GLOBAL HYBRID METHOD FOR COMPUTING THE MINIMUM DISTANCE BETWEEN A POINT AND A PLANE PARAMETRIC CURVE

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ABSTRACT
Global convergent hybrid method is presented for computing the minimum distance between a point and a plane parametric curve. First, It uses a first order geometric iteration method. If iterative parametric value satisfied local Newton convergence condition and convergence in appropriate area, then turning to Newton iteration method. This hybrid method's sensitivity to the choice of initial values is nonexistence. Experimental results show that the algorithms under consideration are robust and efficient.

Keywords: *Point projection, Newton's method, Global convergence, Parametric Curve.*

1. INTRODUCTION

In this paper, we discuss how to compute the minimum distance between a point and a spatial parametric curve and to return the nearest point on the curve as well as its corresponding parameter, which is also called the point projection problem (the point inversion problem)of a spatial parametric curve. It is very interesting for this problem due to its importance in geometric modeling, computer graphics and computer vision[1]. Both projection and inversion are essential for interactively selecting curves[1-2], for the curve fitting problem[1-2], for the reconstructing curves problem[3-5]. It is also a key issue in the ICP(iterative close for construction and rendering of solid models with boundary representation, projecting of a space curve onto a surface for curve surface design[6]. Many algorithms have been developed by using various techniques including turning into solving a root problem of a polynomial equation, geometric methods, subdivision methods, circular clipping algorithm. For more details, see [1-21] and the references therein. In the various methods mentioned above, there are two key issues in the projection and inversion problems: seeking a good initial value, using a Newton-type iterative method for computing the root.

These methods have the same charrators of using a Newton-type iterative method finnally.But it dosen't guarantee to be convergent when using a Newton-type iterative method. In order to avoid the sensitivity of the choice of initial values ,we firstly use a first order geometric iteration method, and secondally decide wethere the condition of partial convergence judge theorem is met by using the partial convergence judge theorem of local Newton-type iterative method and by judging whether the modulus of the minus of the modulus of two parameters' values meet the specific code number. If the conditon is met, call the Newton-type iterative method. So it is convergence globally, and we have raised the speed of convegence. Then we can get good computing results.

2. ALGORITHM ANALYSIS AND REALIZATION

The footpoint of test point *P* employed in first order geometric iteration computation is as follows: Test point *P* is projected onto the tangent line when parametre is t_0 along the plan parametric curve, then the footpoint *q* is created. Now, the footnote q can be demonstrated as $c(t_0)$ and $c'(t_0)$:

$$q = c(t_0) + \Delta t c'(t_0) \qquad (1)$$

If there are two vectors $x, y \in R^2$, the inner product can be demonstrated as $\langle x, y \rangle$ and the norm of vector *x* can be shown as $\|x\|$. So equation (1) can be converted to:

$$\Delta t = \frac{\langle c'(t_0), q - c(t_0) \rangle}{\langle c'(t_0), c'(t_0) \rangle} \qquad (2)$$

This geometric iteration of first order is globally convergent.
The testification is as follows:
Theorem 1: iteration (2) is a first-order and global convergence.
Proof: In order to prove that method (2) is a first-order and global convergence, we first deduce the computation expression footpoint *q*. We suppose parameter α is the corresponding parameter when test point *p* is projected onto the the parametric curve $c(t)$, among which $p = (p_1, p_2)$, $c(t) = (f_1(t), f_2(t))$. According to the

requirement of computing the minimum distance between a point and a plane parametric curve, we can have the following expression of relation:

$$(p-h)\times\vec{n}=0 \qquad (3)$$

Among which, $h=(f_1(\alpha), f_2(\alpha))$ and normal vector $\vec{n}=(-f_2'(\alpha), f_1'(\alpha))$。

Then equation (3) can be rewritenn as:

$$(p_1-f_1(\alpha))f_1'(\alpha)+(p_2-f_2'(\alpha))f_2'(\alpha)=0 \quad (4)$$

From equation (4), we have the relational expression:

$$p_1=\frac{f_1'(\alpha)f_1(\alpha)-f_2'(\alpha)p_2+f_2'(\alpha)f_2(\alpha)}{f_1'(\alpha)}=\frac{a_0a_1-b_1p_2+b_0b_1}{a_1} \quad (5),$$

In expression (5), $a_0=f_1(\alpha), a_1=f_1'(\alpha), b_0=f_2(\alpha), b_1=f_2'(\alpha)$。

Now we begin to deduce the expression of footpoint *q*.

