ON THE FRACTIONAL-ORDER DISTRIBUTED OF A SELF-DEVELOPING MARKET ECONOMY VIA MULTI-STEP DIFFERENTIAL TRANSFORMATION METHOD

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ABSTRACT
In this paper, we research a fractional-order distributed of a self-developing market economy. Furthermore, we performed a detailed analysis on the stability of equilibrium. Multi-step differential transform method (MsDTM) extends to give approximate and analytical solutions of a fractional-order distributed of a self-developing market economy. Numerical simulations are present to verify the reliability and effectiveness results obtained from these methods.

Keywords:*the fractional-order distributed of a self-developing market economy , Multi-step Differential Transformation Method,Stability, Numerical results*

1. INTRODUCTION
For over decades years, scientists in many disciplines have been intensively investigated the properties of chaos. It has been proven to be useful in a variety of disciplines, such as information processing, preventing the collapse of power systems, high-performance circuits and devices, and liquid mixing with low power consumption [1]. The first chaotic attractor was found by the American meteorologist/mathematician Edward N. Lorenz [2]. As he computing numerical solutions to the simple system of three autonomous ordinary differential equations that he came up with, he discovered that initial conditions with small differences eventually produced vastly different solutions. He had observed sensitivity to initial conditions, a characteristic of chaos. The Lorenz model is a dissipative system, which some property of the flow, such as total energy, is conserved. In 1976, Rossler conducted important work that rekindled the interest in low dimensional dissipative dynamical systems [3]. In 1979, Rossler himself proposed an even simpler (algebraic) system [4]. Sprott embarked upon an extensive search [5] for autonomous three-dimensional chaotic systems with fewer than seven terms in the right hand side of the model equations.
On the fractional-order systems have been numerous studies over the last few years. Between their main, in [6]it has been shown that a limit cycle can be generated in the fractional order Wien bridge oscillator. Dynamics of the fractional order Van der Pol oscillator has been studied in [7]. Existence of a limit cycle for the fractional Brusselator has been shown in [8]. Also, it has been found that some fractional order differential systems can demonstrate chaotic behavior. The fractional order Chua circuit [9], the fractional order Duffing system [10], the fractional order jerk model [11], fractional-order Coullet system[12],the fractional order Lorenz system [13], the fractional order Chen system [14], the fractional order Lü system [15], the fractional-order Volta's system[16],the fractional order Rössler system [17], the fractional order Newton–Leipnik system [18], the fractional order Genesio–Tesi system [19], the fractional order Ikeda delay system [20], the fractional-order financial system proposed by Chen in [21] displays many interesting dynamic behaviors, such as fixed points, periodic motions, and chaotic motions and non-integer order cellular neural networks [22] are well known examples from these kinds of systems.

2. DISTRIBUTED OF A SELF-DEVELOPING MARKET ECONOMY
In 1991 N.A. Magnitiskii proposed for description of government controls applied to the development of a market economy. Differential equation system describing the variation of the macroeconomic variables as follow [23-24]:

$$\frac{dx}{dt}=bx\left(\left(1-\beta\right)z-\delta y\right),$$
$$\frac{dy}{dt}=x\left(1-\left(1-\delta\right)y+\beta z\right),$$
$$\frac{dz}{dt}=a\left(y-dx\right), \tag{1}$$
with the initial conditions :
$$x(0)=x_0,\ y(0)=y_0,\ z(0)=z_0.$$

Where fix the parameters γ,θ,η are the organic, production and inventory structure of capital respectively; also fix parameters α,σ and ω. The a,b,d parameters can be defined as follows:
$a=\frac{\alpha\theta}{\beta\eta}, b=\frac{\omega\theta}{(1+\theta+\eta)(1+\gamma)}, d=\frac{\theta+(\eta-1)(1+\gamma)}{\omega\theta}$. For instance, we take $\gamma=1, \theta=12, \eta=2, \sigma=\frac{6\alpha}{7}, \omega=1$ and δ *and* β are parameters that characterize the qualitative behavior of the solutions of system (1). In this paper, the constants are chosen as $a=7, b=0.4, d=1.7$.
In this paper, we consider a fractional-order distributed of a self-developing market economy [23, 24]. After some constraints[23,24] , a three-dimensional model was obtained as follows . The components of the basic three-component model are denoted respectively by $x(t), y(t)$ and $z(t)$. These quantities satisfy

