AN OVERVIEW ON THE BOX-PIERCE PORTMANTEAU TEST

Dakhmouche Meghlaoui [1,*] & Dakhmouche Farida Rosa [2]
[1]Mathematical Modelling and Simulation Laboratory, Department of Mathematics,
Mentouri University of Constantine, Route de Ain El Bey Constantine, 25000, Algeria.
[2] Mathematical Modelling and Simulation Laboratory, Department of English,
Mentouri University of Constantine, Route de Ain El Bey Constantine, 25000, Algeria.

ABSTRACT
The computation of the power of the portmanteau test and of its derivatives is not easy to perform because the distributions of the residuals $\hat{r}_k$ are not available under the alternative hypothesis. In order to quantify the effectiveness of this test, some specialists propose the computation of the expectation and of the variance of the statistic used. But this approach is not sufficient enough to have a clear idea about the power of the test. Therefore, the effectiveness of the portmanteau test and of its derivatives remains an unsolved problem. In this contribution, we propose to use the approximation of Toeplitz-Szegö of the likelihood of a process and to extend it to hypotheses testing problems. Thereby, we have attempted to resume some contributions on the continuous case that we have tried to innovate with the introduction of the Toeplitz-Szegö approximation. If we can determine the Neyman-Pearson test in exponential situations, then this test will be equivalent to the conventional tests such as the portmanteau test. Our main aim is to implement a definite form of Szegö's theorem to obtain some approximation of the Wiener-Hopf equations which are easy to handle in the continuous case, and to extend the results of Dzaparidze and Pisarenko to this case.

Keywords: *Time series, Stochastic processes, Portmanteau test, Toeplitz-Szegö approximation, Likelihood ratio*

1. INTRODUCTION

The Box-Pierce portmanteau test [1,4,5] was built from the following simple idea: if the proposed model was the right one, then the empirical residues $\left(\hat{Z}_t\right)_{t\geq 1}$ computed using the fitted model would be a realization of a sequence $\left(Z_t\right)_{t\geq 1}$ of independent normal random variables with zero mean and variance $\sigma^2_{Z_t}$. It is obvious that the autocorrelations $\hat{r}_k$ of the sequence $\left(\hat{Z}_t\right)_{t\geq 1}$ must be zero for $k \neq 0$. So, the Box-Pierce statistic $S_{BP} = N\sum_k \hat{r}_k^2$ makes the most sense to build a test for the hypothesis $H_0 : "\hat{r}_k = 0, \forall k \neq 0"$ against the hypothesis $H_1 : "\exists k_0 / \hat{r}_{k_0} \neq 0"$, with the following decision rule: H_0 is rejected if $S_{BP} \geq c$ and it is accepted otherwise, where c is an alpha quantile of a χ^2 .

The performance of the Box-Pierce test seems unsatisfactory. Therefore improvements of the portmanteau test have been proposed by many specialists. Ljung and Box [12,13], Li and McLeod [11] and Monti [14] have attempted to revise the normalization constant in the statistic $S_{BP} = N\sum_k \hat{r}_k^2$. Furthermore, Dufour and Roy's [8] contribution is a generalization of the initial portmanteau statistic. Starting from a quadratic form $S = (r-\upsilon)\Sigma^{-1}\,{}^{\tau}(r-\upsilon)$ where r is the vector of the autocorrelations of the residuals, υ is a constant vector and Σ is a positive definite matrix, Dufour and Roy show that Box-Pearce statistic and its derivatives are special cases of their own. Moreover, Kwan and Sim have proposed four types of portmanteau statistics using stabilizing transformations of the variances. Finally, Peña and Rodriguez [15,16] introduce a statistic based on the determinant of the matrix of autocorrelations of the residuals with some variants. Apart from the Ljung-Box test, all the other proposed tests are based on statistics that require more calculations without correcting the alleged deficiencies in the Box-Pierce test. Nevertheless, the portmanteau test is a statistical tool widely used. The problem of the portmanteau test lies in the fact that the null hypothesis is well specified, but the alternative is very vague or poorly specified. The idea which consists to combine the Wiener-Hopf equation of the likelihood ratio with the theory of Toeplitz' operators and Szegö's theorem can lead us to construct a statistical test of Neyman and Pearson type. In the sequel we will mention some results which are useful in the remainder of our paper.