Footpoint q is the intersection of Test point p and parametric curve $c(t)$ when $t=t_0$。 We might as well set tangential equation equation as:

$$\begin{cases} x=f_1(t_0)+f_1'(t_0)w \\ y=f_2(t_0)+f_2'(t_0)w \end{cases} \qquad (6)$$

Here w is the parameter of the tangent line. The linear equation of the straight line which passes test point *p* and be perpendicular to the tangent line can be expressed as:

$$\begin{cases} x=p_1-f_2'(t_0)s \\ y=p_2+f_1'(t_0)s \end{cases} \qquad (7)$$

Combining (6) and (7), we can get the following expression

$$w=\frac{(p_1-f_1(t_0))f_1'(t_0)+(p_2-f_2(t_0))f_2'(t_0)}{f_1'^2(t_0)+f_2'^2(t_0)} \qquad (8)$$

Expression (8) is substituted to(6), then:

$$\begin{cases} q_1=f_1(t_0)+f_1'(t_0)\dfrac{((p_1-f_1(t_0))f_1'(t_0)+(p_2-f_2(t_0))f_2'(t_0))}{f_1'^2(t_0)+f_2'^2(t_0)} \\ q_2=f_2(t_0)+f_2'(t_0)\dfrac{((p_1-f_1(t_0))f_1'(t_0)+(p_2-f_2(t_0))f_2'(t_0))}{f_1'^2(t_0)+f_2'^2(t_0)} \end{cases} \qquad (9)$$

Then we substitute (9) to(2), the simplified relational expression is:

$$\Delta t=\frac{(p_1-f_1(t_0))f_1'(t_0)+(p_2-f_2(t_0))f_2'(t_0)}{f_1'^2(t_0)+f_2'^2(t_0)} \qquad (10)$$

Now, we set $e_n=t_n-\alpha, a_i=(1/i!)(f_1^{(i)}(\alpha)), b_i=(1/i!)(f_2^{(i)}(\alpha)), i=0,1,2,3,...,$ About $f_1(t), f_2(t)$, we use Taylor the expression:

$$f_1(t_n)=a_0+a_1e_n+a_2e_n^2+o(e_n^3) \qquad (11)$$

$$f_2(t_n)=b_0+b_1e_n+b_2e_n^2+o(e_n^3) \qquad (12)$$

Further more, we have:

$$f_1'(t_n)=a_1+2a_2e_n+o(e_n^2) \qquad (13)$$

$$f_2'(t_n)=b_1+2b_2e_n+o(e_n^2) \qquad (14)$$

Now, expressions (11)-(14) are brought into (10), and Taylor expansion from Maple is employed to get the following:

$$e_{n+1} = -\frac{(-2a_1b_2p_2 + 2b_2a_1b_0 + b_1^2a_1 + 2b_1a_2p_2 - 2b_1a_2b_0 + a_1^3)}{a_1(a_1^2 + b_1^2)} e_n + o(e_n^2) \tag{15}$$

The demonstration above is the proof that iteration (2) is first order and convergent.
Now we begin to illustrate that iteration (2) is globally convergent, that is to say, this convergence method is freefrom sensitivity of choosing initial iteration point. Our demonstration is similar to that of literature [21-22].If iteration begins from the left side of parameter α , footpoint q is definitely at the right side of iteration point. So Δt is a positive real number. Thereupon, the following iteration sequence is defined through expression (2).

$$t_n = t_{n-1} + \Delta t_{n-1}, \tag{16}$$

Here $\Delta t = \frac{\langle c'(t_{n-1}), q - c(t_{n-1})\rangle}{\langle c'(t_{n-1}), c'(t_{n-1})\rangle}$ 。 When $t_n < \alpha,$ sequence t_n is a strict, monotonic and increasing sequence. When $t_n > \alpha,$ it is several times of iteration, sequence t_n is converged to α . This kind of iteration sequence is similar to weakening simple pendulum. Of course, if initial iteration point begins from the right side of parameter α , the convergence state is similar to that begins from the left side. Although this geometric iteration is first order and globally convergent, it is not so efficient. We find that when parametric iteration value approaches the end value, the iteration step length minifies to the minimum, and the number of iteration times magnifies to the maximum. Besides, when initial iteration point is far from actual value, the corresponding iteration step is very large. Expression (17) is Newton iteration method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{17}$$

This is a second order iteration method, the convergence speed of which is obviously faster than that of first order iteration convergence. However, the limitation of Newton iteration method is that it is sensitive to the initial point. Newton iteration method is the most efficient and effective only when the conditions are satisfied. Therefore, we present the judge theorem of partial convergence of Newton iteration method.

