# NUMERICAL SOLUTION OF PRICING OF EUROPEAN CALL OPTION WITH STOCHASTIC VOLATILITY

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**ABSTRACT**
We propose a transformation that allows to build an explicit finite difference scheme for option pricing in stochastic volatility models. The scheme is second order in space and first order in time. We present conditions of positivity and monotonicity of the scheme. To test conditional stability results in the sense of von Neumann performing a Fourier analysis of the problem and follows the convergence of our scheme. We present some numerical experimental results for European call option pricing.

**Keywords:** Option pricing, explicit finite difference scheme, positivity, monotonicity.

## 1. INTRODUCTION
The central model of option pricing theory is the Black-Scholes model (1973), which shows that, without making assumptions about the preferences of investors, one can obtain an expression of the value of options that not directly dependent on the expected performance of the underlying stock or the option. This is achieved through dynamic hedging argument in a free market perfect arbitrage.
The assumptions of the Black-Scholes model form an ideal scenario, in which the continuous trading is possible, in perfect markets, in which the interest rate is constant risk free and the price of the underlying asset behaves like a geometric Brownian motion. However, some empirical studies have shown that these considerations are unrealistic and do not explain a significant impact on financial markets such as volatility changes.
In this direction, there are sophisticated models that incorporate more accurate volatility as a random variable that is set up as a second factor of risk in financial markets because not only the returns of assets are at risk.
This class of models known as stochastic volatility models. The most representative work in this regard is the model of Heston (1993). This model is based on a system of two coupled stochastic differential equations that represent the dynamic behavior of the underlying asset and the other dynamics of volatility and which are correlated Brownian motions. Following the description in Düring and Fournié (2012), in such systems can be represented as

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dZ_1 \tag{1}$$

$$dV_t = a(V_t)dt + b(V_t)dZ_2 \tag{2}$$

$$dZ_1(t)dZ_2(t) = \rho dt \tag{3}$$

where $\mu$ is the trend term of the asset and $a(V_t)$ and $b(V_t)$ are respectively the coefficients of the diffusion and trend of the stochastic volatility and $\rho$ is the correlation factor.
Similar arguments set in Black-Scholes (1973), allow to find the partial differential equation

$$F_t + \frac{1}{2}S^2 V F_{SS} + \rho b(V)\sqrt{V} S F_{SV} + \frac{1}{2}b(V)^2 F_{vv} + a(V)F_v + rSF_s - rF = 0 \tag{4}$$

Where r is the free risk interest rate.
Equation (4) has been solved for $S, V > 0$, $0 \le t \le T$ subject to the boundary conditions depending on the specific type of option.
In general, the model Heston when the coefficients are not constant, equation (4) must be solved numerically. Moreover, for the case where the option is the American type, must be solve a free boundary problem with a restriction for the early exercise constraint for the option price. Also for this problem has to resort to numerical approximations.

In the mathematical literature, there are many articles about numerical methods for option pricing, especially addressing the case of a single risk factor, also second-order finite difference methods and more recently, high order finite difference schemes. Other approaches include finite element, finite volume and spectral methods. (See, for example, Düring and Fournié (2012) and references therein).
Other finite difference approaches used are standard methods of low order (second order in space) for option pricing in stochastic volatility models. In D.Y. Tangman, A. Gopaul, and M. Bhuruth (2008) is considered a higher order compact scheme (HOC) for parabolic partial differential equations to discretize the quasi-linear Black-Scholes PDE in the numerical evaluation of European and American options. Also show that the system (HOC) with a grid stretching along the asset price dimension, gives approximate numerical solutions for European type options under stochastic volatility. In Rana and Ahmad (2011) proposes a finite difference scheme for option pricing with stochastic volatility incorporating a GARCH model in context of Indian financial market that is solved by the Crank-Nicolson method. Four division of type schemes Alternate Direction implicit (ADI): Douglas scheme, the Craig-Sneyd scheme, the modified Craig-Sneyd scheme, and the scheme Hundsdörfer-Verwer, each of which contains a free parameter, was proposed by K. J. In 't Hout and S. Foulon (2010) which develops a semi- discretization of Heston PDE, using finite difference schemes with nonuniform mesh, resulting in large systems stiff ordinary differential equations.
This paper presents an explicit finite difference scheme for option pricing models of European type with stochastic volatility. Though our presentation is focused on the Heston model can be easily adapted to other models with stochastic volatility. It proposes a transformation of the differential equation Heston which reduces the number of terms to obtain an approximation scheme for a second-order in the space and first-order in time. It also establish, positivity and monotonicity conditions for the numerical scheme. To test the results on conditional stability in the sense of von Neumann performing a Fourier analysis of the problem and the derivation of the convergence is conducted by the Lax-Richtmyer theorem.
The paper is organized as follows. In the first section, we will make a description of the model of Heston (1993) and the closed-form solution for the case of constant coefficients. The transformation of the partial differential equation in a simpler equation by introducing new independent variables is described in section 3. Section 4 presents the deduction of the numerical scheme, establishing the conditions of positivity and monotonicity, we analyze the stability and follows a result of . Numerical results of European call options and the error plots are presented in Section 5.