$$\begin{aligned}
&\frac{d^{q_1}x}{dt^{q_1}}=bx\left((1-\beta)z-\delta y\right),\\
&\frac{d^{q_2}y}{dt^{q_2}}=x\left(1-(1-\delta)y+\beta z\right),\\
&\frac{d^{q_3}z}{dt^{q_3}}=a\left(y-dx\right),\\
&\text{with the initial conditions:}\\
&x(0)=x_0, y(0)=y_0, z(0)=z_0.
\end{aligned} \tag{2}$$

This was done with the standard parameter values given above and initial values $x(0)=1.31, y(0)=1.53$ and $z(0)=0.5$ for the three-component model.
The goal of this paper is to extend application to classical DTM and multi-step DTM for obtained approximant analytical solution and stability of the aboved mentioned the fractional-order distributed of a self-developing market economy. The differential transform method (DTM) was first proposed by Zhou [25]. The interested reader can see the Refs. [26-34] for development of DTM. This technique has been employed to solve a large variety of linear and nonlinear problems. One can see the Refs. [34-45] for more applications of the differential transformation method in various problems of physics and engineering.

The paper is organized as follows. In Section 3, the fractional calculus and some useful stability theorems of the fractional order systems are briefly introduced. Section 4 is about multi-step fractional differential transform method. Section 5 regards equilibrium points and their asymptotic stability of a fractional order distributed of a self-developing market economy. Section 6 focuses on the MsDTM to nonlinear chaotic fractional order ordinary differential Equation systems as the fractional-order distributed of a self-developing market economy. The conclusions are given in Section 7.

3. FRACTIONAL DIFFERENTIAL TRANSFORM METHOD

Consider a general system of fractional differential equations[28,42, 44]:

$$\begin{aligned}
&D_*^{\alpha_1}x_1(t)+h_1\left(t,x_1,x_2,\ldots,x_m\right)=g_1(t),\\
&D_*^{\alpha_2}x_2(t)+h_2\left(t,x_1,x_2,\ldots,x_m\right)=g_2(t),\\
&\vdots\\
&D_*^{\alpha_m}x_m(t)+h_m\left(t,x_1,x_2,\ldots,x_m\right)=g_m(t),
\end{aligned} \tag{3}$$

where $D_*^{\alpha_i}$ is the derivative of x_i of order α_i in the sense of Caputo and $0<\alpha_i\le 1, (i=1,2,...,m)$ subject to the initial conditions

$$x_1(t_0)=d_1,\quad x_2(t_0)=d_2,\cdots,x_m(t_0)=d_m. \tag{4}$$

In this paper, we introduce the multi-step fractional differential transform method used in this paper to obtain approximate analytical solutions for the system of fractional differential equations (3). This method has been developed in[46] as follows:

$$D_{x_0}^q f(x)=\frac{1}{\Gamma(m-q)}\frac{d^m}{dx^m}\left[\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1+q-m}}dt\right], \tag{5}$$

For $m-1\le q<m, m\in Z^+, x>x_0$. Let us expand the analytical and continuous function $f(x)$ in terms of fractional power series as follows:

$$f(x)=\sum_{k=0}^{\infty}F(k)(x-x_0)^{\frac{k}{\alpha}} \tag{6}$$

Where α is the order of fraction and $F(k)$ is the fractional differential transform of $f(x)$ .in order to avoid fractional initial and boundary conditions, we define the fractional derivative in the Caputo sense. The relation between the Riemann-Liouville operator and Caputo operator is given by

$$D_{*x_0}^{q}f(x)=D_{x_0}^{q}\left[f(x)-\sum_{k=0}^{m-1}\frac{1}{k!}(x-x_0)^k f^{(k)}(x_0)\right] \tag{7}$$

Setting $f(x)-\sum_{k=0}^{m-1}\frac{1}{k!}(x-x_0)^k f^{(k)}(x_0)$ in Eq.(5) and using Eq. (7), we obtain fractional derivative in the Caputo sense [46] as follows:

$$D_{*x_0}^{q}f(x)=\frac{1}{\Gamma(m-q)}\frac{d^m}{dx^m}\left[\int_{x_0}^{x}\left\{\frac{f(t)-\sum_{k=0}^{m-1}\frac{1}{k!}(t-x_0)^k f^{(k)}(x_0)}{(x-t)^{1+q-m}}\right\}dt\right] \tag{8}$$

