STUDY THE BEHAVIOR OF THE SOLUTION AND ASYMPTOTIC BEHAVIORS OF EIGENVALUES OF A SIX ORDER BOUNDARY VALUE PROBLEM

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ABSTRACT
In this paper we study the behavior of the solution and asymptotic behavior of eigenvalues of a six order boundary value problem of the form:

$$-y^{6}(x)+p_4(x)y^{4}(x)+p_3(x)y''(x)+p_2(x)y'(x)+p_1(x)y(x)=\lambda^6\rho(x)y(x),\quad 0<x<a,$$

With boundary conditions

$$U_j(y)=y^{(j)}(0)=0\,,\qquad j=0,1,2,3$$
$$U_j(y)=y^{(5-j)}(a,\lambda)=0\,,\qquad j=4,5,$$

Where $p_4(x),\ p_3(x),\ p_2(x),\ p_1(x)$and $\rho(x)$ are real functions and $\rho(x)>0$, and λ is spectral parameter in which$\lambda=\sigma+i\tau\,,\sigma,\tau\in R,i=\sqrt{-1}\,,\tau\neq 0$

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1. INTRODUCTION

The investigation of boundary value problems for which the eigenvalues parameter appears in both the equation and boundary conditions originates from the works of G.D. Birkhoff [2-3]. There are many papers and books, where the spectral properties of such problems are investigates (see, for example [1] and [4-9]). But in this paper we study the behavior of the solution and asymptotic behaviors of eigenvalues of a six order boundary value problemS_1 which is defined by:

$$-y^{6}(x)+p_4(x)y^{4}(x)+p_3(x)y''(x)+p_2(x)y'(x)+p_1(x)y(x)=\lambda^6\rho(x)y(x),\quad 0<x<a, \tag{1.1}$$

with boundary conditions

$$U_j(y)=y^{(j)}(0)=0\,,\qquad j=0,1,2,3$$
$$U_j(y)=y^{(5-j)}(a,\lambda)=0\,,\qquad j=4,5,$$

where $p_4(x),\ p_3(x),\ p_2(x),\ p_1(x)$and $\rho(x)$ are real functions and $\rho(x)>0$, and λ is spectral parameter in which$\lambda=\sigma+i\tau\,,\sigma,\tau\in R,i=\sqrt{-1}\,,\tau\neq 0$. Here we assume that $p_4(x)\in C^4[0,a],p_3(x)\in C^2[0,a],p_2(x)\in C^1[0,a],p_1(x)\in C\ \ [0,a],\rho(x)\in C^6[0,a]$. We have introduced the sectors T_k and their conjugates$\overline{T_k}$(relative to the x-axis). Let λ located in some fixed sector T_k or $\overline{T_k}$, and let w_j's (j=0,1,2,3,4,5)be different roots of unity of degree 6, and ordered so that for all $\lambda\in T_k$ (or $\overline{T_k}$) satisfied the inequality:

$$\text{Re}(iw_o\lambda)\leq\text{Re}(iw_1\lambda)\leq\text{Re}(iw_2\lambda)\leq\text{Re}(iw_3\lambda)\leq\text{Re}(iw_4\lambda)\leq\text{Re}(iw_5\lambda),$$

Numbering depends, the selected sector contains λ. Entire complex plane of $\lambda=\sigma+i\tau$divided into 12 sectors T_k and $(\overline{T_k})$ in the plane λ determined by the inequalities $\frac{k\pi}{3}\leq\arg\lambda\leq\frac{k\pi}{3}+\frac{\pi}{6}$,
$k=0,1,2,3,4,5$, and we assume that $w_k=\sqrt[6]{1}$, and $\emptyset_k=iw_k\sqrt[6]{\rho(x)}$. (1.2)

2. STUDY THEBEHAVIOR OF THE SOLUTION OF SIXTH ORDER BOUNDARY VALUE PROBLEM

The aim of this section is to estimate the behavior of the solution to the given sixth order boundary value problem S_1 and finding their coefficients $A_i(x), i = 0,1,2,3,4,5,6$.

Theorem2.1:

Suppose $\lambda \in T_k$ or $\lambda \in \overline{T_k}$ $(k = 0,1,2,3,4,5)$ and $\emptyset_k(k = 0,1,2,3,4,5)$ satisfies the inequality (1.2), then there exist linear independent solutions $y_k \ (x,\lambda), (k = 0,1,2,3,4,5)$ of equation (1.1) regular with sufficient large $|\lambda\,|$ such that when $j = 0,1,2,3,4,5$ uniformly in $0 \le x \le a$ satisfy the relation

$$y_k{}^{(j)}(x,\lambda) = (\emptyset_k(x)\,\lambda)^j e^{\lambda \int_0^x \emptyset_k(t)dt}\left[\sum_{i=0}^{6}\frac{A_i(x)}{\lambda^i} + O\left(\frac{1}{\lambda^7}\right)\right],$$

such that

$$A_o(x) = \rho(x)^{-\frac{5}{12}},$$

$$A_1(x) = A_o(x)\int_0^x\left[-\frac{5}{2}\frac{A_o''(t)}{\emptyset_k} - 10\frac{\emptyset_k'}{\emptyset_k^2}A_o'(t) - \left(\frac{15}{2}\frac{(\emptyset_k')^2}{\emptyset_k^3} + \frac{10}{3}\frac{\emptyset_k''}{\emptyset_k^2} - \frac{p_4(t)}{6\emptyset_k}\right)A_o(t)\right]\frac{dt}{A_o(t)},$$

$$A_2(x) = A_o(x)\int_0^x [-\frac{5}{2}\frac{A_1''(t)}{\emptyset_k} - 10\frac{\emptyset_k'}{\emptyset_k^2}A_1'(t) - \left(\frac{15}{2}\frac{(\emptyset_k')^2}{\emptyset_k^3} + \frac{10}{3}\frac{\emptyset_k''}{\emptyset_k^2} - \frac{p_4(t)}{6\emptyset_k}\right)A_1(t) - \frac{10}{3}\frac{A_o^3(t)}{\emptyset_k^2} - 15\frac{\emptyset_k'}{\emptyset_k^3}A_o''(t)$$
$$-\left(10\frac{\emptyset_k''}{\emptyset_k^3} + 15\frac{(\emptyset_k')^2}{\emptyset_k^4} - \frac{2}{3}\frac{p_4(t)}{\emptyset_k^2}\right)A_o'(t)$$
$$-(\frac{5}{2}\frac{(\emptyset_k{}^{(3)}}{\emptyset_k^3} + 10\frac{\emptyset_k'\emptyset_k''}{\emptyset_k^4} + \frac{5}{2}\frac{(\emptyset_k')^3}{\emptyset_k^5} - \frac{\emptyset_k'}{\emptyset_k^3}p_4(t))\,A_o(t)\,]\ \frac{dt}{A_o(t)},$$

$$A_3(x) = A_o(x)\int_0^x [-\frac{5}{2}\frac{A_2''(t)}{\emptyset_k} - 10\frac{\emptyset_k'}{\emptyset_k^2}A_2'(t) - \left(\frac{15}{2}\frac{(\emptyset_k')^2}{\emptyset_k^3} + \frac{10}{3}\frac{\emptyset_k''}{\emptyset_k^2} - \frac{p_4(t)}{6\emptyset_k}\right)A_2(t) - \frac{10}{3}\frac{A_1^3(t)}{\emptyset_k^2} - 15\frac{\emptyset_k'}{\emptyset_k^3}A_1''(t)$$
$$-\left(10\frac{\emptyset_k''}{\emptyset_k^3} + 15\frac{(\emptyset_k')^2}{\emptyset_k^4} - \frac{2}{3}\frac{p_4(t)}{\emptyset_k^2}\right)A_1'(t)$$
$$-(\frac{5}{2}\frac{\emptyset_k{}^{(3)}}{\emptyset_k^3} + 10\frac{\emptyset_k'\emptyset_k''}{\emptyset_k^4} + \frac{5}{2}\frac{(\emptyset_k')^3}{\emptyset_k^5} - \frac{\emptyset_k'}{\emptyset_k^3}p_4(t)\,)\,A_1(t) - \frac{5}{2}\frac{A_o^{(4)}(t)}{\emptyset_k^3} - 10\frac{\emptyset_k'}{\emptyset_k^4}A_o^{(3)}(t)$$
$$-\left(10\frac{\emptyset_k''}{\emptyset_k^4} + \frac{15}{2}\frac{(\emptyset_k')^2}{\emptyset_k^5} - \frac{p_4(t)}{\emptyset_k^3}\right)A_o''(t) - \left(5\frac{\emptyset_k{}^{(3)}}{\emptyset_k^4} + 10\frac{\emptyset_k'\emptyset_k''}{\emptyset_k^5} - 2\frac{\emptyset_k'}{\emptyset_k^4}p_4(t)\right)A_o'(t)$$
$$-(\frac{\emptyset_k{}^{(4)}}{\emptyset_k^4} + \frac{5}{2}\frac{\emptyset_k'\emptyset_k^{(3)}}{\emptyset_k^5} + \frac{5}{3}\frac{(\emptyset_k'')^2}{\emptyset_k^5} - \frac{1}{2}\frac{(\emptyset_k')^2}{\emptyset_k^5}p_4(t) - -\frac{2}{3}\frac{\emptyset_k''}{\emptyset_k^4}p_4(t) - \frac{p_3(t)}{6\emptyset_k^3})\,A_o(t)\,]\ \frac{dt}{A_o(t)},$$