2. THE WIENER-HOPF EQUATION OF THE LIKELIHOOD RATIO

In this section we will focus especially on the equation of the likelihood ratio of two Gaussian processes associated with Gaussian measures that are absolutely continuous with respect to each other.

Let P_0 and P_1 be two Gaussian distributions centered, stationary and equivalent. Let Δ_N be a finite subset of the interval [0,T] and let $X_t, t \in \Delta_N$ be a sequence of realizations of the process X_t. It is obvious that if $P_0 \sim P_1$ then their restrictions that are P_0^N and P_1^N of P_0 and P_1 over the σ- algebra generated by $X_t, t \in \Delta_N$ are such that $P_0^N \sim P_1^N$. Then the likelihood ratio of P_0^N and P_1^N is defined as:

$$\frac{dP_0^N}{dP_1^N} = p_N$$

The logarithm of the likelihood ratio of two Gaussian processes [2,9,18] is defined as:

$$\ln p_N = -\frac{1}{2}\ln\det\left(B_{1,\Delta_N} B^{-1}{}_{0,\Delta_N}\right) + {}^{\tau}X^{\Delta_N}\left(B^{-1}{}_{1,\Delta_N} - B^{-1}{}_{0,\Delta_N}\right)X^{\Delta_N} \qquad (1)$$

where X^{Δ_N} is the vector $X_t, t \in \Delta_N$ and B_{i,Δ_N} is the covariance matrix of X^{Δ_N} under the distribution $P_i,\ i = 0,1$.

Let $d(s,t),\ s,t \in \Delta_N$ be the general term of the matrix $B^{-1}{}_{1,\Delta_N} - B^{-1}{}_{0,\Delta_N}$. We put:

$$\varphi_N(\lambda,\mu) = \sum_{s,t} d(s,t)\exp\{i(\lambda s - \mu t)\} \qquad (2)$$

Substituting X_t by its spectral representation in (1) [2], we have:

$$\ln p_N = -\frac{1}{2}\ln\det\left(B_{1,\Delta_N} B^{-1}{}_{0,\Delta_N}\right) + \iint \varphi_N(\lambda,\mu) F_0(d\lambda) F_1(d\mu)$$

where $F_0(d\lambda)$ and $F_1(d\mu)$ are spectral measures respectively associated with P_0 and P_1.

Moreover, according to the relation (2) we have [2,6,7]:

$$\iint \exp\{-i(\lambda s - \mu t)\}\varphi_N(\lambda,\mu) dF_0(\lambda) dF_1(\mu) = \iint \sum_{s',t'} d(s',t')\exp\{i(\lambda s' - \mu t') - i(\lambda s - \mu t)\} dF_0(\lambda) dF_1(\mu)$$

$$= \sum_{s',t'} B_{0,(s'-s)}\left(B^{-1}_{1,(s',t')} - B^{-1}_{0,(s',t')}\right) B_{0,(t'-t)} = b(s,t)$$

Therefore, we can consider that φ_N is a solution of:

$$\iint \exp\{-i(\lambda s - \mu t)\}\varphi_N(\lambda,\mu) dF_0(\lambda) dF_1(\mu) = b(s,t) \quad s,t \in \Delta_N \qquad (3)$$

The equation (3) is a Wiener-Hopf equation of finite type [2,3].