## 2. HESTON MODEL

For the development of this presentation we will focus at the Heston model. It is a stochastic volatility model: such a model assumes that the volatility of the asset is not constant, nor even deterministic, but follows a random process.
We begin by asuming that the spot asset at time $t$ folows the diffusion:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dZ_1(t) \tag{5}$$

where $Z_1(t)$ is a Wiener process. If the volatility folows an Ornstein Uhlenbeck process:

$$d\sqrt{V_t} = -\beta\sqrt{V_t}dt + \delta dZ_2(t) \tag{6}$$

then Ito's lemma shows that the variance $V_t$ folows the process:

$$dV_t = [\delta^2 - 2\beta V_t]dt + 2\delta\sqrt{V_t}dZ_2(t) \tag{7}$$

this can be written as

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}dZ_2(t) \tag{8}$$

All of this for $0 \le t \le T$ with $S_0, V_0 > 0$ and $\mu, k, \sigma$ and $\theta$ the drift, the mean reversion speed, the volatility of volatility and the long run mean of $V_t$ respectively and also $k = 2\beta$, $\theta = \dfrac{\delta^2}{2\beta}$ y $\sigma = 2\delta$.

$Z_2$ (t) has correlation $\rho$ with english $Z_1$ (t)

$$dZ_1(t)dZ_2(t) = \rho dt \tag{9}$$

The Heston model says that the value of any asset $F(S_t, V_t, t)$ must satisfy the partial differential equation :

$$\frac{\partial F}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 F}{\partial S^2} + \rho V\sigma S\frac{\partial^2 F}{\partial S\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2 F}{\partial V^2} + rS\frac{\partial F}{\partial S} + \left\{k(\theta - V) - \Lambda(S,V,t)\sigma\sqrt{V}\right\}\frac{\partial F}{\partial V} - rF = 0 \tag{10}$$

where $\Lambda(S,V,t)$ represents the market price of volatility risk and Heston assumes that the market price of volatility risk is proportional to volatility, i.e.

$\exists a$ constant:

$$\begin{aligned} &\Lambda(S,V,t) = a\sqrt{V_t} \\ &\Lambda(S,V,t)\sigma\sqrt{V} = a\sigma V_t \\ &= \lambda(S,V,t) \end{aligned} \tag{11}$$

After, with (10) and (11)

$$\frac{\partial F}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 F}{\partial S^2} + \rho V\sigma S\frac{\partial^2 F}{\partial S\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2 F}{\partial V^2} + rS\frac{\partial F}{\partial S} + \{[k(\theta - V) - \lambda(S,V,t)]\}\frac{\partial F}{\partial V} - rF = 0 \tag{12}$$

An European call option with strike price K and maturing at time T satisfies the equation (12) and the problem is completed, subject to the following boundary conditions

$$F(S,V,T) = Max(0, S-K) \tag{13}$$

$$F(0,V,t) = 0 \tag{14}$$

$$F(\infty,V,t) = 1 \tag{15}$$

$$rS\frac{\partial F}{\partial S}(S,0,t) + k\theta\frac{\partial F}{\partial V}(S,0,t) - rF(S,0,t) + F(S,0,t) = 0 \tag{16}$$

$$F(S,\infty,t) = S \tag{17}$$

After defining this, important to review the effects of stochastic volatility in the option price and make the valuation of price in a risk neutral world where the variance follows a square root process moving from a real world measure to an EMM (Equivalent Martingale Measure) is achieved by Girsavov's Theorem (see englishMao, X.(1997)).
In particular, we have

$$d\hat{Z}_1(t) = dZ_1(t) + \phi_t dt \tag{18}$$

$$d\hat{Z}_2(t) = dZ_2(t) + \Lambda(S,V,t)dt \tag{19}$$

$$\begin{aligned} &\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\{-\frac{1}{2}\int_0^t(\varphi_s^2 + \Lambda(S,V,s)^2)ds \\ &-\int_0^t \phi_s dZ_1(s) - \int_0^t \Lambda(S,V,s)dZ_2(s)\} \end{aligned} \tag{20}$$

$$\phi_t = \frac{\mu - r}{\sqrt{V_t}} \tag{21}$$

Where $\mathbb{P}$ is the real world measure and $\left\{\hat{Z}_1(t)\right\}_{t\geq 0}$ and english $\left\{\hat{Z}_2(t)\right\}_{t\geq 0}$ are $\mathbb{Q}$ -Brownian Motions.
Under measure $\mathbb{Q}$ (5) and (8) become

$$dS_t = rS_t dt + \sqrt{V_t} S_t d\hat{Z}_1(t) \tag{22}$$

$$dV_t = k^*(\theta^* - V_t)dt + \sigma\sqrt{V_t}\hat{Z}_2(t) \tag{23}$$

$$\rho dt = dZ_1(t)dZ_2(t) \tag{24}$$

where the modified parameters arespanish

$$k^* = k + \lambda; \qquad \theta^* = \frac{k\theta}{k+\lambda}$$

## 3. TRANSFORMATION OF THE PROBLEM

For the sake of convenience the equation (12) will be transformed into an equivalent nonlinear model using the following transformation

$$H = e^{r(T-t)}F; \quad X = e^{r(T-t)}S; \quad \tau = \frac{v_M}{2}(T-t); \quad v = v \tag{25}$$