$$A_4(x) = A_o(x)\int_0^x [-\frac{5}{2}\frac{A_3''(t)}{\emptyset_k} - 10\frac{\emptyset_k'}{\emptyset_k^2}A_3'(t) - \left(\frac{15}{2}\frac{(\emptyset_k')^2}{\emptyset_k^3} + \frac{10}{3}\frac{\emptyset_k''}{\emptyset_k^2} - \frac{p_4(t)}{6\emptyset_k}\right)A_3(t)$$
$$-\frac{10}{3}\frac{A_2^{(3)}(t)}{\emptyset_k^2} - 15\frac{\emptyset_k'}{\emptyset_k^3}A_2''(t) - \left(10\frac{\emptyset_k''}{\emptyset_k^3} + 15\frac{(\emptyset_k')^2}{\emptyset_k^4} - \frac{2}{3}\frac{p_4(t)}{\emptyset_k^2}\right)A_2'(t) - (\frac{5}{2}\frac{\emptyset_k{}^{(3)}}{\emptyset_k^3}$$
$$+10\frac{\emptyset_k'\emptyset_k''}{\emptyset_k^4} + \frac{5}{2}\frac{(\emptyset_k')^3}{\emptyset_k^5} - \frac{\emptyset_k'}{\emptyset_k^3}p_4(t)\,)\,A_2(t) - \frac{5}{2}\frac{A_1^{(4)}(t)}{\emptyset_k^3} - 10\frac{\emptyset_k'}{\emptyset_k^4}A_1^{(3)}(t) - (10\frac{\emptyset_k''}{\emptyset_k^4}$$
$$+\frac{15}{2}\frac{(\emptyset_k')^2}{\emptyset_k^5} - \frac{p_4(t)}{\emptyset_k^3})A_1''(t) - \left(5\frac{\emptyset_k{}^{(3)}}{\emptyset_k^4} + 10\frac{\emptyset_k'\emptyset_k''}{\emptyset_k^5} - 2\frac{\emptyset_k'}{\emptyset_k^4}p_4(t)\right)A_1'(t) - (\frac{\emptyset_k{}^{(4)}}{\emptyset_k^4}$$

$$+\frac{5}{2}\frac{\phi_k'\phi_k^{(3)}}{\phi_k^5}+\frac{5}{3}\frac{(\phi_k'')^2}{\phi_k^5}-\frac{1}{2}\frac{(\phi_k')^2}{\phi_k^5}p_4(t)-\frac{2}{3}\frac{\phi_k''}{\phi_k^4}p_4(t)-\frac{p_3(t)}{6\phi_k^3}\,)\,A_1(t)-\frac{A_o^{(5)}(t)}{\phi_k^4}$$

$$-\frac{5}{2}\frac{\phi_k'}{\phi_k^5}A_o^{(4)}(t)-\left(\frac{10}{3}\frac{\phi_k''}{\phi_k^5}-\frac{2}{3}\frac{p_4(t)}{\phi_k^4}\right)A_o^{(3)}(t)-\left(\frac{5}{2}\frac{\phi_k^{(3)}}{\phi_k^5}-\frac{\phi_k'}{\phi_k^5}p_4(t)\right)A_o''(t)-(\frac{\phi_k^{(4)}}{\phi_k^5}$$

$$-\frac{2}{3}\frac{\phi_k''}{\phi_k^5}p_4(t)-\frac{1}{3}\frac{p_3(t)}{\phi_k^4})A_o'(t)-(\frac{\phi_k^{(5)}}{6\phi_k^5}-\frac{\phi_k^{(3)}}{6\phi_k^5}p_4(t)-\frac{\phi_k'}{6\phi_k^5}p_3(t)-\frac{p_2(t)}{6\phi_k^4}\,)\,A_o(t)]\;\frac{dt}{A_o(t)},$$

$$A_5(x)=A_o(x)\int_0^x[-\frac{5}{2}\frac{A_4''(t)}{\phi_k}-10\frac{\phi_k'}{\phi_k^2}A_4'(t)-\left(\frac{15}{2}\frac{(\phi_k')^2}{\phi_k^3}+\frac{10}{3}\frac{\phi_k''}{\phi_k^2}-\frac{p_4(t)}{6\phi_k}\right)A_4(t)$$

$$-\frac{10}{3}\frac{A_3^{(3)}(t)}{\phi_k^2}-15\frac{\phi_k'}{\phi_k^3}A_3''(t)-\left(10\frac{\phi_k''}{\phi_k^3}+15\frac{(\phi_k')^2}{\phi_k^4}-\frac{2}{3}\frac{p_4(t)}{\phi_k^2}\right)A_3'(t)-(\frac{5}{2}\frac{\phi_k^{(3)}}{\phi_k^3}$$

$$+10\frac{\phi_k'\phi_k''}{\phi_k^4}+\frac{5}{2}\frac{(\phi_k')^3}{\phi_k^5}-\frac{\phi_k'}{\phi_k^3}p_4(t)\,)\,A_3(t)-\frac{5}{2}\frac{A_2^{(4)}(t)}{\phi_k^3}-10\frac{\phi_k'}{\phi_k^4}A_2^{(3)}(t)-(10\frac{\phi_k''}{\phi_k^4}$$

$$+\frac{15}{2}\frac{(\phi_k')^2}{\phi_k^5}-\frac{p_4(t)}{\phi_k^3})A_2''(t)-\left(5\frac{\phi_k^{(3)}}{\phi_k^4}+10\frac{\phi_k'\phi_k''}{\phi_k^5}-2\frac{\phi_k'}{\phi_k^4}p_4(t)\right)A_2'(t)-(\frac{\phi_k^{(4)}}{\phi_k^4}$$

$$+\frac{5}{2}\frac{\phi_k'\phi_k^{(3)}}{\phi_k^5}+\frac{5}{3}\frac{(\phi_k'')^2}{\phi_k^5}-\frac{1}{2}\frac{(\phi_k')^2}{\phi_k^5}p_4(t)-\frac{2}{3}\frac{\phi_k''}{\phi_k^4}p_4(t)-\frac{p_3(t)}{6\phi_k^3}\,)\,A_2(t)-\frac{A_1^{(5)}(t)}{\phi_k^4}$$

$$-\frac{5}{2}\frac{\phi_k'}{\phi_k^5}A_1^{(4)}(t)-\left(\frac{10}{3}\frac{\phi_k''}{\phi_k^5}-\frac{2}{3}\frac{p_4(t)}{\phi_k^4}\right)A_1^{(3)}(t)-\left(\frac{5}{2}\frac{\phi_k^{(3)}}{\phi_k^5}-\frac{\phi_k'}{\phi_k^5}p_4(t)\right)A_1''(t)-(\frac{\phi_k^{(4)}}{\phi_k^5}$$

$$-\frac{2}{3}\frac{\phi_k''}{\phi_k^5}p_4(t)-\frac{1}{3}\frac{p_3(t)}{\phi_k^4})A_1'(t)-\left(\frac{\phi_k^{(5)}}{6\phi_k^5}-\frac{\phi_k^{(3)}}{6\phi_k^5}p_4(t)-\frac{\phi_k'}{6\phi_k^5}p_3(t)-\frac{p_2(t)}{6\phi_k^4}\right)A_1(t)$$

