ON THE LOCAL PROPERTY OF $\left|\bar{N}, p_n, \alpha_n; \delta\right|_k$ SUMMABILITY OF A FACTORED FOURIER SERIES

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ABSTRACT

In this paper we have established a theorem on the local property of $\left|\bar{N}, p_n, \alpha_n; \delta\right|_k$ summability of a factored Fourier series.

Key Words: $\left|\bar{N}, p_n\right|_k$ - summability, $\left|\bar{N}, p_n, \alpha_n\right|_k$ - summability, $\left|\bar{N}, p_n, \alpha_n; \delta\right|_k$ - summability and Fourier series.

AMS Classification No: 40D25

1. INTRODUCTION

Let $\sum a_n$ be a given infinite series with sequence of partial sums $\{s_n\}$.Let $\{p_n\}$be a sequence of positive real constants such that

(1.1) $$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty \text{ as } n \to \infty \; (P_{-i} = p_{-i} = 0, i \geq 1)$$

The sequence –to-sequence transformation

(1.2) $$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu$$

defines the sequence $\{t_n\}$ of the $\left|\bar{N}, p_n\right|$-means of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$.The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if

(1.3) $$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|t_n - t_{n-1}\right|^k < \infty .$$

For k=1, $\left|\bar{N}, p_n\right|_k$ -summability is same as $\left|\bar{N}, p_n\right|$ -summability.

When $p_n = 1$ *for all n and* $k = 1$, $\left|\bar{N}, p_n\right|_k$ -summability is same as $\left|C,1\right|$ -summability.

Also if we take $k = 1$ and $p_n = \frac{1}{(n+1)}$,summability is equivalent to the summability $\left|R, \log n, 1\right|$.

For any sequence $\{c_n\}$ we use the following notation

$$\Delta c_n = c_n - c_{n-1} \ , \ \Delta^2 c_n = \Delta(\Delta c_n) \ .$$

A sequence $\{\lambda_n\}$ is said to be convex if $\Delta^2 \lambda_n \geq 0$ for every positive integer 'n'.

Let $\{\alpha_n\}$ be any sequence of positive numbers. The series $\sum a_n$ is said to be summable $\left|\overline{N},\ p_n, \alpha_n\right|_k, \quad k \geq 1$, if

(1.4) $$\sum_{n=1}^{\infty} \alpha_n^{k-1} \left| t_n - t_{n-1} \right| < \infty,$$

where $\{t_n\}$ is as defined in (1.2).

Let $\{\alpha_n\}$ be any sequence of positive numbers .The series $\sum a_n$ is said to be $\left|\overline{N}, p_n, \alpha_n; \delta\right|_k, k \geq 1, \delta \geq 0$, summable if

(1.5) $$\sum_{n=1}^{\infty} \alpha_n^{\delta k + k - 1} \left| t_n - t_{n-1} \right|^k < \infty \ .$$

Let $f(t)$ be a periodic function with period 2π , integrable in the sense of Lebesgue over $(-\pi, \pi)$ .Without loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

(1.6) $$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$

2. KNOWN THEOREMS

Dealing with the $\left|\overline{N}, p_n\right|_k$ -summability of an infinite series Bor**[1]** proved the following theorem:

2.1. THEOREM:

Let $k \geq 1$ and let the sequences $\{p_n\}$ *and* $\{\lambda_n\}$ be such that

(2.1.1) $$\Delta X_n = O\left(\frac{1}{n}\right),$$

(2.1.2) $$\sum_{n=1}^{\infty} X_n^{k-1} \frac{\left|\lambda_n\right|^k + \left|\lambda_{n+1}\right|^k}{n} < \infty,$$

(2.1.3) $$\sum_{n=1}^{\infty} (X_n^k + 1) \left|\Delta \lambda_n\right| < \infty,$$

Where $X_n = \dfrac{P_n}{np_n}$. Then the summability $\left|\overline{N}, p_n\right|_k$ of the series $\sum_{n=1}^{\infty} A_n(t) \lambda_n X_n$ at a point can be ensured by the local property.

2.2. REMARK:

It is known that if $\{\lambda_n\}$ is a convex sequence and $\sum n^{-1}\lambda_n$ is convergent, then

$\lambda_n \geq \lambda_{n+1} \geq 0,\ \ \lambda_n \log n = O(1)$ *and* $\sum \log n \, \Delta\lambda_n < \infty$.

In the present paper, we have proved a theorem on the local property of $\left|\overline{N}, p_n, \alpha_n; \delta\right|_k$ -summability of a factored Fourier series.

