the local property.

**2.2. Remark:**

It is known that if  $\{\lambda_n\}$  is a convex sequence and  $\sum n^{-1} \lambda_n$  is convergent, then

$$\lambda_n \geq \lambda_{n+1} \geq 0, \lambda_n \log n = O(1) \text{ and } \sum \log n \Delta \lambda_n < \infty.$$

So that (2.3) is a natural condition to impose.

In the present paper, we have proved a theorem on the local property of  $|\overline{N}, p_n, \alpha_n|_k$ -summability of a factored Fourier series.

**3. MAIN THEOREM**

Let  $k \geq 1$ . Suppose  $\{\lambda_n\}$  be a convex sequence such that  $\sum n^{-1} \lambda_n$  is convergent and  $\{p_n\}$  be a sequence such that

$$(3.1) \quad \Delta X_n = O\left(\frac{1}{n}\right),$$

$$(3.2) \quad \sum_{n=v+1}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \left(\frac{P_n}{P_n}\right)^k \left(\frac{1}{P_{n-1}}\right) = O\left(\frac{1}{P_v}\right),$$

$$(3.3) \quad \sum_{n=1}^{\infty} X_n^{k-1} \frac{|\lambda_n|^k}{n} < \infty,$$

$$(3.4) \quad \sum_{n=1}^{\infty} X_n^k |\Delta \lambda_n| < \infty,$$

and

$$(3.5) \quad \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \frac{|\lambda_n|^k}{n^k} < \infty,$$

where  $X_n = \frac{P_n}{nP_n}$ . Then the summability  $\left[\bar{N}, p_n, \alpha_n; f(\delta)\right]_k, k \geq 1$  of the series  $\sum_{n=1}^{\infty} A_n(t) \lambda_n X_n$  at a Point can be ensured by the local property where  $\{\alpha_n\}$  is any sequence of positive numbers.

In order to prove the above theorem we require the following lemma:

**Lemma:**

Let  $k \geq 1$ . Suppose  $\{\lambda_n\}$  be a convex sequence such that  $\sum n^{-1} \lambda_n$  is convergent and  $\{p_n\}$  be a sequence such that the conditions (3.1)-(3.5) are satisfied. If  $\{s_n\}$  is bounded then the series  $\sum_{n=1}^{\infty} a_n \lambda_n X_n$  is  $\left[\bar{N}, p_n, \alpha_n; f(\delta)\right]_k, k \geq 1$ -summable when  $\{\alpha_n\}$  is any sequence of positive numbers.

**4. PROOF OF THE LEMMA**

Let  $\{T_n\}$  denote the  $\left[\bar{N}, p_n\right]$ -mean of the series  $\sum_{n=1}^{\infty} a_n \lambda_n X_n$ . Then by definition we have

$$\begin{aligned} T_n &= \frac{1}{P_n} \sum_{v=0}^n p_v \sum_{r=0}^v a_r \lambda_r X_r \\ &= \frac{1}{P_n} \sum_{v=0}^n (P_v - P_{v-1}) a_v \lambda_v X_v, X_0 = 0. \end{aligned}$$

Then for  $n \geq 1$ , we have

$$T_n - T_{n-1} = \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} a_v \lambda_v X_v.$$

$$T_n - T_{n-1} = -\frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v X_v + \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v X_v \Delta \lambda_v$$

So,

$$+ \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \lambda_{v+1} \Delta X_v + \frac{P_n s_n \lambda_n X_n}{P_n}.$$

(by Abel's transformation)

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \quad (say)$$

To complete the proof of the Lemma by Minokowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \{f(\delta)\}^k (\alpha_n)^{k-1} |T_{n,i}|^k < \infty \text{ for } i = 1,2,3,4.$$

Now, we have

$$\begin{aligned} & \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} |T_{n,1}|^k \\ &= \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \left| \frac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} p_{\nu} s_{\nu} \lambda_{\nu} X_{\nu} \right|^k \\ &\leq \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \left( \frac{P_n}{P_n} \right)^k \left( \frac{1}{P_{n-1}} \right) \left( \sum_{\nu=1}^{n-1} p_{\nu} |\lambda_{\nu}|^k |s_{\nu}|^k X_{\nu}^k \right) \left( \frac{1}{P_{n-1}} \sum_{\nu=1}^{n-1} p_{\nu} \right)^{k-1} \\ &= O(1) \sum_{\nu=1}^m p_{\nu} |\lambda_{\nu}|^k |s_{\nu}|^k X_{\nu}^k \sum_{n=\nu+1}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \left( \frac{P_n}{P_n} \right)^k \left( \frac{1}{P_{n-1}} \right)^{k-1} \\ &= O(1) \sum_{\nu=1}^m |\lambda_{\nu}|^k X_{\nu}^k \frac{P_{\nu}}{P_{\nu}} \text{ , by (3.2)} \\ &= O(1) \sum_{\nu=1}^m |\lambda_{\nu}|^k X_{\nu}^{k-1} \frac{P_{\nu}}{P_{\nu}} \frac{P_{\nu}}{\nu P_{\nu}} \text{ , as } X_n = \frac{P_n}{n p_n} \\ &= O(1) \sum_{\nu=1}^m X_{\nu}^{k-1} \frac{|\lambda_{\nu}|^k}{\nu} \\ &= O(1) \text{ as } m \rightarrow \infty \text{ , by (3.3).} \end{aligned}$$