So

$$H = H(X, v, \tau) \tag{26}$$

then

$$F = e^{-r(T-t)}H \tag{27}$$

After the equation (12) become

$$\frac{\partial H}{\partial \tau} = X^2 \frac{\partial^2 H}{\partial X^2} + 2\rho\sigma X \frac{\partial^2 H}{\partial X \partial v} + \sigma^2 \frac{\partial^2 H}{\partial v^2} + \frac{2}{v}[k^*(\theta^* - v)]\frac{\partial H}{\partial v} \tag{28}$$

where

$$(X, v, \tau) \in ]0, \infty[ \times [v_m, v_M] \times [0, \frac{v_M}{2}T] \tag{29}$$

with the initial condition

$$H(X, v, 0) = f(X); \ \ X \geq 0 \tag{30}$$

## 4. NUMERICAL SCHEME CONSTRUCTION

As a domain of equation (12) is unbounded and to the numerical approximation is important to have a bounded domain such that it is possible to compute the solution. The bounded numerical domain can be chosen according with diferent criteria; see R.Kangro et. al (2000) for instance.

Let us denote $[0, b]$ the domain for asset variable X, where b is chosen such that the interval includes the exercise price and initial price and denote $[c, d]$ the domain for variance variable $v$, where c and d are chosen such that the interval includesthe minimum and maximum possible variance.

Then we define the numerical domain as:

$$(X, v, \tau) \in [0, b] \times [c, d] \times [0, \frac{d}{2}T] \tag{31}$$

with the nodes

$$X_i = ih_1; \quad 0 \leq i \leq N_x$$

$$v_j = c + jh_2; \quad 0 \leq j \leq N_v$$

$$\tau^n = nk; \quad 0 \leq n \leq N_\tau$$

$$N_x h_1 = b; \quad N_v h_2 = d - c; \quad N_\tau k = \frac{d}{2}T$$

The numerical aproximation of exact solution $H(x_i, v_j, \tau^n)$ is denoted by $U_{ij}^n$.

Then the approximations for the partial derivatives are given by

$$\begin{aligned}
&\frac{\partial H}{\partial \tau}(x_i, v_j, \tau^n) = \frac{U_{ij}^{n+1} - U_{ij}^n}{k} + O(k) \\
&\frac{\partial H}{\partial v}(x_i, v_j, \tau^n) = \frac{U_{ij+1}^n - U_{ij-1}^n}{2h_2} + O(h_2) \\
&\frac{\partial^2 H}{\partial v^2}(x_i, v_j, \tau^n) = \frac{U_{ij-1}^n - 2U_{ij}^n + U_{ij+1}^n}{h_2^2} + O(h_2^2) \\
&= \Delta_j^n(U) + O(h_2^2) \\
&\frac{\partial^2 H}{\partial X \partial v}(x_i, v_j, \tau^n) = \frac{U_{i+1j+1}^n + U_{i-1j-1}^n - U_{i-1j+1}^n - U_{i+1j-1}^n}{4h_1 h_2} \\
&= \Delta_{ij}^n(U) + O(h_1 h_2) \\
&\frac{\partial^2 H}{\partial X^2}(x_i, v_j, \tau^n) = \frac{U_{i-1j}^n - 2U_{ij}^n + U_{i+1j}^n}{h_1^2} + O(h_1^2) \\
&= \Delta_i^n(U) + O(h_1^2)
\end{aligned} \tag{32}$$

Note that due to the use of centered aproximations of the derivates at $X_0 = 0$, $X_{N_x} = b$ y $v_0 = c$, $v_{N_v} = d$ external fictitious nodes appear $X_{-1} = -h_1$, $X_{N_x+1} = (N_x + 1)h_1$, $v_{-1} = c - h_2$ y $v_{N_v+1} = c + (N_v + 1)h_2$.

The aproximations $U_{0,-1}^n$, $U_{0,N_v+1}^n$, $U_{N_x,-1}^n$, $U_{N_x,N_v+1}^n$, $U_{-1,0}^n$, $U_{N_x+1,0}^n$, $U_{-1,N_v}^n$, $U_{N_x+1,N_v}^n$ are obteined by using linear extrapolation throughout the aproximations obtained in closest interior nodes of numerical domain.