3. MAIN THEOREM

Let $k \geq 1$. Suppose $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent. Let $\{\alpha_n\}$ and $\{p_n\}$ be a sequence of positive numbers such that

(3.1) $$\Delta X_n = O\left(\frac{1}{n}\right),$$

(3.2) $$\sum_{n=\nu+1}^{m+1} \alpha_n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k \left(\frac{1}{P_{n-1}}\right) = O\left(\frac{1}{P_\nu}\right),$$

(3.3) $$\sum_{n=1}^{\infty} X_n^{k-1} \frac{|\lambda_n|^k + |\lambda_{n+1}|^k}{n} < \infty ,$$

(3.4) $$\sum_{n=1}^{\infty} (X_n^k + 1)|\Delta\lambda_n| < \infty,$$

and

(3.5) $$\sum_{n=2}^{\infty} \alpha_n^{\delta k+k-1} \frac{|\lambda_n|^k}{n^k} < \infty,$$

where $X_n = \dfrac{P_n}{np_n}$. Then the summability $\left|\overline{N}, p_n, \alpha_n; \delta\right|_k, k \geq 1, \delta \geq 0$ of the series $\sum_{n=1}^{\infty} A_n(t)\lambda_n X_n$ at a point can be ensured by the local property.

In order to prove the above theorem we require the following lemma:

4. LEMMA

Let $k \geq 1$. Suppose $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and $\{p_n\}$ be a sequence such that the conditions **(3.1)-(3.5)** are satisfied. If $\{s_n\}$ is bounded then the series $\sum_{n=1}^{\infty} a_n\lambda_n X_n$ is $\left|\overline{N}, p_n, \alpha_n; \delta\right|_k, k \geq 1, \delta \geq 0$ -summable when $\{\alpha_n\}$ is any sequence of positive numbers.

5. PROOF OF THE LEMMA

Let $\{T_n\}$ denote the $\left|\overline{N}, p_n\right|$ -mean of the series $\sum_{n=1}^{\infty} a_n\lambda_n X_n$ .Then by definition we have

$$T_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu \sum_{r=0}^{\nu} a_r \lambda_r X_r$$

$$= \frac{1}{P_n}\sum_{\nu=0}^{n}(P_\nu - P_{\nu-1})a_\nu \lambda_\nu X_\nu \ , X_0 = 0.$$

For $n \geq 1$,we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n} P_{\nu-1} a_\nu \lambda_\nu X_\nu .$$

So,

$$T_n - T_{n-1} = -\frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n-1} p_\nu s_\nu \lambda_\nu X_\nu + \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n-1} P_\nu s_\nu X_\nu \Delta\lambda_\nu$$

$$+ \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n-1} P_\nu s_\nu \lambda_{\nu+1} \Delta X_\nu + \frac{p_n s_n \lambda_n X_n}{P_n}.$$

(by Abel's transformation)

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \quad (say)$$

To complete the proof of the Lemma using Minokowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \alpha_n^{\delta k+k-1} \left|T_{n,i}\right|^k < \infty \quad for \ i = 1,2,3,4.$$

Now, we have

$$\sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \left|T_{n,1}\right|^k$$

$$= \sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} p_\nu s_\nu \lambda_\nu X_\nu \right|^k$$

$$\leq \sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \left(\frac{p_n}{P_n}\right)^k \left(\frac{1}{P_{n-1}}\right) \left(\sum_{\nu=1}^{n-1} p_\nu |\lambda_\nu|^k |s_\nu|^k X_\nu^k\right) \left(\frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1} p_\nu\right)^{k-1}$$

$$= O(1)\sum_{\nu=1}^{m} p_\nu |\lambda_\nu|^k |s_\nu|^k X_\nu^k \sum_{n=\nu+1}^{m+1} \alpha_n^{\delta k+k-1} \left(\frac{p_n}{P_n}\right)^k \left(\frac{1}{P_{n-1}}\right)$$

$$= O(1)\sum_{\nu=1}^{m} |\lambda_\nu|^k X_\nu^k \frac{p_\nu}{P_\nu} \quad \textbf{, by (3.2)}$$

$$= O(1)\sum_{\nu=1}^{m} |\lambda_\nu|^k X_\nu^k \frac{p_\nu}{P_\nu} \frac{P_\nu}{\nu\, p_\nu}, \text{as } X_n = \frac{P_n}{np_n}$$

$$= O(1)\sum_{\nu=1}^{m} X_\nu^{k-1} \frac{|\lambda_\nu|^k}{\nu}$$

$$= O(1) \quad as \ m \to \infty \textbf{, by (3.3).}$$

Again,

$$\sum_{n=2}^{m+1}\alpha_n^{\delta k+k-1}\left|T_{n,2}\right|^k$$

$$=\sum_{n=2}^{m+1}\alpha_n^{\delta k+k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n-1}P_\nu s_\nu X_\nu \Delta\lambda_\nu\right|^k$$

$$\leq\sum_{n=2}^{m+1}\alpha_n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k\left(\frac{1}{P_{n-1}}\right)\left(\sum_{\nu=1}^{n-1}P_\nu\left|\Delta\lambda_\nu\right|\ \left|s_\nu\right|^k X_\nu^k\right)\left(\frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}P_\nu\left|\Delta\lambda\right|\right)^{k-1}$$