Again,

$$\begin{aligned} & \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} |T_{n,2}|^k \\ &= \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \left| \frac{P_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_{\nu} s_{\nu} X_{\nu} \Delta \lambda_{\nu} \right|^k \\ &\leq \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \left( \frac{P_n}{P_n} \right)^k \left( \frac{1}{P_{n-1}} \right) \left( \sum_{\nu=1}^{n-1} P_{\nu} |\Delta \lambda_{\nu}| |s_{\nu}|^k X_{\nu}^k \right) \left( \frac{1}{P_{n-1}} \sum_{\nu=1}^{n-1} P_{\nu} |\Delta \lambda| \right)^{k-1} \end{aligned}$$

Since

$$\begin{aligned} \sum_{r=1}^{n-1} P_r |\Delta \lambda_r| &\leq P_{n-1} \sum_{\nu=1}^{n-1} |\Delta \lambda_{\nu}| \Rightarrow \frac{1}{P_{n-1}} \sum_{r=1}^{n-1} P_r |\Delta \lambda_r| \leq \sum_{\nu=1}^{n-1} |\Delta \lambda_{\nu}| = O(1) \\ \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} |T_{n,2}|^k &= O(1) \sum_{\nu=1}^m P_{\nu} |\Delta \lambda_{\nu}| X_{\nu}^k \sum_{n=\nu+1}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \left( \frac{P_n}{P_n} \right)^k \left( \frac{1}{P_{n-1}} \right)^{k-1} \\ &= O(1) \sum_{\nu=1}^m |\Delta \lambda_{\nu}| X_{\nu}^k \text{ , by (3.2)} \\ &= O(1) \text{ as } m \rightarrow \infty \text{ , by (3.4).} \end{aligned}$$

Further,

$$\sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} |T_{n,3}|^k$$

$$\begin{aligned}
 &= \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \left| \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \lambda_{v+1} \Delta X_v \right|^k \\
 &= O(1) \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \left( \frac{P_n}{P_n P_{n-1}} \right)^k \left\{ \sum_{v=1}^{n-1} P_v |\lambda_{v+1}| s_v \left| \frac{1}{v} \right|^k \right\}^k, \text{ by (3.1)} \\
 &= O(1) \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \left( \frac{P_n}{P_n P_{n-1}} \right)^k \left\{ \sum_{v=1}^{n-1} P_v X_v |\lambda_{v+1}| \right\}^k \\
 &= O(1) \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \left( \frac{P_n}{P_n} \right)^k \left( \frac{1}{P_{n-1}} \right)^k \left\{ \sum_{v=1}^{n-1} P_v X_v^k |\lambda_{v+1}|^k \right\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \right\}^{k-1} \\
 &= O(1) \sum_{v=1}^m P_v |\lambda_{v+1}|^k X_v^k \sum_{n=v+1}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \left( \frac{P_n}{P_n} \right)^k \left( \frac{1}{P_{n-1}} \right)^k \\
 &= O(1) \sum_{v=1}^m \frac{P_v}{P_v} |\lambda_{v+1}|^k X_v^k, \text{ by (3.2)} \\
 &= O(1) \sum_{v=1}^m |\lambda_{v+1}|^k X_v^{k-1} \frac{P_v}{P_v} \frac{P_v}{v P_v}, \text{ as } X_n = \frac{P_n}{n P_n} \\
 &= O(1) \sum_{v=1}^m X_v^{k-1} \frac{|\lambda_{v+1}|^k}{v} \\
 &= O(1) \text{ as } m \rightarrow \infty, \text{ by (3.3).}
 \end{aligned}$$

Now,

$$\begin{aligned}
 &\sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} |T_{n,4}|^k \\
 &= \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \left| \frac{P_n s_n \lambda_n X_n}{P_n} \right|^k \\
 &= O(1) \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} X_n^k |\lambda_n|^k \left( \frac{P_n}{P_n} \right)^k \\
 &= O(1) \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \left( \frac{P_n}{n P_n} \right)^k |\lambda_n|^k \left( \frac{P_n}{P_n} \right)^k, \text{ as } X_n = \frac{P_n}{n P_n} \\
 &= O(1) \sum_{n=2}^{m+1} \{f(\delta)\}^k (\alpha_n)^{k-1} \frac{|\lambda_n|^k}{n^k} \\
 &= O(1) \text{ as } m \rightarrow \infty, \text{ by (3.5).}
 \end{aligned}$$

This completes the proof of the Lemma.

### 5. PROOF OF THE THEOREM

Since the behavior of the Fourier series, as far as convergence is concerned, for a particular value of  $x$  depends on the behavior of the function in the immediate neighborhood of this point only, hence the truth of the theorem is necessary consequence of the Lemma.

### 6. REFERENCES

[1]. H.Bor., 1992, On the Local Property of  $\left[ \overline{N}, p_n \right]_k$  Summability of factored Fourier series, journal of mathematical analysis and applications- **163** (220-226).