$$-\frac{A_o^{(6)}(t)}{6\phi_k^5}+\frac{p_4(t)}{6\phi_k^5}A_o^{(4)}(t)+\frac{p_3(t)}{6\phi_k^5}A_o''(t)+\frac{p_2(t)}{6\phi_k^5}A_o'(t)+\frac{p_1(t)}{6\phi_k^5}A_o(t)]\;\frac{dt}{A_o(t)},$$

$$A_6(x)=A_o(x)\int_0^x[-\frac{5}{2}\frac{A_5''(t)}{\phi_k}-10\frac{\phi_k'}{\phi_k^2}A_5'(t)-\left(\frac{15}{2}\frac{(\phi_k')^2}{\phi_k^3}+\frac{10}{3}\frac{\phi_k''}{\phi_k^2}-\frac{p_4(t)}{6\phi_k}\right)A_5(t)$$

$$-\frac{10}{3}\frac{A_4^{(3)}(t)}{\phi_k^2}-15\frac{\phi_k'}{\phi_k^3}A_4''(t)-\left(10\frac{\phi_k''}{\phi_k^3}+15\frac{(\phi_k')^2}{\phi_k^4}-\frac{2}{3}\frac{p_4(t)}{\phi_k^2}\right)A_4'(t)-(\frac{5}{2}\frac{\phi_k^{(3)}}{\phi_k^3}$$

$$+10\frac{\phi_k'\phi_k''}{\phi_k^4}+\frac{5}{2}\frac{(\phi_k')^3}{\phi_k^5}-\frac{\phi_k'}{\phi_k^3}p_4(t)\,)\,A_4(t)-\frac{5}{2}\frac{A_3^{(4)}(t)}{\phi_k^3}-10\frac{\phi_k'}{\phi_k^4}A_3^{(3)}(t)-(10\frac{\phi_k''}{\phi_k^4}$$

$$+\frac{15}{2}\frac{(\phi_k')^2}{\phi_k^5}-\frac{p_4(t)}{\phi_k^3})A_3''(t)-\left(5\frac{\phi_k^{(3)}}{\phi_k^4}+10\frac{\phi_k'\phi_k''}{\phi_k^5}-2\frac{\phi_k'}{\phi_k^4}p_4(t)\right)A_3'(t)-(\frac{\phi_k^{(4)}}{\phi_k^4}$$

$$+\frac{5}{2}\frac{\phi_k'\phi_k^{(3)}}{\phi_k^5}+\frac{5}{3}\frac{(\phi_k'')^2}{\phi_k^5}-\frac{1}{2}\frac{(\phi_k')^2}{\phi_k^5}p_4(t)-\frac{2}{3}\frac{\phi_k''}{\phi_k^4}p_4(t)-\frac{p_3(t)}{6\phi_k^3}\,)\,A_3(t)-\frac{A_2^{(5)}(t)}{\phi_k^4}$$

$$-\frac{5}{2}\frac{\phi_k'}{\phi_k^5}A_2^{(4)}(t)-\left(\frac{10}{3}\frac{\phi_k''}{\phi_k^5}-\frac{2}{3}\frac{p_4(t)}{\phi_k^4}\right)A_2^{(3)}(t)-\left(\frac{5}{2}\frac{\phi_k^{(3)}}{\phi_k^5}-\frac{\phi_k'}{\phi_k^5}p_4(t)\right)A_2''(t)-(\frac{\phi_k^{(4)}}{\phi_k^5}$$

$$-\frac{2}{3}\frac{\emptyset_k''}{\emptyset_k^5}p_4(t)-\frac{1}{3}\frac{p_3(t)}{\emptyset_k^4})A_2'(t)-\left(\frac{\emptyset_k^{(5)}}{6\emptyset_k^5}-\frac{\emptyset_k^{(3)}}{6\emptyset_k^5}p_4(t)-\frac{\emptyset_k'}{6\emptyset_k^5}p_3(t)-\frac{p_2(t)}{6\emptyset_k^4}\right)A_2(t)$$
$$-\frac{A_1^{(6)}(t)}{6\emptyset_k^5}+\frac{p_4(t)}{6\emptyset_k^5}A_1^{(4)}(t)+\frac{p_3(t)}{6\emptyset_k^5}A_1''(t)+\frac{p_2(t)}{6\emptyset_k^5}A_1'(t)+\frac{p_1(t)}{6\emptyset_k^5}A_1(t)]\ \frac{dt}{A_o(t)}.$$

Proof:

From [9], he proved the solutions of equation (1.1) for sufficient large $|\lambda|$ which are can be written of the form

$$y_k\ (x,\lambda)=e^{\lambda\int_0^x\emptyset_k(t)dt}\left[\sum_{i=0}^{6}\frac{A_i(x)}{\lambda^i}+O\left(\frac{1}{\lambda^7}\right)\right] \qquad (2.1)$$

By derivation (2.1) up to sixth order with respect to x, the following relations are obtained:

$$y'(x)=(\emptyset_k(x)\lambda)\,e^{\lambda\int_0^x\emptyset_k(t)dt}\left[A_o(x)+\sum_{i=0}^{5}\frac{1}{\lambda^{i+1}}(A_{i+1}(x)+\frac{A_i'(x)}{\emptyset_k})\ +O\left(\frac{1}{\lambda^7}\right)\right], \qquad (2.2)$$

$$y''(x)=(\emptyset_k(x)\lambda)^2e^{\lambda\int_0^x\emptyset_k(t)dt}\ [A_o(x)+\frac{1}{\lambda}\left(A_1(x)+3\frac{A'_o(x)}{\emptyset_k}+3\frac{\emptyset'}{\emptyset^2}A_o(x)\right)+$$
$$\sum_{i=0}^{4}\frac{1}{\lambda^{i+2}}(A_{i+2}(x)+2\frac{A_{i+1}'(x)}{\emptyset_k}+\frac{\emptyset'}{\emptyset^2}A_{i+1}(x)+\frac{A''_i(x)}{\emptyset_k^2})\ +O\left(\frac{1}{\lambda^7}\right)], \qquad (2.3)$$

$$y^{(3)}(x)=(\emptyset_k(x)\lambda)^3e^{\lambda\int_0^x\emptyset_k(t)dt}\ [A_o(x)+\frac{1}{\lambda}\left(A_1(x)+3\frac{A'_o(x)}{\emptyset_k}+3\frac{\emptyset'}{\emptyset^2}A_o(x)\right)+$$
$$\frac{1}{\lambda^2}\left(A_2(x)+3\frac{A'_1(x)}{\emptyset_k}+3\frac{\emptyset'}{\emptyset^2}A_1(x)+3\frac{A_o(x)}{\emptyset^2}+3\frac{\emptyset'}{\emptyset^3}A_o'(x)+\frac{\emptyset''}{\emptyset^3}A_o(x)\right)+$$
$$\sum_{i=0}^{3}\frac{1}{\lambda^{i+3}}(A_{i+3}(x)+3\frac{A'_{i+2}(x)}{\emptyset_k}+3\frac{\emptyset'}{\emptyset^2}A_{i+2}(x)+3\frac{A_{i+1}(x)}{\emptyset^2}+3\frac{\emptyset'}{\emptyset^3}A_{i+1}'(x)+\frac{\emptyset''}{\emptyset^3}A_{i+1}(x)$$
$$+\frac{A_i^{(3)}(x)}{\emptyset^3})+O\left(\frac{1}{\lambda^7}\right)], \qquad (2.4)$$