Thus

$$\begin{aligned}
U_{0,-1}^n &= 2U_{0,0}^n - U_{0,1}^n \\
U_{N_x,-1}^n &= 2U_{N_x 0}^n - U_{N_x,1}^n \\
U_{-1,0}^n &= 2U_{0,0}^n - U_{1,0}^n \\
U_{-1,N_v}^n &= 2U_{0,N_v}^n - U_{1,N_v}^n \\
U_{0,N_v+1}^n &= 2U_{0,N_v}^n - U_{0,N_v-1}^n \\
U_{N_x,N_v+1}^n &= 2U_{N_x,N_v}^n - U_{N_x,N_v-1}^n \\
U_{N_x+1,0}^n &= 2U_{N_x,0}^n - U_{N_x-1,0}^n \\
U_{N_x+1,N_v}^n &= 2U_{N_x,N_v}^n - U_{N_x-1,N_v}^n
\end{aligned} \tag{33}$$

and from (32) one gets

$$\Delta_{i,0}^n U = \Delta_{i,N_v}^n U = \Delta_{0,j}^n U = \Delta_{N_x,j}^n U = 0, \quad 0 \le n \le N_\tau$$

By replacing the partial derivatives of equation (28) by the aproximations given in (32) one gets the numerical scheme

$$U_{ij}^{n+1} = b_i U_{i-1j-1}^n + a_i U_{i-1j}^n - b_i U_{i-1j+1}^n + c_j U_{ij-1}^n + d_i U_{ij}^n + e_j U_{ij+1}^n - b_i U_{i+1j-1}^n + a_i U_{i+1j}^n + b_i U_{i+1j+1}^n \tag{34}$$

Where

$$a_i = ki^2 \tag{35}$$

$$b_i = \frac{\rho\sigma ik}{2h_2} \tag{36}$$

$$c_j = k\left(\frac{\sigma^2}{h_2^2} - \lambda_j\right) \tag{37}$$

$$d_i = \left(1 - 2a_i - \frac{2k}{h_2^2}\sigma^2\right) \tag{38}$$

$$e_j = k\left(\frac{\sigma^2}{h_2^2} + \lambda_j\right) \tag{39}$$

$$\lambda_j = \frac{1}{h_2(c + jh_2)}[k^*(\theta^* - (c + jh_2))] \tag{40}$$

Using the extrapolation and the numerical scheme (34) at the boundaries, we obtain:

$$\begin{aligned} U_{00}^{n+1} &= U_{00}^n = \ldots = U_{00}^0 = f(X_0) = f(0) \\ U_{0N_v}^{n+1} &= U_{0N_v}^n = \ldots = U_{0N_v}^0 = f(X_0) = f(0) \\ U_{N_x0}^{n+1} &= U_{N_x0}^n = \ldots = U_{N_x0}^0 = f(X_{N_x}) = f(b) \\ U_{N_xN_v}^{n+1} &= U_{N_xN_v}^n = \ldots = U_{N_xN_v}^0 = f(X_{N_x}) = f(b) \end{aligned} \tag{41}$$

So the (30) (29) and (28), we obtain:

$$U_{ij}^{n+1} = k\lambda_j\left(U_{ij+1}^n - U_{ij-1}^n\right) + U_{ij}^n \tag{42}$$

for $i = 0$ , $i = N_x$ and $j = 1 \ldots N_v - 1$, or for $j = 0$, $j = N_v$ and $i = 1 \ldots N_x - 1$.

For the sake of convenience the numerical aproximation will be write in matrix form $U^n$ .

$$U^{n+1} = \begin{bmatrix} U_{00}^{n+1} & U_{01}^{n+1} & \cdots & U_{0N_v}^{n+1} \\ U_{10}^{n+1} & U_{11}^{n+1} & \cdots & U_{1N_v}^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ U_{N_x0}^{n+1} & U_{N_x1}^{n+1} & \cdots & U_{N_xN_v}^{n+1} \end{bmatrix}_{(N_x+1)(N_v+1)} \tag{43}$$

$$U^n = \begin{bmatrix} U_{00}^n & U_{01}^n & \cdots & U_{0N_v}^n \\ U_{10}^n & U_{11}^n & \cdots & U_{1N_v}^n \\ \vdots & \vdots & \ddots & \vdots \\ U_{N_x 0}^n & U_{N_x 1}^n & \cdots & U_{N_x N_v}^n \end{bmatrix}_{(N_{x+1})(N_{v+1})} \quad (44)$$

Where

$$U^{n+1}; \quad 1 \le j \le N_{v-1}, \quad 1 \le i \le N_{x-1}$$

$U_{ij}^{n+1} = Traza(U^n A_{(ij)}^n)$ where $[A_{(ij)}^n]_{(N_v+1)(N_x+1)}$ is

$$A_{(ij)}^n = \begin{cases} A_{kl}^n, & k = j+p, \quad l = i+q \\ 0 & \text{otherwise} \end{cases} \quad (45)$$

where

$$A_{(kl)}^n = \begin{bmatrix} b_i & c_j & -b_i \\ a_i & d_i & a_i \\ -b_i & e_j & b_i \end{bmatrix}$$