Pisarenko [17] has treated the case of a stationary centred Gaussian process $X_t,\ t \in [0,T]$. He has considered the σ- algebra of parts of Ω on which he defines two Gaussian probabilities P_0 and P_1. Let $R_i(s,t) = E_{P_i}(X_s X_t)$ be the autocorrelation function of X_t for $i = 0,1$. The stationarity of X_t insures us that $R_0(s,t) = R_0(s-t)$ and $R_1(s,t) = R_1(s-t)$, and assuming that P_0 and P_1 are equivalent, the author shows that $b(s,t) = R_0(s-t) - R_1(s-t)$ can be written as:

$$b(s,t) = \iint \exp\{-i(\lambda s - \mu t)\}\varphi(\lambda,\mu) f_0(\lambda) f_1(\mu) d\lambda d\mu \quad 0 \le s,t \le T \qquad (4)$$

The equation (4) is an integral equation of the Wiener-Hopf type of the function $\varphi(\lambda,\mu)$. The solution of this type of equations considered as an element of $L^2_{[0,T]}$ is often unique [7,9]. But the author does not find out explicitly a statistic on which one can build a test.

Moreover, Rozanov [18] has proposed a solution of the Wiener-Hopf equation of the likelihood ratio of two random processes of rational spectral densities [4,15].

Indeed, if $f_0(\lambda)$ and $f_1(\mu)$ are the densities of P_0 and P_1 with respect to the Lebesgue measure, we can always reduce the problem to the resolution of two one-dimensional equations by setting:

$$K_s(\mu) = \int_{\mathbb{R}} \exp(i\lambda s)\varphi(\lambda,\mu) f_0(\lambda) d\lambda$$

And

$$\int_{\mathbb{R}} \exp(i\mu t) K_s(\mu) f_1(\mu) d\mu = b(s-t)$$

In other words, one needs only know how to solve any equation like:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(i\lambda t)\varphi(\lambda)\left[\frac{Q(i\lambda)}{P(i\lambda)}\right]^2 d\lambda = x(t) \qquad 0\le t\le T$$

where the polynomial P of degree r, the polynomial Q of degree s and the function $x\in L^2$ are given. If x has derivatives of order $r-s$ in L^2, we set up the problem in the classical situation of the distribution theory. In the sequel, we will resume some works on the continuous case that we will try to innovate with the introduction of the Toeplitz-Szegö's approximation of the likelihood. Our objective is to apply some form of Szegö's theorem [10] to the continuous case to achieve approximations of the Wiener-Hopf equations easy to handle [2,9].

3. RESULTS

In this last section we adopt the Toeplitz-Szegö's approximation [2] of the likelihood of a random process. The advantage of this representation is to relocate the study of random processes in the elegant setting of harmonic analysis. The main objective is to apply the form of the theorem of Szegö stated by LeKAC [2,3,6] in the continuous case to obtain approximations easy to handle.

Let F and G be two stationary and centered Gaussian measures on $\mathbb{R}^{\mathbb{Z}}$. We note by X^N the observation vector $(X_1, X_2, ..., X_N)$ of a random process X_t. Let us assume that for all N large enough, F and G admit densities with respect to the Lebesgue measure on $\mathbb{R}^N$. To insure this, it is necessary and sufficient that F and G have spectral densities f and g such that $\ln f$ and $\ln g$ belong to L_T^1.

In what follows we set $L_N(t)=\frac{1}{N}\ln\varphi_N(t)$ where $\varphi_N(t)=E_F\exp\left(t\ln\frac{dG}{dF}\right)$ $t\in[0,1]$, is the Laplace transform of the likelihood ratio of F and G. Let $(T_N f)_N$ and $(T_N g)_N$ be the sequences of Toeplitz matrices [2, 9, 10] associated with f and g respectively. So we have:

$$\ln\frac{dG}{dF}=\frac{1}{2}\ln\det\left(T_N^{-1}gT_N f\right)-\frac{1}{2}{}^tX^N\left(T_N^{-1}f-T_N^{-1}g\right)X^N$$

Moreover, as the covariance of $T_N^{-\frac{1}{2}}fX^N$ is the identity matrix I_N, then denoting by $\left(\mu_i^N\right)_{i=1,2,...,N}$ the eigenvalues of $A_N=I_N-T_N^{-\frac{1}{2}}(f)T_N^{-1}(g)T_N^{\frac{1}{2}}(f)$, we get:

$$NL_N(t)=\frac{t}{2}\ln\det\left(T_N^{-1}(g)T_N(f)\right)-\frac{1}{2}\sum_{i=1}^{N}\ln\left(1-\mu_i^N t\right)$$