$$y^{(4)}(x)=(\emptyset_k(x)\lambda)^4e^{\lambda\int_0^x\emptyset_k(t)dt}\ [A_o(x)+\frac{1}{\lambda}\left(A_1(x)+4\frac{A'_0(x)}{\emptyset_k}+6\frac{\emptyset'}{\emptyset^2}A_o(x)\right)+$$
$$\frac{1}{\lambda^2}\left(A_2(x)+4\frac{A'_1(x)}{\emptyset_k}+6\frac{\emptyset'}{\emptyset^2}A_1(x)+6\frac{A''_o(x)}{\emptyset_k^2}+12\frac{\emptyset'}{\emptyset^3}A_o'(x)+(3\frac{(\emptyset')^2}{\emptyset^4}+4\frac{\emptyset''}{\emptyset^3})A_o(x)\right)+$$
$$\frac{1}{\lambda^3}\ (A_3(x)+4\frac{A'_2(x)}{\emptyset_k}+6\frac{\emptyset'}{\emptyset^2}A_2(x)+6\frac{A''_1(x)}{\emptyset_k^2}+12\frac{\emptyset'}{\emptyset^3}A_1'(x)+(3\frac{(\emptyset')^2}{\emptyset^4}+4\frac{\emptyset''}{\emptyset^3})A_1(x)+$$
$$4\frac{A^{(3)}{}_o(x)}{\emptyset_k^3}+6\frac{\emptyset'}{\emptyset^4}A_o''(x)+4\frac{\emptyset''}{\emptyset^4}A_o'(x)+\frac{\emptyset^{(3)}}{\emptyset^4}A_o(x))+\sum_{i=0}^{2}\frac{1}{\lambda^{i+4}}(A_{i+4}(x)+4\frac{A'_{i+3}(x)}{\emptyset_k}+$$
$$6\frac{\emptyset'}{\emptyset^2}A_{i+3}(x)+6\frac{A''_{i+2}(x)}{\emptyset_k^2}+12\frac{\emptyset'}{\emptyset^3}A_{i+2}'(x)+(3\frac{(\emptyset')^2}{\emptyset^4}+4\frac{\emptyset''}{\emptyset^3})A_{i+2}(x)+4\frac{A^{(3)}{}_{i+1}(x)}{\emptyset_k^3}$$
$$+6\frac{\emptyset'}{\emptyset^4}A_{i+1}''(x)+4\frac{\emptyset''}{\emptyset^4}A_{i+1}'(x)+\frac{\emptyset^{(3)}}{\emptyset^4}A_{i+1}(x)+\frac{A_i^{(4)}(x)}{\emptyset^4}+O\left(\frac{1}{\lambda^7}\right)], \qquad (2.5)$$

$$y^{(5)}(x)=(\emptyset_k(x)\lambda)^5e^{\lambda\int_0^x\emptyset_k(t)dt}\ [A_o(x)+\frac{1}{\lambda}\left(A_1(x)+5\frac{A'_o(x)}{\emptyset_k}+10\frac{\emptyset'}{\emptyset^2}A_o(x)\right)+$$

$$\frac{1}{\lambda^2}\,(\,A_2(x)+5\frac{A'_1(x)}{\phi_k}+10\frac{\phi'}{\phi^2}A_1(x)+10\frac{A''_o(x)}{\phi_k^2}+30\frac{\phi'}{\phi^3}A'_o(x)+(15\frac{(\phi')^2}{\phi^4}+10\frac{\phi''}{\phi^3})$$
$$A_o(x)\,)+\frac{1}{\lambda^3}\,\,(\,A_3(x)+5\frac{A'_2(x)}{\phi_k}+10\frac{\phi'}{\phi^2}A_2(x)+10\frac{A''_1(x)}{\phi_k^2}+30\frac{\phi'}{\phi^3}A'_1(x)+(15\frac{(\phi')^2}{\phi^4}$$
$$+10\frac{\phi''}{\phi^3})A_1(x)+10\frac{A^{(3)}{}_o(x)}{\phi_k^3}+30\frac{\phi'}{\phi^4}A''_o(x)+20\frac{\phi''}{\phi^4}A'_o(x)+5\frac{\phi^{(3)}}{\phi^4}A_o(x)$$
$$+15\frac{(\phi')^2}{\phi^5}A'_o(x)+10\frac{\phi'\phi''}{\phi^5}A_o(x)\,)+\frac{1}{\lambda^4}\,\,(\,A_4(x)+5\frac{A'_3(x)}{\phi_k}+10\frac{\phi'}{\phi^2}A_3(x)+10\frac{A''_2(x)}{\phi_k^2}$$
$$+30\frac{\phi'}{\phi^3}A'_2(x)+(15\frac{(\phi')^2}{\phi^4}+10\frac{\phi''}{\phi^3})A_2(x)+10\frac{A^{(3)}{}_1(x)}{\phi_k^3}+30\frac{\phi'}{\phi^4}A''_1(x)+20\frac{\phi''}{\phi^4}A'_1(x)$$
$$+5\frac{\phi^{(3)}}{\phi^4}A_1(x)+15\frac{(\phi')^2}{\phi^5}A'_1(x)+10\frac{\phi'\phi''}{\phi^5}A_1(x)+5\frac{A^{(4)}{}_o(x)}{\phi_k^4}+10\frac{\phi'}{\phi^5}A_o^{(3)}(x)+$$
$$10\frac{\phi''}{\phi^5}A''_o(x)+5\frac{\phi^{(3)}}{\phi^5}A'_o(x)\frac{\phi^{(4)}}{\phi^5}A_o(x)\,)+\sum_{i=0}^{1}\frac{1}{\lambda^{i+5}}\,(\,A_{i+5}(x)+5\frac{A'_{i+4}(x)}{\phi_k}$$
$$+10\frac{\phi'}{\phi^2}A_{i+4}(x)+10\frac{A''_{i+3}(x)}{\phi_k^2}+30\frac{\phi'}{\phi^3}A'_{i+3}(x)+(15\frac{(\phi')^2}{\phi^4}+10\frac{\phi''}{\phi^3})A_{i+3}(x)$$
$$+10\frac{A^{(3)}{}_{i+2}(x)}{\phi_k^3}+30\frac{\phi'}{\phi^4}A''_{i+2}(x)+20\frac{\phi''}{\phi^4}A'_{i+2}(x)+5\frac{\phi^{(3)}}{\phi^4}A_{i+2}(x)+15\frac{(\phi')^2}{\phi^5}A'_{i+2}(x)$$
$$+10\frac{\phi'\phi''}{\phi^5}A_{i+2}(x)+5\frac{A^{(4)}{}_{i+1}(x)}{\phi_k^4}+10\frac{\phi'}{\phi^5}A_{i+1}^{(3)}(x)+30\frac{\phi'}{\phi^3}A'_2(x)+10\frac{\phi''}{\phi^5}A''_{i+1}(x)$$
$$+5\frac{\phi^{(3)}}{\phi^5}A'_{i+1}(x)+\frac{\phi^{(4)}}{\phi^5}A_{i+1}(x)+\frac{A_i^{(5)}(x)}{\phi^5})+O\left(\frac{1}{\lambda^7}\right)\,],$$

$$y^{(6)}(x)=(\phi_k(x)\,\lambda)^6e^{\lambda\int_0^x\phi_k(t)dt}\,\,[\,A_o(x)+\frac{1}{\lambda}\left(A_1(x)+6\frac{A'_O(x)}{\phi_k}+15\frac{\phi'}{\phi^2}A_o(x)\right)+$$
$$\frac{1}{\lambda^2}\,(\,A_2(x)+6\frac{A'_1(x)}{\phi_k}+15\frac{\phi'}{\phi^2}A_1(x)+15\frac{A''_o(x)}{\phi_k^2}+60\frac{\phi'}{\phi^3}A'_o(x)+(45\frac{(\phi')^2}{\phi^4}+20\frac{\phi''}{\phi^3})$$
$$A_o(x)\,)+\frac{1}{\lambda^3}\,\,(\,A_3(x)+6\frac{A'_2(x)}{\phi_k}+15\frac{\phi'}{\phi^2}A_2(x)+15\frac{A''_1(x)}{\phi_k^2}+60\frac{\phi'}{\phi^3}A'_1(x)+(45\frac{(\phi')^2}{\phi^4}+$$
$$20\frac{\phi''}{\phi^3})A_1(x)+20\frac{A^{(3)}{}_o(x)}{\phi_k^3}+90\frac{\phi'}{\phi^4}A''_o(x)+60\frac{\phi''}{\phi^4}A'_o(x)+15\frac{\phi^{(3)}}{\phi^4}A_o(x)$$
$$+90\frac{(\phi')^2}{\phi^5}A'_o(x)+60\frac{\phi'\phi''}{\phi^5}A_o(x)+15\frac{(\phi')^3}{\phi^6}A_o\,\,(x)\,)+\frac{1}{\lambda^4}\,\,(\,A_4(x)+6\frac{A'_3(x)}{\phi_k}$$
$$+15\frac{\phi'}{\phi^2}A_3(x)+15\frac{A''_2(x)}{\phi_k^2}+60\frac{\phi'}{\phi^3}A'_2(x)+(45\frac{(\phi')^2}{\phi^4}+20\frac{\phi''}{\phi^3})A_2(x)+20\frac{A^{(3)}{}_1(x)}{\phi_k^3}+$$
$$90\frac{\phi'}{\phi^4}A''_1(x)+60\frac{\phi''}{\phi^4}A'_1(x)+15\frac{\phi^{(3)}}{\phi^4}A_1(x)+90\frac{(\phi')^2}{\phi^5}A'_1(x)+60\frac{\phi'\phi''}{\phi^5}A_1(x)+$$
$$15\frac{(\phi')^3}{\phi^6}A_1\,\,(x)+15\frac{A^{(4)}{}_o(x)}{\phi_k^4}+60\frac{\phi'}{\phi^5}A_o^{(3)}(x)+60\frac{\phi''}{\phi^5}A''_o(x)+30\frac{\phi^{(3)}}{\phi^5}A'_o(x)$$
$$+6\frac{\phi^{(4)}}{\phi^5}A_o(x)+45\frac{(\phi')^2}{\phi^6}A''_o(x)+60\frac{\phi'\phi''}{\phi^6}A'_o(x)+15\frac{\phi'\phi^{(3)}}{\phi^6}A_o(x)$$
$$+10\,\,\frac{(\phi'')^2}{\phi^6}A_o(x)\,)+\frac{1}{\lambda^5}\,\,(\,A_5(x)+6\frac{A'_4(x)}{\phi_k}+15\frac{\phi'}{\phi^2}A_4(x)+15\frac{A''_3(x)}{\phi_k^2}$$
$$+60\frac{\phi'}{\phi^3}A'_3(x)+(45\frac{(\phi')^2}{\phi^4}+20\frac{\phi''}{\phi^3})A_3(x)+20\frac{A^{(3)}{}_2(x)}{\phi_k^3}+90\frac{\phi'}{\phi^4}A''_2(x)$$