Theorem 2: Let $f:[a,b] \to [a,b]$ be a defferentable function, as a result, when $\forall x \in [a,b]$, we have

$$\left| \frac{f(x)f''(x)}{f'^2(x)} \right| < 1 \tag{18}$$

So there exists a fixed point $l_0 \in [a,b]$ in Newton iteration expression (17). At the same time, the iteration sequence $\{x_n\}$ generated from expression (17) can be converged to the fixed point l_0 when $\forall x_0 \in [a,b]$.
The justification of this theorem can be achieved by the convergence judge theorem for fixed-point iteration method. Comparing the advantages and disadvantages of first order geometric iteration and second order Newton iteration, we put forward the mixed method. That is, we begin with the first order iteration method, and change into the second order Newton iteration when the parameter value of iteration matches the condition of partial convergence of Newton iteration, and when the first order iteration step has not varied a lot, or the variation is comparatively stable. Theoretically speaking, the iteration parameter value is sometimes not convergent, although it is convergent when it matches the conditions of partial convergence of second order Newton iteration. When the first order geometric iteration step has not varied a lot, or the variation is comparatively stable, it guaranteed that the iteration parameter value is within the range of partial convergence of Newton iteration.
The following computing method is employed to realize our hybrid algorithm.

hybrid algorithm:

Set an Error tolerance ι_1, a Code number ι_2, the number of iteration times *n,* an initial estimated value x_0, a first variable bUseNewton indicating whether it is appliable to Newton iteration method, a second variable bIsCloser indicating whether it is within the range of using Newton method:
bool bUseNewton =false; //Newton method is not used at first
bool bIsCloser =false; //Assumed that it is not within the range of use Newton mehod
n=0;

do
{
(1) If ((bUserNewton==true)&&(bIsCloser==true))
x_n =Newton(x_{n-1});// use Newton method
Else
x_n =firstOrder(x_{n-1});// use first order convergence method

(2) To judge whether Newton method can be used

Step 1: $x_0 = x_n$;

Step 2:computing $\eta = \left| \frac{f(x_0)}{f'(x_0)} \right|$;

Step 3：Computing $K = \left| \frac{f''(x_0)}{f'(x_0)} \right|$;

Step 4：$h = K\eta$

Step 5：If(h<1)
bUseNewton =true;
else
bUseNewton =false;

(3) To judge whether it is within the appropriate area of using Newton iteration method

Step 1: x1= x_n ,x2= x_{n-1} ;

Step 2: If($\| x_1 | - | x_2 \|$ <code number ι_2)
bIsCloser = true;// it is within the appropriate area of using Newton method.
else
bIsCloser = false;

}While($\| x_n | - | x_{n-1} \|$ > error tolerance ι_1)

In this paper, the mixed global algorithm is discussed on the basis of plane parametric curve. Actually, this algorithm can be applied to any n-dimension Euclidean space in which the computation of the minimum distance between a point and a parametric curve is needed.

3. EXAMPLE

Professor Hu provided an iteration method of second order global convergence in reference [7] (now Hu-iteration for short). Here, we consider the plane parametric curve $c(t) = (t, \sin(t))$, and code number is 0.2. We compared the different methods of first order geometric iteration, second order Newton iteration, second order Hu-iteration and the mixed iteration method we proposed. Table 1 shows the results of adopting different methods to computing the different initial iteration parameter value t_0 . In this table, NC means that it cannot be converged to the needed root. From table 1, we can find that the mixed method we used is faster in convergence speed than first order geometric method and Hu-iteration method, while Newton method is sensitive and unstabel to initial point.

Tabe 1: comparison of the numer of iteration times by emplying different methods to compute different initial iteration parameter value

$p=(1,2)$	$t_0=-5000.0$	$t_0=-4.0$	$t_0=5.0$	$t_0=7.0$	$t_0=8.0$	$t_0=10.0$	$t_0=11.0$	$t_0=40000.0$
First order method	351	353	352	352	349	350	350	355
Hu-iteration method	400	30	32	32	33	29	31	2523
Newton method	NC	NC	NC	NC	NC	NC	NC	NC
Mixed method	15	19	17	17	15	17	15	23

4. CONCLUSION

The present paper discussed the issue of the distance of a point projected to a plane parametric curve and the reverse. Data has shown that the hybrid method we proposed is robust and effective. Our next objective is to give a more-high-level global convergence method to compute the minimum distance between a point and a plane parametric curve, and we try to present the strict relational expression of parametric curve and code number.

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