For N fixed, and for $|t|<\frac{1}{\sup_{1\le i\le N}\left|\mu_i^N\right|}$, we have:

$$\sum_{i=1}^{N}\ln\left(1-\mu_i^N t\right)=-\sum_{p=1}^{\infty}\frac{t^p}{p}Tr\left(A_N^p\right)$$

$$=-\sum_{p=1}^{\infty}(-1)^l\frac{1}{l!}Tr\left(T_N(f)T_N^{-1}(g)\right)^l\left(\frac{t}{1-t}\right)^l+N\ln(1-t)$$

Let $z\in\mathbb{C}$. As $T_N(f)$ and $T_N(g)$ are regular, there is an open neighbourhood of 0 where $T_N(zf+g)$ is regular and therefore, according to a result in [2, 6], we can show:

$$\frac{d}{dz}\ln\det\left(T_N(zf+g)\right)=Tr\left(T_N(f)T_N^{-1}(zf+g)\right)$$

and

$$\frac{d}{dz^l}\ln\det\left(T_N(zf+g)\right)\Big|_{z=0}=(-1)^{l-1}(l-1)!\,Tr\left(T_N(f)T_N^{-1}(zf+g)\right)^l$$

Hence

$$\sum_{i=1}^{N}\ln\left(1-\mu_i^N t\right)=\sum_{l=1}^{\infty}\frac{1}{l!}\ln\det\left(T_N(zf+g)\right)\Big|_{z=0}^{(l)}\left(\frac{t}{1-t}\right)^l+N\ln(1-t)$$

From the analyticity of the function $z \to \ln\det\left(T_N\left(zf+g\right)\right)$ in the neighbourhood of 0, we deduce:

$$\sum_{i=1}^{N}\ln\left(1-\mu_i^N t\right)=\ln\det\left(\frac{t}{1-t}f+g\right).$$

Thus

$$NL_N\left(t\right)=\frac{t}{2}\ln\det\left(T_N^{-1}(g)T_N(f)\right)-\frac{1}{2}\left[\ln\det\left(T_N\left(\frac{t}{1-t}f+g\right)\right)-\ln\left(\det T_N\left(g\right)\right)+N\ln\left(1-t\right)\right]$$

Finally

$$NL_N\left(t\right)=\frac{1}{2}\left[t\ln\det\left(T_N(f)\right)+\left(1-t\right)\ln\det\left(T_N\left(g\right)\right)-\ln\det T_N\left(tf+\left(1-t\right)g\right)\right]$$

This formula is valid in the neighbourhood of 0.
The function $z\to\ln\det\left(zf+\left(1-z\right)g\right)$ is analytic in a $\mathbb{C}$-neighbourhood of any point of $\left]0,1\right[$, because $\ln\left(tf+\left(1-t\right)g\right)\in L^1$ by convexity, so $T_N\left(tf+\left(1-t\right)g\right)$ is regular for $t\in\left[0,1\right]$ as stated in ([7]). Consequently, by analytic extension, the previous formula is valid for $t\in\left]0,1\right[$ and, by continuity, it is valid for $t\in\left[0,1\right]$.
Using Szegö's theorem, if $\ln(f)\in L^1$, then

$$\frac{1}{N}\ln\det\left(T_N(f)\right)\to\frac{1}{2\pi}\int_{[0,T]}\ln f(\theta)d\theta$$

when $N\to+\infty$.