$$+60\frac{\emptyset''}{\emptyset^4}A_2'(x)+15\frac{\emptyset^{(3)}}{\emptyset^4}A_2(x)+90\frac{(\emptyset')^2}{\emptyset^5}A_2'(x)+60\frac{\emptyset'\emptyset''}{\emptyset^5}A_2(x)+15\frac{A^{(4)}{}_1(x)}{\emptyset_k^4}$$
$$+60\frac{\emptyset'}{\emptyset^5}A_1^{(3)}(x)+60\frac{\emptyset''}{\emptyset^5}A_1''(x)+30\frac{\emptyset^{(3)}}{\emptyset^5}A_1'(x)+6\frac{\emptyset^{(4)}}{\emptyset^5}A_1(x)+6\frac{A^{(5)}{}_o(x)}{\emptyset_k^5}$$
$$+15\ \frac{(\emptyset'\ )^3}{\emptyset^6}A_2(x)+45\frac{(\emptyset')^2}{\emptyset^6}A_1''(x)+60\frac{\emptyset'\emptyset''}{\emptyset^6}A_1'(x)+15\frac{\emptyset'\emptyset^{(3)}}{\emptyset^6}A_1(x)$$
$$+10\ \frac{(\emptyset'')^2}{\emptyset^6}A_1(x)+15\frac{\emptyset'}{\emptyset^6}A_o^{(4)}(x)+20\frac{\emptyset''}{\emptyset^6}A_o^{(3)}(x)+15\frac{\emptyset^{(3)}}{\emptyset^6}A_o''(x)+6\frac{\emptyset^{(4)}}{\emptyset^6}A_o'(x)$$
$$+\frac{\emptyset^{(5)}}{\emptyset^6}A_o\ (x)\ )\ +\frac{1}{\lambda^6}\ (\ A_6(x)+6\frac{A'{}_5(x)}{\emptyset_k}+15\frac{\emptyset'}{\emptyset^2}A_5(x)+15\frac{A''{}_4(x)}{\emptyset_k^2}+60\frac{\emptyset'}{\emptyset^3}A_4'(x)+$$
$$+(45\frac{(\emptyset')^2}{\emptyset^4}+20\frac{\emptyset''}{\emptyset^3})A_4(x)+20\frac{A^{(3)}{}_3(x)}{\emptyset_k^3}+90\frac{\emptyset'}{\emptyset^4}A_3''(x)+60\frac{\emptyset''}{\emptyset^4}A_3'(x)+$$
$$15\frac{\emptyset^{(3)}}{\emptyset^4}A_3(x)+90\frac{(\emptyset')^2}{\emptyset^5}A_3'(x)+60\frac{\emptyset'\emptyset''}{\emptyset^5}A_3(x)+15\frac{A^{(4)}{}_2(x)}{\emptyset_k^4}+60\frac{\emptyset'}{\emptyset^5}A_2^{(3)}(x)+$$
$$60\frac{\emptyset''}{\emptyset^5}A_2''(x)+30\frac{\emptyset^{(3)}}{\emptyset^5}A_2'(x)+6\frac{\emptyset^{(4)}}{\emptyset^5}A_2(x)+6\frac{A_1^{(5)}(x)}{\emptyset^5}+15\frac{(\emptyset')^3}{\emptyset^6}A_3\ (x)+$$
$$45\frac{(\emptyset')^2}{\emptyset^6}A_2''(x)+60\frac{\emptyset'\emptyset''}{\emptyset^6}A_2'(x)+15\frac{\emptyset'\emptyset^{(3)}}{\emptyset^6}A_2(x)+10\ \frac{(\emptyset'')^2}{\emptyset^6}A_2(x)+15\frac{\emptyset'}{\emptyset^6}A_1^{(4)}(x)$$
$$+20\frac{\emptyset''}{\emptyset^6}A_1^{(3)}(x)+15\frac{\emptyset^{(3)}}{\emptyset^6}A_1''(x)+6\frac{\emptyset^{(4)}}{\emptyset^6}A_1'(x)+\frac{\emptyset^{(5)}}{\emptyset^6}A_1\ (x)+\frac{A_o^{(6)}(x)}{\emptyset^6}\ )$$
$$+O\left(\frac{1}{\lambda^7}\right)\ ]. \qquad (2.6)$$

Now we substitute the resulting equations (2.2) - (2.6) with equation (2.1) in equation (1.1), by long computations, we obtain that

$$A_o(x)=\ \rho(x)^{-\frac{5}{12}},$$
$$A_1(x)=A_o(x)\int_0^x\left[-\frac{5}{2}\frac{A_o''(t)}{\emptyset_k}-10\frac{\emptyset_k'}{\emptyset_k^2}A_o'(t)-\left(\frac{15}{2}\frac{(\emptyset_k')^2}{\emptyset_k^3}+\frac{10}{3}\frac{\emptyset_k''}{\emptyset_k^2}-\frac{p_4(t)}{6\emptyset_k}\right)A_o(t)\right]$$
$$\frac{dt}{A_o(t)},$$
$$A_2(x)=A_o(x)\int_0^x[-\frac{5}{2}\frac{A_1''(t)}{\emptyset_k}-10\frac{\emptyset_k'}{\emptyset_k^2}A_1'(t)-\left(\frac{15}{2}\frac{(\emptyset_k')^2}{\emptyset_k^3}+\frac{10}{3}\frac{\emptyset_k''}{\emptyset_k^2}-\frac{p_4(t)}{6\emptyset_k}\right)A_1(t)$$
$$-\frac{10}{3}\frac{A_o^3(t)}{\emptyset_k^2}-15\frac{\emptyset_k'}{\emptyset_k^3}A_o''(t)-\left(10\frac{\emptyset_k''}{\emptyset_k^3}+15\frac{(\emptyset_k')^2}{\emptyset_k^4}-\frac{2}{3}\frac{p_4(t)}{\emptyset_k^2}\right)A_o'(t)-(\frac{5}{2}\frac{(\emptyset_k{}^{(3)}}{\emptyset_k^3}$$
$$+10\frac{\emptyset_k'\emptyset_k''}{\emptyset_k^4}+\frac{5}{2}\frac{(\emptyset_k')^3}{\emptyset_k^5}-\frac{\emptyset_k'}{\emptyset_k^3}p_4(t))\ A_o(t)\ ]\ \frac{dt}{A_o(t)},$$