**5. SCHEME ANALYSIS**

Supose that

$$0 \le c \le \theta^* \le d;; \quad h_1 > 0 \quad (46)$$

$$If \qquad 2cd \le \theta^*(d+c)$$

$$Then\ h_2 \le \frac{\sigma^2 c}{k^*[\theta^* - c]}; \quad k \le \frac{1}{2\left[\left(\frac{b}{h_1}\right)^2 + \left(\frac{\sigma}{h_2}\right)^2\right]}$$

In consecuence the coeficients $a_i, b_i, c_j, d_i, e_j$ are nonnegatives for $0 \le i \le N_x$ and $0 \le j \le N_v$.

**5.1 Positivity**

A suitable property of the numerical scheme for the pricing equation is positivity.

**Definition 1.**

Define $\underset{i}{\Delta}{}_j^n = U_{i+1j}^n - U_{ij}^n$; $\quad \Delta_i^n = U_{ij+1}^n - U_{ij}^n$ .

Note that english

$$\underset{i}{\Delta}{}_j^n \ge 0, \quad \underset{j}{\Delta}{}_i^n \ge 0 \quad and \quad \Delta_j^n(U) = \underset{j}{\dot{\Delta}}{}_i^n - \underset{j-1}{\dot{\Delta}}{}_i^n \ge 0$$

$$\underset{i}{\dot{\Delta}}{}_j^n = U_{i+1j}^n - U_{i-1j}^n \ge 0 \quad (47)$$

$$\underset{j}{\dot{\Delta}}{}_i^n = U_{ij+1}^n - U_{ij-1}^n \ge 0 \quad (48)$$

Then, for our scheme it is true that

$$If\quad \dot{\underset{i}{\Delta}}{}_{j}^{n} - \dot{\underset{i}{\Delta}}{}_{j-1}^{n} \geq 0; \qquad \dot{\underset{j}{\Delta}}{}_{i}^{n} - \dot{\underset{j}{\Delta}}{}_{i-1}^{n} \geq 0 \tag{49}$$

and the restrictions (46) are met, one gets that for a nonegative payoff $U_{ij}^{0}$ the numerical solution english $U_{ij}^{n}$ is nonegative for

$0 \leq n \leq N_\tau,\ 0 \leq i \leq N_x \ \ 0 \leq j \leq N_v$.

### 5.2 Monotonicity

For the sake of clarity in the presentation we introduce the following definition of monotonicity-preserving numerical scheme (see Xiao et. al (1996)).

**Definition 2.**

Consider the scheme $W(U_{ij}^{n}) = 0$ , $i \in I,\ \ j \in J,\ \ n \in L$. where J and L are sets of nonnegative integers.

We say that the scheme is i-monotonicity-preserving if assuming that $\underset{i}{\Delta}{}_{j}^{n} \geq 0$ then it occurs that $\underset{i}{\Delta}{}_{j}^{n+1} \geq 0$. So it, is j-monotonicity-preserving if $\underset{j}{\Delta}{}_{i}^{n} \geq 0$ then $\underset{j}{\Delta}{}_{i}^{n+1} \geq 0$.

**Proposition 1.**

Under hypotheses (46) and (49) the numerical scheme (34) is i-monotonicity-preserving and j-monotonicity-preserving, with $0 \leq n \leq N_\tau,\ 0 \leq i \leq N_x$ y $0 \leq j \leq N_v$.

*Proof.* Let us write

$$\begin{aligned} U_{i+1j}^{n+1} - U_{ij}^{n+1} &= (U_{i+1j}^{n+1} - U_{i+1j}^{n}) + (U_{i+1j}^{n} - U_{ij}^{n}) - (U_{ij}^{n+1} - U_{ij}^{n}) = \\ &(b_{i+1}U_{ij-1}^{n} + a_{i+1}U_{ij}^{n} - b_{i+1}U_{ij+1}^{n} + c_j U_{i+1j-1}^{n} + d_{i+1}U_{i+1j}^{n} + e_j U_{i+1j+1}^{n} - b_{i+1}U_{i+2j-1}^{n} + a_{i+1}U_{i+2j}^{n} \\ &- b_{i+1}U_{i+2j+1}^{n} - U_{i+1j}^{n}) + (U_{i+1j}^{n} - U_{ij}^{n}) - (b_i U_{i-1j-1}^{n} + a_i U_{i-1j}^{n} - b_i U_{i-1j+1}^{n} + c_j U_{ij-1}^{n} + d_i U_{ij}^{n} + e_j U_{ij+1}^{n} \\ &- b_i U_{i+1j-1}^{n} + a_i U_{i+1j}^{n} + b_i U_{i+1j+1}^{n} - U_{ij}^{n}) \end{aligned} \tag{50}$$