Proposition 3.1 If $\ln\left(f\right),\ \ln\left(g\right)\in L^1_{[0,T]}$ et si $t\in\left[0,1\right]$, then $L_N\left(t\right)\to L\left(t\right)$ when $N\to+\infty$ with

$$L\left(t\right)=\frac{1}{4\pi}\int_{-\pi}^{+\pi}\left[t\ln f(\theta)+\left(1-t\right)\ln g(\theta)-\ln\left(tf(\theta)+\left(1-t\right)g(\theta)\right)\right]d\theta$$

From this result, we can deduce as it is stated in [2,6] the Chernoff formula and the asymptotic properties of the Neyman-Pearson lemma.
In the continuous case, we proceed in the same way as developed above. We calculate the Laplace transform of the likelihood ratio. Thus, for N large enough, we can use the formula to determine the Chernoff bound for $P\left[\ln\left(\frac{dF}{dG}\right)>\lambda\right]$. The exact computation of the probability distribution of $\ln\left(\frac{dF}{dG}\right)$ is sophisticated since the function φ_N is a solution of the Wiener-Hopf equation. However, for sufficiently large N and X_t fairly regular, it seems reasonable to discretize the problem and to apply the previous method to the process of the type $\left(X_{\frac{kT}{N}}\right)_{k=1,2,\ldots,N}$. This is also what Pisarenko [17] and Rozanov [18] have attempted to achieve.

4. REFERENCES

[1]. M.A. Arranz, Portmanteau Test Statistics in Time Series, Tol-Project, Time Oriented Language http://www.tol-project.org/.
[2]. R Azencot, D. Dacunha-Castelle, "Séries d'Observations Irrégulières: Modélisation et Prévision", Masson, Paris (1984).
[3]. H. A. Baxter, Norm Inequality for a "Finite Section" Wiener-Hopf Equation, Illinois J. Math. **7**, 97-103 (1963).
[4]. G.E.P. Box, G.M. Jenkins, Time Series Analysis: Forecasting and Control. San Francisco, Holden-day;, San Francisco (1994).
[5]. G.E.P. Box, D.A. Pierce, Distribution of Residual Autocorrelations in the Autoregressive-Integrated Moving Average Time Series Models, J. Amer. Statist. Assoc. **65**, 1509–1526 (1970).
[6]. R.B. Davies, Asymptotic Inference in Stationary Time Series, Adv. Appl. Prob. **5**, 469-97 (1973).
[7]. N. Davies, C.M. Triggs, P. Newbold, Significance Levels of the Box-Pierce Portmanteau Statistic in Finite Samples, Biometrika, **64**, **3**, 517-22 (1977).
[8]. J.M. Dufour, R. Roy, Generalized Portmanteau Statistics and Tests of Randomness, J. Comm. Statist. Theory Methods, **15**, 2953-72 (1986).

[9]. K. Dzaparidze, “Parameter Estimation and Hypothesis Testing in Spectral Analysis of Stationary Time Series”, Springer-Verlag, Tokyo (1986).
[10]. U. Grenander, G. Szegö, “Toeplitz Forms and their Applications”, Univ. of Calif. Press, Berkeley (1968).
[11]. W. Li, A. McLeod, Distribution of the Residual Autocorrelations in Multivariate ARMA Time Series Models, J. Roy. Statist. Soc. Series B, **43**, 231–9 (1981).
[12]. G. Ljung, Diagnostic Testing of Univariate Time Series Models, Biometrika, **65**, 297–303 (1986).
[13]. G. Ljung, G.E.P. Box, On a Measure of Lack of Fit in Time Series Models, Biometrika, **65**, 297–303 (1978).
[14]. A.C. Monti, A Proposal for Residual Autocorrelation Test in Linear Models, Biometrika, **81**, 776–80 (1994).
[15]. D. Peña, J. Rodriguez, A Powerful Portmanteau Test of Lack of Fit for Time Series, J. J. Amer. Statist. Assoc. **97**, 601-10 (2002).
[16]. D. Peña, J. Rodriguez, The Log of the Determinant of the Autocorrelation Matrix for Testing the Goodness of Fit in Time Series, Journal of Statistical Planning and Inference (2004).
[17]. V.F. Pisarenko, On the Computation of the Likelihood Ratio for Gaussian Processes with a Rational Spectrum, Theory Prob. Applications, **10**, 299-303 (1965).
[18]. Y.A. Rosanov, On Gaussian Distribution Densities and Wiener-Hopf Integral Equations, Theory Prob. Applications, **11**, 152-60 (1966).