$$A_3(x)=A_o(x)\int_0^x[-\frac{5}{2}\frac{A_2''(t)}{\emptyset_k}-10\frac{\emptyset_k'}{\emptyset_k^2}A_2'(t)-\left(\frac{15}{2}\frac{(\emptyset_k')^2}{\emptyset_k^3}+\frac{10}{3}\frac{\emptyset_k''}{\emptyset_k^2}-\frac{p_4(t)}{6\emptyset_k}\right)A_2(t)$$
$$-\frac{10}{3}\frac{A_1^3(t)}{\emptyset_k^2}-15\frac{\emptyset_k'}{\emptyset_k^3}A_1''(t)-\left(10\frac{\emptyset_k''}{\emptyset_k^3}+15\frac{(\emptyset_k')^2}{\emptyset_k^4}-\frac{2}{3}\frac{p_4(t)}{\emptyset_k^2}\right)A_1'(t)-(\frac{5}{2}\frac{\emptyset_k{}^{(3)}}{\emptyset_k^3}$$
$$+10\frac{\emptyset_k'\emptyset_k''}{\emptyset_k^4}+\frac{5}{2}\frac{(\emptyset_k')^3}{\emptyset_k^5}-\frac{\emptyset_k'}{\emptyset_k^3}p_4(t)\ )\ A_1(t)-\ \frac{5}{2}\frac{A_o^{(4)}(t)}{\emptyset_k^3}-10\frac{\emptyset_k'}{\emptyset_k^4}A_o^{(3)}(t)$$

$$-\left(10\frac{\phi_k''}{\phi_k^4}+\frac{15}{2}\frac{(\phi_k')^2}{\phi_k^5}-\frac{p_4(t)}{\phi_k^3}\right)A_o''(t)-\left(5\frac{\phi_k^{(3)}}{\phi_k^4}+10\frac{\phi_k'\phi_k''}{\phi_k^5}-2\frac{\phi_k'}{\phi_k^4}p_4(t)\right)A_o'(t)$$
$$-(\frac{\phi_k^{(4)}}{\phi_k^4}+\frac{5}{2}\frac{\phi_k'\phi_k^{(3)}}{\phi_k^5}+\frac{5}{3}\frac{(\phi_k'')^2}{\phi_k^5}-\frac{1}{2}\frac{(\phi_k')^2}{\phi_k^5}p_4(t)--\frac{2}{3}\frac{\phi_k''}{\phi_k^4}p_4(t)-\frac{p_3(t)}{6\phi_k^3})\,A_o(t)]$$
$$\frac{dt}{A_o(t)},$$

$$A_4(x)=A_o(x)\int_0^x[-\frac{5}{2}\frac{A_3''(t)}{\phi_k}-10\frac{\phi_k'}{\phi_k^2}A_3'(t)-\left(\frac{15}{2}\frac{(\phi_k')^2}{\phi_k^3}+\frac{10}{3}\frac{\phi_k''}{\phi_k^2}-\frac{p_4(t)}{6\phi_k}\right)A_3(t)$$
$$-\frac{10}{3}\frac{A_2^{(3)}(t)}{\phi_k^2}-15\frac{\phi_k'}{\phi_k^3}A_2''(t)-\left(10\frac{\phi_k''}{\phi_k^3}+15\frac{(\phi_k')^2}{\phi_k^4}-\frac{2}{3}\frac{p_4(t)}{\phi_k^2}\right)A_2'(t)-(\frac{5}{2}\frac{\phi_k^{(3)}}{\phi_k^3}$$
$$+10\frac{\phi_k'\phi_k''}{\phi_k^4}+\frac{5}{2}\frac{(\phi_k')^3}{\phi_k^5}-\frac{\phi_k'}{\phi_k^3}p_4(t)\;)\;A_2(t)-\frac{5}{2}\frac{A_1^{(4)}(t)}{\phi_k^3}-10\frac{\phi_k'}{\phi_k^4}A_1^{(3)}(t)-(10\frac{\phi_k''}{\phi_k^4}$$
$$+\frac{15}{2}\frac{(\phi_k')^2}{\phi_k^5}-\frac{p_4(t)}{\phi_k^3})A_1''(t)-\left(5\frac{\phi_k^{(3)}}{\phi_k^4}+10\frac{\phi_k'\phi_k''}{\phi_k^5}-2\frac{\phi_k'}{\phi_k^4}p_4(t)\right)A_1'(t)-(\frac{\phi_k^{(4)}}{\phi_k^4}$$
$$+\frac{5}{2}\frac{\phi_k'\phi_k^{(3)}}{\phi_k^5}+\frac{5}{3}\frac{(\phi_k'')^2}{\phi_k^5}-\frac{1}{2}\frac{(\phi_k')^2}{\phi_k^5}p_4(t)-\frac{2}{3}\frac{\phi_k''}{\phi_k^4}p_4(t)-\frac{p_3(t)}{6\phi_k^3})\;A_1(t)-\frac{A_o^{(5)}(t)}{\phi_k^4}$$
$$-\frac{5}{2}\frac{\phi_k'}{\phi_k^5}A_o^{(4)}(t)-\left(\frac{10}{3}\frac{\phi_k''}{\phi_k^5}-\frac{2}{3}\frac{p_4(t)}{\phi_k^4}\right)A_o^{(3)}(t)-\left(\frac{5}{2}\frac{\phi_k^{(3)}}{\phi_k^5}-\frac{\phi_k'}{\phi_k^5}p_4(t)\right)A_o''(t)-(\frac{\phi_k^{(4)}}{\phi_k^5}$$
$$-\frac{2}{3}\frac{\phi_k''}{\phi_k^5}p_4(t)-\frac{1}{3}\frac{p_3(t)}{\phi_k^4})A_o'(t)-(\frac{\phi_k^{(5)}}{6\phi_k^5}-\frac{\phi_k^{(3)}}{6\phi_k^5}p_4(t)-\frac{\phi_k'}{6\phi_k^5}p_3(t)-\frac{p_2(t)}{6\phi_k^4})\;A_o(t)]$$
$$\frac{dt}{A_o(t)},$$

$$A_5(x)=A_o(x)\int_0^x[-\frac{5}{2}\frac{A_4''(t)}{\phi_k}-10\frac{\phi_k'}{\phi_k^2}A_4'(t)-\left(\frac{15}{2}\frac{(\phi_k')^2}{\phi_k^3}+\frac{10}{3}\frac{\phi_k''}{\phi_k^2}-\frac{p_4(t)}{6\phi_k}\right)A_4(t)$$
$$-\frac{10}{3}\frac{A_3^{(3)}(t)}{\phi_k^2}-15\frac{\phi_k'}{\phi_k^3}A_3''(t)-\left(10\frac{\phi_k''}{\phi_k^3}+15\frac{(\phi_k')^2}{\phi_k^4}-\frac{2}{3}\frac{p_4(t)}{\phi_k^2}\right)A_3'(t)-(\frac{5}{2}\frac{\phi_k^{(3)}}{\phi_k^3}$$
$$+10\frac{\phi_k'\phi_k''}{\phi_k^4}+\frac{5}{2}\frac{(\phi_k')^3}{\phi_k^5}-\frac{\phi_k'}{\phi_k^3}p_4(t)\;)\;A_3(t)-\frac{5}{2}\frac{A_2^{(4)}(t)}{\phi_k^3}-10\frac{\phi_k'}{\phi_k^4}A_2^{(3)}(t)-(10\frac{\phi_k''}{\phi_k^4}$$
$$+\frac{15}{2}\frac{(\phi_k')^2}{\phi_k^5}-\frac{p_4(t)}{\phi_k^3})A_2''(t)-\left(5\frac{\phi_k^{(3)}}{\phi_k^4}+10\frac{\phi_k'\phi_k''}{\phi_k^5}-2\frac{\phi_k'}{\phi_k^4}p_4(t)\right)A_2'(t)-(\frac{\phi_k^{(4)}}{\phi_k^4}$$
$$+\frac{5}{2}\frac{\phi_k'\phi_k^{(3)}}{\phi_k^5}+\frac{5}{3}\frac{(\phi_k'')^2}{\phi_k^5}-\frac{1}{2}\frac{(\phi_k')^2}{\phi_k^5}p_4(t)-\frac{2}{3}\frac{\phi_k''}{\phi_k^4}p_4(t)-\frac{p_3(t)}{6\phi_k^3})\;A_2(t)-\frac{A_1^{(5)}(t)}{\phi_k^4}$$
$$-\frac{5}{2}\frac{\phi_k'}{\phi_k^5}A_1^{(4)}(t)-\left(\frac{10}{3}\frac{\phi_k''}{\phi_k^5}-\frac{2}{3}\frac{p_4(t)}{\phi_k^4}\right)A_1^{(3)}(t)-\left(\frac{5}{2}\frac{\phi_k^{(3)}}{\phi_k^5}-\frac{\phi_k'}{\phi_k^5}p_4(t)\right)A_1''(t)-(\frac{\phi_k^{(4)}}{\phi_k^5}$$
$$-\frac{2}{3}\frac{\phi_k''}{\phi_k^5}p_4(t)-\frac{1}{3}\frac{p_3(t)}{\phi_k^4})A_1'(t)-\left(\frac{\phi_k^{(5)}}{6\phi_k^5}-\frac{\phi_k^{(3)}}{6\phi_k^5}p_4(t)-\frac{\phi_k'}{6\phi_k^5}p_3(t)-\frac{p_2(t)}{6\phi_k^4}\right)A_1(t)$$