after some algebraic procedures

$$\begin{aligned} U_{i+1j}^{n+1} - U_{ij}^{n+1} &= b_i \left( \underset{i+1}{\Delta}{}_{j+1}^{n} - \underset{i+1}{\Delta}{}_{j-1}^{n} \right) - b_i \left( \underset{i-1}{\Delta}{}_{j+1}^{n} - \underset{i-1}{\Delta}{}_{j-1}^{n} \right) + a_i \left( \underset{i+1}{\Delta}{}_{j}^{n} + \underset{i-1}{\Delta}{}_{j}^{n} \right) + c_j \underset{i}{\Delta}{}_{j-1}^{n} + e_j \underset{i}{\Delta}{}_{j+1}^{n} \\ &+ d_i \underset{i}{\Delta}{}_{j}^{n} + (2ki + k)\left( \underset{i+1}{\Delta}{}_{j}^{n} - \underset{i}{\Delta}{}_{j}^{n} \right) + \frac{\rho\sigma k}{2h_2}\left( \left( \underset{i}{\Delta}{}_{j+1}^{n} - \underset{i}{\Delta}{}_{j-1}^{n} \right) - \left( \underset{i+1}{\Delta}{}_{j+1}^{n} - \underset{i+1}{\Delta}{}_{j-1}^{n} \right) \right) \end{aligned} \tag{51}$$

Asumming (47) and (48) and from (34), one can easily show that

$$b_i (\underset{i+1}{\Delta}{}_{j+1}^{n} - \underset{i+1}{\Delta}{}_{j-1}^{n}) - b_i (\underset{i-1}{\Delta}{}_{j+1}^{n} - \underset{i-1}{\Delta}{}_{j-1}^{n}) \geq 0. \tag{52}$$

then $U_{i+1j}^{n+1} - U_{ij}^{n+1} \geq 0.$ ■

In an analogous way result that $U_{ij+1}^{n+1} - U_{ij}^{n+1} \geq 0.$

Then, the numerical scheme (34) is i-monotonicity-preserving and j-monotonicity-preserving, with $0 \leq n \leq N_\tau,\ 0 \leq i \leq N_x$ y $0 \leq j \leq N_v$.

**Corollary 1.**

Under hypothesess (46) and (49) and the notation of last proposition, assuming that the payoff function f(x) is nondecreasing and nonnegative with f(0)=0, then the scheme (34) is nondecreasing and nonnegative in $i, j$ for each time stage $n$.

### 5.3 Consistency

Consistency of a numerical scheme with respect to a partial differential equation means that the exact solution of the finite diference scheme aproximates the exact solution of the PDE (see Smith (1985)).

**Theorem 1.**
For any fixed parameters, the scheme (34) is consistent with the partial differential equation.

*Proof.* Trivial by construction (See section 4).

### 5.4 Stability

To analyse teh linear Von Neumann stability of the scheme (34)rewrite

$$U^n(i,j) = A^n \exp(I[\overline{k_1}m + \overline{k_2}n]) \tag{53}$$

where I is imaginary unit, $A^n$ is the amplitude at time level $n$, $\overline{k_i} = \dfrac{2\pi h}{\lambda_i}$ are phases angles with wavelengths $\lambda_i$, and $i\Delta x = m$ y $j\Delta v = n$.

Then, $\xi = \dfrac{A^{n+1}}{A^n}$ the amplification factor satisfies

$$\xi = -4b[\sin(\overline{k_1}\Delta x)\sin(\overline{k_2}\Delta v)] + 2a\cos(\overline{k_1}\Delta x) + d + c[\cos(\overline{k_2}\Delta v) - I\sin(\overline{k_2}\Delta v)] + e[\cos(\overline{k_2}\Delta v) - I\sin(\overline{k_2}\Delta v)] \tag{54}$$

Thus

$$\lim_{\Delta x,\Delta v \to 0} |-4b[\sin(\overline{k_1}\Delta x)\sin(\overline{k_2}\Delta v)] + 2a\cos(\overline{k_1}\Delta x) + d + c[\cos(\overline{k_2}\Delta v) - I\sin(\overline{k_2}\Delta v)] + e[\cos(\overline{k_2}\Delta v) - I\sin(\overline{k_2}\Delta v)]| \tag{55}$$

$$= \left|2a_i + d_i + c_j + e_j\right| \tag{56}$$

Is clear that the coefficients of the numerical scheme (34) are $2a_i + d_i + c_j + e_j = 1$.

$$|\xi| = \left|2a_i + d_i + c_j + e_j\right| = 1 \tag{57}$$

Therefore, the scheme (34) is conditionally stable.

### 5.5 Convergence

Finally using the Lax-Richtmyer equivalence theorem with the theorem 1 and condition (57)we can conclude convergence of the scheme (34).

## 6. RESULTS

In this section we check the properties of the proposed numerical scheme (34).

### 6.1 Example.

Consider the european call option (so that f(s)=max(s-K,0)).