Since

$$\sum_{r=1}^{n-1}P_\nu\left|\Delta\lambda_r\right|\leq P_{n-1}\sum_{\nu=1}^{n-1}\left|\Delta\lambda_\nu\right|\Rightarrow\frac{1}{P_{n-1}}\sum_{r=1}^{n-1}P_\nu\left|\Delta\lambda_r\right|\leq\sum_{\nu=1}^{n-1}\left|\Delta\lambda_\nu\right|=O(1)$$

$$=O(1)\sum_{\nu=1}^{m}P_\nu\left|\Delta\lambda_\nu\right|X_\nu^k\sum_{n=\nu+1}^{m+1}\alpha_n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k\left(\frac{1}{P_{n-1}}\right)$$

$$=O(1)\sum_{\nu=1}^{m}\left|\Delta\lambda_\nu\right|X_\nu^k \text{ , by (3.2)}$$

$$=O(1)\quad as\ m\to\infty \text{ , by (3.4).}$$

Further,

$$\sum_{n=2}^{m+1}\alpha_n^{\delta k+k-1}\left|T_{n,3}\right|^k$$

$$=\sum_{n=2}^{m+1}\alpha_n^{\delta k+k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n-1}P_\nu s_\nu \lambda_{\nu+1}\Delta X_\nu\right|^k$$

$$=O(1)\sum_{n=2}^{m+1}\alpha_n^{\delta k+k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left\{\sum_{\nu=1}^{n-1}P_\nu\left|\lambda_{\nu+1}\right|\left|s_\nu\right|\frac{1}{\nu}\right\}^k \quad \text{, by (3.1)}$$

$$=O(1)\sum_{n=2}^{m+1}\alpha_n^{\delta k+k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k\left\{\sum_{\nu=1}^{n-1}p_\nu X_\nu\left|\lambda_{\nu+1}\right|\right\}^k$$

$$=O(1)\sum_{n=2}^{m+1}\alpha_n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k\left(\frac{1}{P_{n-1}}\right)\left\{\sum_{\nu=1}^{n-1}p_\nu X_\nu^k\left|\lambda_{\nu+1}\right|^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}p_\nu\right\}^{k-1}$$

$$=O(1)\sum_{\nu=1}^{m}p_\nu\left|\lambda_{\nu+1}\right|^k X_\nu^k\sum_{n=\nu+1}^{m+1}\alpha_n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k\left(\frac{1}{P_{n-1}}\right)$$

$$=O(1)\sum_{\nu=1}^{m}\frac{p_\nu}{P_\nu}\left|\lambda_{\nu+1}\right|^k X_\nu^k \text{ , by (3.2)}$$

$$=O(1)\sum_{\nu=1}^{m}\left|\lambda_{\nu+1}\right|^k X_\nu^k\frac{p_{\nu r}}{P_\nu}\ \frac{P_\nu}{\nu\ p_\nu}\text{, as }\ X_n=\frac{P_n}{np_n}$$

$$=O(1)\sum_{\nu=1}^{m}X_\nu^{k-1}\frac{\left|\lambda_{\nu+1}\right|^k}{\nu}$$

$$=O(1)\quad as\ m\to\infty \text{ , by (3.3).}$$

Now,

$$\sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \left|T_{n,4}\right|^k$$

$$= \sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \left|\frac{p_n s_n \lambda_n X_n}{P_n}\right|^k$$

$$= O(1)\sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} X_n^k \left|\lambda_n\right|^k \left(\frac{p_n}{P_n}\right)^k$$

$$= O(1)\sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \left(\frac{P_n}{np_n}\right)^k \left|\lambda_n\right|^k \left(\frac{p_n}{P_n}\right)^k \quad \text{, as } X_n = \frac{P_n}{np_n}$$

$$= O(1)\sum_{n=2}^{m+1} \alpha_n^{\delta k+k-1} \frac{\left|\lambda_n\right|^k}{n^k}$$

$$= O(1) \quad as\ m \to \infty \text{, by (3.5) .}$$

This completes the proof of the Lemma.

6. PROOF OF THE THEOREM

Since the behavior of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behavior of the function in the immediate neighborhood of this point only, the truth of the theorem is necessary consequence of the Lemma.

REFERENCES

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