$$-\frac{A_o^{(6)}(t)}{6\phi_k^5}+\frac{p_4(t)}{6\phi_k^5}A_o^{(4)}(t)+\frac{p_3(t)}{6\phi_k^5}A_o''(t)+\frac{p_2(t)}{6\phi_k^5}A_o'(t)+\frac{p_1(t)}{6\phi_k^5}A_o(t)]\ \frac{dt}{A_o(t)},$$

$$A_6(x)=A_o(x)\int_0^x[-\frac{5}{2}\frac{A_5''(t)}{\phi_k}-10\frac{\phi_k'}{\phi_k^2}A_5'(t)-\left(\frac{15}{2}\frac{(\phi_k')^2}{\phi_k^3}+\frac{10}{3}\frac{\phi_k''}{\phi_k^2}-\frac{p_4(t)}{6\phi_k}\right)A_5(t)$$
$$-\frac{10}{3}\frac{A_4^{(3)}(t)}{\phi_k^2}-15\frac{\phi_k'}{\phi_k^3}A_4''(t)-\left(10\frac{\phi_k''}{\phi_k^3}+15\frac{(\phi_k')^2}{\phi_k^4}-\frac{2}{3}\frac{p_4(t)}{\phi_k^2}\right)A_4'(t)-(\frac{5}{2}\frac{{\phi_k}^{(3)}}{\phi_k^3}$$
$$+10\frac{\phi_k'\phi_k''}{\phi_k^4}+\frac{5}{2}\frac{(\phi_k')^3}{\phi_k^5}-\frac{\phi_k'}{\phi_k^3}p_4(t)\ )\ A_4(t)-\frac{5}{2}\frac{A_3^{(4)}(t)}{\phi_k^3}-10\frac{\phi_k'}{\phi_k^4}A_3^{(3)}(t)-(10\frac{\phi_k''}{\phi_k^4}$$
$$+\frac{15}{2}\frac{(\phi_k')^2}{\phi_k^5}-\frac{p_4(t)}{\phi_k^3})A_3''(t)-\left(5\frac{{\phi_k}^{(3)}}{\phi_k^4}+10\frac{\phi_k'\phi_k''}{\phi_k^5}-2\frac{\phi_k'}{\phi_k^4}p_4(t)\right)A_3'(t)-(\frac{{\phi_k}^{(4)}}{\phi_k^4}$$
$$+\frac{5}{2}\frac{\phi_k'\phi_k^{(3)}}{\phi_k^5}+\frac{5}{3}\frac{(\phi_k'')^2}{\phi_k^5}-\frac{1}{2}\frac{(\phi_k')^2}{\phi_k^5}p_4(t)-\frac{2}{3}\frac{\phi_k''}{\phi_k^4}p_4(t)-\frac{p_3(t)}{6\phi_k^3}\ )\ A_3(t)-\frac{A_2^{(5)}(t)}{\phi_k^4}$$
$$-\frac{5}{2}\frac{\phi_k'}{\phi_k^5}A_2^{(4)}(t)-\left(\frac{10}{3}\frac{\phi_k''}{\phi_k^5}-\frac{2}{3}\frac{p_4(t)}{\phi_k^4}\right)A_2^{(3)}(t)-\left(\frac{5}{2}\frac{\phi_k^{(3)}}{\phi_k^5}-\frac{\phi_k'}{\phi_k^5}p_4(t)\right)A_2''(t)-(\frac{{\phi_k}^{(4)}}{\phi_k^5}$$
$$-\frac{2}{3}\frac{\phi_k''}{\phi_k^5}p_4(t)-\frac{1}{3}\frac{p_3(t)}{\phi_k^4})A_2'(t)-\left(\frac{{\phi_k}^{(5)}}{6\phi_k^5}-\frac{\phi_k^{(3)}}{6\phi_k^5}p_4(t)-\frac{\phi_k'}{6\phi_k^5}p_3(t)-\frac{p_2(t)}{6\phi_k^4}\right)A_2(t)$$
$$-\frac{A_1^{(6)}(t)}{6\phi_k^5}+\frac{p_4(t)}{6\phi_k^5}A_1^{(4)}(t)+\frac{p_3(t)}{6\phi_k^5}A_1''(t)+\frac{p_2(t)}{6\phi_k^5}A_1'(t)+\frac{p_1(t)}{6\phi_k^5}A_1(t)]\ \frac{dt}{A_o(t)}.$$

Hence the proof is completed.

3. ASYMPTOTIC BEHAVIORS OF EIGENVALUES TO THE PROBLEM S_1 IN IRREGULAR CASE

The aim of this section is to study the asymptotic behaviors of eigenvalues of the problem S_1 by the following theorem:

Theorem 3.1:
Asymptotic behavior of eigenvalues for sufficiently large λ of the spectral boundary value problem S_1 in the case of irregular and in the sector T_k has the form:

$$\lambda_m=\frac{1}{I}\big(m\pi-if(\lambda)\big)+O\left(\frac{1}{\lambda}\right),$$

And in the sector $\overline{T_k}$ asymptotic behavior of spectral has the form:

$$\lambda_m=\frac{1}{I}\big(m\pi-if(\lambda)\big)+O\left(\frac{1}{\lambda}\right),$$

where $I=\int_0^a\rho(t)dt$ and $f(\lambda)=\frac{1}{2}\ Ln\ \frac{k_2}{3k_1+k_1e^{-i\lambda I}}$.

Proof:
Consider the equation of $y_k(x,\lambda)=e^{\lambda\int_0^x\phi_k(t)dt}\left[A_o(x)+O\left(\frac{1}{\lambda}\right)\right]$

Or

$y_k(x,\lambda)=e^{i\lambda w_k\int_0^x\sqrt[6]{\rho(t)}\,dt}\left[\rho(x)^{-\frac{5}{12}}+O\left(\frac{1}{\lambda}\right)\right]$. (Using equation (1.2))

The three successive derivatives of this equation with respect to x are:

$$y_k'(x,\lambda) = (i\,\lambda\, w_k)\, e^{i\lambda w_k \int_0^x \sqrt[6]{\rho(t)}\,dt}\left[\rho(x)^{-\frac{5}{12}} + O\left(\frac{1}{\lambda}\right)\right],$$

$$y_k''(x,\lambda) = (i\,\lambda\, w_k)^2\, e^{i\lambda w_k \int_0^x \sqrt[6]{\rho(t)}\,dt}\left[\rho(x)^{-\frac{5}{12}} + O\left(\frac{1}{\lambda}\right)\right],$$

$$y_k^{(3)}(x,\lambda) = (i\,\lambda\, w_k)^3\, e^{i\lambda w_k \int_0^x \sqrt[6]{\rho(t)}\,dt}\left[\rho(x)^{-\frac{5}{12}} + O\left(\frac{1}{\lambda}\right)\right],$$

Now consider the characteristic determinant $\Delta(\lambda) = \left|U_j(y_k)\right|_{k,j=0,1,2,3,4,5}$ (3.1)