Figures show the computed price with numerical method, and the results obtained with $h_1 = 5$ (Figure 1) and $h_1 = 1$ (Figure 2).

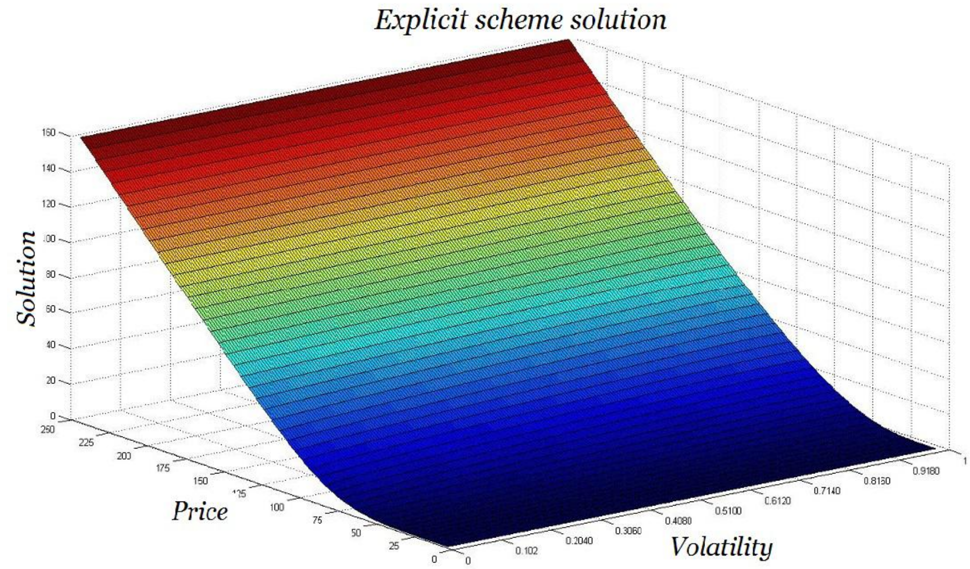


Figure 1: Explicit scheme solution $(h_1 = 5)$. Parameters: r=0.05,T=$\frac{1}{2}$, $\sigma = 0.1$, $k^* = 2$, $\theta^* = 0.011$, c=0.01, $\rho = 1$, d=1, K=80,b=240, $S_0 = 100$.

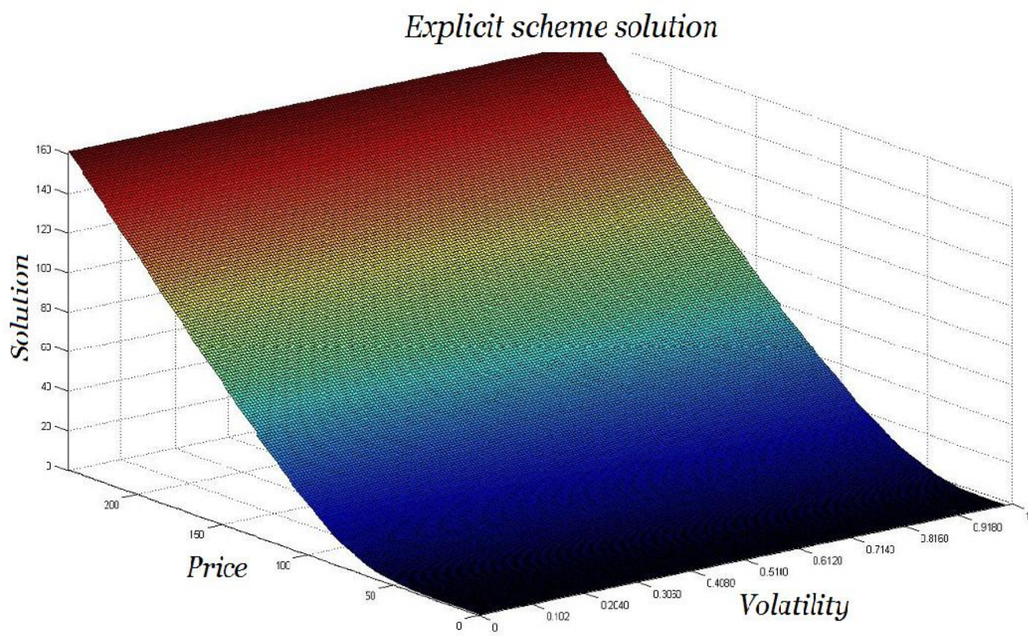


Figure 2: Explicit scheme solution $(h_1 = 1)$. Parameters: r=0.05,T=$\frac{1}{2}$, $\sigma = 0.1$, $k^* = 2$, $\theta^* = 0.011$, c=0.01, $\rho = 1$, d=1, K=80,b=240, $S_0 = 100$

In addition, Figure 3 shows the mesh convergence analysis with the solution corresponding to i = 16 and j = 207 for each of the proposed mesh, is evident that the mesh will converge to the solution and the difference decreases.