Using the given boundary conditions in our spectral boundary value problem S_1, and substituting all expressions into characteristic determinant $\Delta(\lambda)(3.1)$, we get that

$$\Delta(\lambda) = \begin{vmatrix} \frac{1}{\sqrt[12]{(\rho(0))^5}} & \frac{1}{\sqrt[12]{(\rho(0))^5}} & \frac{1}{\sqrt[12]{(\rho(0))^5}} & \frac{1}{\sqrt[12]{(\rho(0))^5}} & \frac{1}{\sqrt[12]{(\rho(0))^5}} & \frac{1}{\sqrt[12]{(\rho(0))^5}} \\ \frac{i\lambda w_o}{\sqrt[4]{\rho(0)}} & \frac{i\lambda w_1}{\sqrt[4]{\rho(0)}} & \frac{i\lambda w_2}{\sqrt[4]{\rho(0)}} & \frac{i\lambda w_3}{\sqrt[4]{\rho(0)}} & \frac{i\lambda w_4}{\sqrt[4]{\rho(0)}} & \frac{i\lambda w_5}{\sqrt[4]{\rho(0)}} \\ \frac{-\lambda^2 w_o^2}{\sqrt[12]{\rho(0)}} & \frac{-\lambda^2 w_1^2}{\sqrt[12]{\rho(0)}} & \frac{-\lambda^2 w_2^2}{\sqrt[12]{\rho(0)}} & \frac{-\lambda^2 w_3^2}{\sqrt[12]{\rho(0)}} & \frac{-\lambda^2 w_4^2}{\sqrt[12]{\rho(0)}} & \frac{-\lambda^2 w_5^2}{\sqrt[12]{\rho(0)}} \\ -i\lambda^3 w_o^3 \sqrt[12]{\rho(0)} & -i\lambda^3 w_1^3 \sqrt[12]{\rho(0)} & -i\lambda^3 w_2^3 \sqrt[12]{\rho(0)} & -i\lambda^3 w_3^3 \sqrt[12]{\rho(0)} & -i\lambda^3 w_4^3 \sqrt[12]{\rho(0)} & -i\lambda^3 w_5^3 \sqrt[12]{\rho(0)} \\ \frac{i\lambda^2 w_o e^{i\lambda\ w_o I}}{\sqrt[4]{\rho(a)}} & \frac{i\lambda^2 w_1 e^{i\lambda\ w_1 I}}{\sqrt[4]{\rho(a)}} & \frac{i\lambda^2 w_2 e^{i\lambda\ w_2 I}}{\sqrt[4]{\rho(a)}} & \frac{i\lambda^2 w_3 e^{i\lambda\ w_3 I}}{\sqrt[4]{\rho(a)}} & \frac{i\lambda^2 w_4 e^{i\lambda\ w_4 I}}{\sqrt[4]{\rho(a)}} & \frac{i\lambda^2 w_5 e^{i\lambda\ w_5 I}}{\sqrt[4]{\rho(a)}} \\ \frac{e^{i\lambda\ w_o I}}{\sqrt[12]{(\rho(0))^5}} & \frac{e^{i\lambda\ w_1 I}}{\sqrt[12]{(\rho(0))^5}} & \frac{e^{i\lambda\ w_2 I}}{\sqrt[12]{(\rho(0))^5}} & \frac{e^{i\lambda\ w_3 I}}{\sqrt[12]{(\rho(0))^5}} & \frac{e^{i\lambda\ w_4 I}}{\sqrt[12]{(\rho(0))^5}} & \frac{e^{i\lambda\ w_5 I}}{\sqrt[12]{(\rho(0))^5}} \end{vmatrix}$$

We now turn to finding the zeros of the determinant $\Delta(\lambda)$in the irregular case. Through the properties of the determinant we expand the above determinant $\Delta(\lambda)$, andafter simplifying algebraic steps, we obtain:

$$\left(1+i\sqrt{3}\right)e^{-\sqrt{3}\,\lambda\, I} + \left(1-i\sqrt{3}\right) + \left(\left(3-i3\sqrt{3}\right) + \left((3+i3\sqrt{3}\right)e^{i\,\lambda\, I} + \left(1-i\sqrt{3}\right)e^{i2\,\lambda\, I}\right)$$
$$e^{(i\lambda\, I-\sqrt{3}\lambda\, I)} + \left(\left(3+i3\sqrt{3}\right) + \left(3-i3\sqrt{3}\right)e^{i\,\lambda\, I} + \left(1+i\sqrt{3}\right)e^{i2\,\lambda\, I}\right)e^{i\,\lambda\, I} -$$

$$2\,\left(\left(e^{-\sqrt{3}\,\lambda\, I} + e^{\sqrt{3}\,\lambda\, I}\right)e^{i\,\lambda\, I} + 3\left(1+e^{i\,2\lambda\, I}\right)\right)e^{\frac{1}{2}i\,\lambda\, I-\frac{\sqrt{3}}{2}\lambda\, I} + O\left(\frac{1}{\lambda}\right) = 0.$$

Let $\left(1-i\sqrt{3}\right) = k$ and $\left(1+i\sqrt{3}\right) = k_1$, then the above equations reduces to:

$$k_1 e^{-\sqrt{3}\,\lambda\, I} + k + \left(3k + 3k_1 e^{i\,\lambda\, I} + k e^{i2\,\lambda\, I}\right)e^{(i\lambda\, I-\sqrt{3}\lambda\, I)} + \left[3k_1 + 3k e^{i\,\lambda\, I} + k_1 e^{i2\,\lambda\, I}\right]e^{i\,\lambda\, I}$$
$$-2\,\left(\left(e^{-\sqrt{3}\,\lambda\, I} + e^{\sqrt{3}\,\lambda\, I}\right)e^{i\,\lambda\, I} + 3\left(1+e^{i\,2\lambda\, I}\right)\right)e^{\frac{1}{2}i\,\lambda\, I-\frac{\sqrt{3}}{2}\lambda\, I} + O\left(\frac{1}{\lambda}\right) = 0.$$

Or

$$k_1 e^{-\sqrt{3}\,\lambda\, I} + k + \left(3k + 3k_1 e^{i\,\lambda\, I} + k e^{i2\,\lambda\, I}\right)e^{(i-\sqrt{3})\,\lambda\, I} + \left(3k_1 + 3k e^{i\,\lambda\, I} + k_1 e^{i2\,\lambda\, I}\right)e^{i\,\lambda\, I}$$
$$-2\left(\left(e^{-\sqrt{3}\,\lambda\, I} + e^{\sqrt{3}\,\lambda\, I}\right)e^{i\,\lambda\, I} + 3\left(1+e^{i\,2\lambda\, I}\right)e^{-\frac{\sqrt{3}}{2}\lambda\, I}\right)e^{\frac{1}{2}i\,\lambda\, I} + O\left(\frac{1}{\lambda}\right) = 0.$$

By multiplying both sides by $e^{-i\,\lambda\, I}$, we conclude:

$$(k_1 e^{-\sqrt{3}\,\lambda\, I} + k + (3k + 3k_1 e^{i\,\lambda\, I} + k e^{i2\,\lambda\, I}) e^{(i-\sqrt{3})\,\lambda\, I})\, e^{-i\,\lambda\, I} +$$
$$e^{i\,\lambda\, I}\left(3k_1 + 3k e^{i\,\lambda\, I} + k_1 e^{i2\,\lambda\, I}\right) - 2\left(\left(e^{-\sqrt{3}\,\lambda\, I} + e^{\sqrt{3}\,\lambda\, I}\right)e^{i\,\lambda\, I} + 3\left(1 + e^{i\,2\lambda\, I}\right)e^{-\frac{\sqrt{3}}{2}\lambda\, I}\right)$$
$$e^{-\frac{1}{2} i\,\lambda\, I} + O\left(\frac{1}{\lambda}\right) = 0$$

In the last equation, if $\lambda \to \infty$, we deduce that:

$$3k_1 + 3k e^{i\,\lambda\, I} + k_1 e^{i2\,\lambda\, I} + O\left(\frac{1}{\lambda}\right) = 0 \;\; \to e^{i2\,\lambda\, I} = \frac{k_2}{3k + k_1 e^{-i\,\lambda\, I}} + O\left(\frac{1}{\lambda}\right),$$
where $k_2 = -3k_1$.

Taking the initial approximation $\lambda_o = \frac{m\pi}{I}$,and using the method of successive approximations [4] we obtain:

$$\lambda_m = \frac{1}{I}\left(m\pi - if(\lambda)\right) + O\left(\frac{1}{\lambda}\right),$$

where $f(\lambda) = \frac{1}{2} Ln\, \frac{k_2}{3k + k_1 e^{-i\,\lambda\, I}}$, $I = e^{\int_0^a \rho(t)dt}$, $m = N, N+1, N+2, \ldots$ (N is a natural number).

Thus in the case of irregular and in the sector T_k asymptotic behavior of spectrum has the form:

$\lambda_m = \frac{1}{I}\left(m\pi - if(\lambda)\right) + O\left(\frac{1}{\lambda}\right)$, where $m = N, N+1, N+2, \ldots$ (N is a natural number),

and in the sector $\overline{T_k}$

$\lambda_m = \frac{1}{I}\left(m\pi + if(\lambda)\right) + O\left(\frac{1}{\lambda}\right)$,where $m = N, N+1, N+2, \ldots$ (N is a natural number).

Thus the theorem is proved.

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