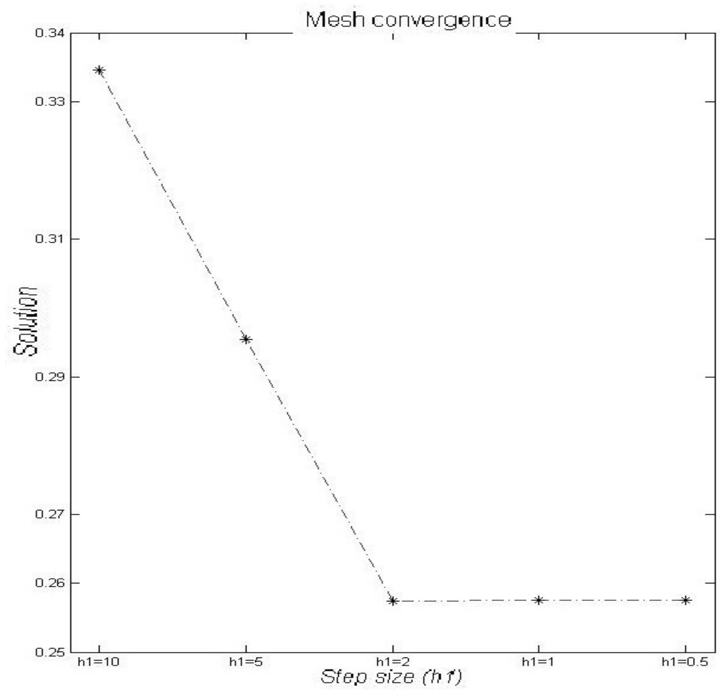


Figure 3: Mesh convergence. (Solution for different step sizes)

Is possible then ensure that when the mesh is refined and taken $\Delta S = h_1$ smaller the relative error respect to the exact solution decreases, we can specify that the method is convergent experimentally, see Figure 4

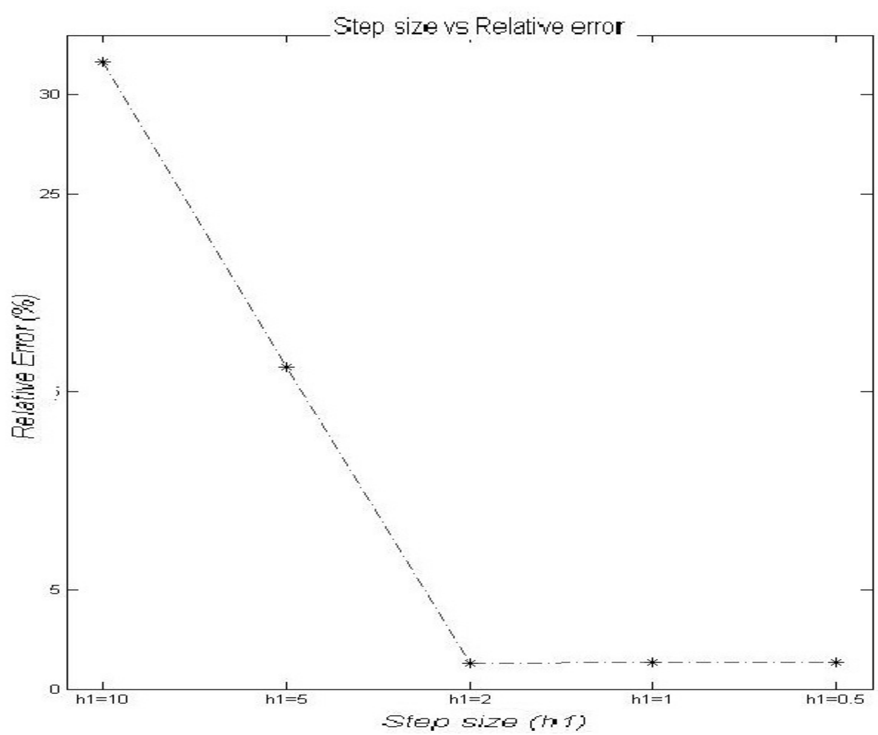


Figure 4: Relative Error. (Relative error compared to the exact solution for different step sizes)

## 7. CONCLUSIONS

In this paper we have constructed a explicit finite difference numerical scheme that is consistent for the equation (28), which was obtained by a transformation of variables in equation (12). The sufficient conditions for the step sizes of the discretization in volatility and time are obtained depending on the step size of the asset price in order to ensure the positivity of the coefficients and therefore of the solution in addition to stability of the scheme for general payment convex functions. Our numerical scheme avoids inappropriate oscillations of the numerical solution because it is monotonous - conservative.

The computational implementation of this numerical scheme is rather simple with a low computational cost and provides desired solutions that are non-decreasing in the underlying asset and in the volatility direction from a non-decreasing function of initial payments.

Some numerical computational results are performed to graphically illustrate the convergence of the scheme and the approximation error.

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