Since the initial conditions are implemented to the integer order derivatives, the transformation of the initial conditions are defined as follows:

$$F(k)=\begin{cases} if\ \frac{k}{\alpha}\in Z^+, & \frac{1}{\left(\frac{k}{\alpha}\right)!}\left[\frac{d^{\frac{k}{\alpha}}f(x)}{dx^{\frac{k}{\alpha}}}\right]x=x_0\ for\ k=0,1,2,\ldots,(q\alpha-1) \\ if\ \frac{k}{\alpha}\notin Z^+ & 0, \end{cases} \tag{9}$$

where, q is the order of fractional differential equation considered.

Lemma 3.1([53]).Autonomous system $D_t^q x(t)=Ax,\ x(0)=x_0$ is asymptotically stable if and only if

$$|\arg(eig(A))|>\frac{q\pi}{2}. \tag{10}$$

In this case, each component of the states decays toward 0 like t^{-q}. Also, this system is stable if and only if $|\arg(eig(A))|\geq\frac{q\pi}{2}$ and those critical eigenvalues that satisfy $|\arg(eig(A))|=\frac{q\pi}{2}$ have geometric multiplicity one.

Lemma 3.2([55]).Consider the following n-dimensional linear fractional order system:

$$\begin{aligned} D_t^{q_1}x_1&=a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n\\ D_t^{q_2}x_2&=a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n\\ &\vdots\\ D_t^{q_n}x_n&=a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n \end{aligned} \tag{11}$$

Where all q_i are the rational numbers between 0 and 1. Assume M to be lowest common multiple of the denominators u_i's of q_i's, where $q_i=\frac{v_i}{u_i},(u_i,v_i)=1,\ u_i,v_i\in Z^+, i=1...n.$ Then the zero solution system (11) is Lyapunov globally asymptotically stable if all the roots of equation

$$\det(\Delta(\lambda))=\det\left(diag\left(\left[\lambda^{Mq_1},\lambda^{Mq_2},\cdots,\lambda^{Mq_n}\right]\right)-(a_{ij})_{nxn}\right)=0 \text{ satisfy } |\arg(\lambda)|>\frac{\pi}{2M}. \tag{12}$$

From Lemma 3.1, we can see that an equilibrium point is asymptotically stable if the condition $\frac{\pi}{2M}-\min_i|\arg(\lambda_i)|<0$ is satisfied. The term $\frac{\pi}{2M}-\min_i|\arg(\lambda_i)|$ is called the instability measure for equilibrium points in fractional order systems (IMFOS). It should be noticed that some authors have proved by numerical simulations that the condition $\text{IMFOS} \geq 0$ is only a necessary condition and not the sufficient one for the system to show chaos [55].

The following theorems that can be deduced from Eqs.(5) and (6) are given below, for proofs and detailed see[48].

Theorem 3.1. If $z(t) = x(t) \pm y(t)$, then $Z(k) = X(k) \pm Y(k)$.

Theorem 3.2. If $z(t) = cy(t)$, then $Z(k) = cY(k)$.

Theorem 3.3. If $z(t) = x(t)y(t)$, then $Z(k) = \sum_{k_1=0}^{k} X(k_1)Y(k-k_1)$.

Theorem 3.4. If $z(t) = (t-t_0)^n$, then $Z(k) = \delta(k-\alpha p)$ where,

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Theorem 3.5. If $z(t) = D_{t_0}^{q}\left[g(t)\right]$ then $Z(k) = \dfrac{\Gamma\left(q+1+\frac{k}{\alpha}\right)}{\Gamma\left(1+\frac{k}{\alpha}\right)} G(k+\alpha q)$.

According to fractional DTM, by taking differential transformed both sides of the systems of equations given Eqs.(3) and (4) is transformed as follows:

$$\begin{aligned}
&\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} X_1(k+1) + H_1(k) = G_1(k), \\
&\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} X_2(k+1) + H_2(k) = G_2(k), \\
&\vdots \\
&\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} X_m(k+1) + H_m(k) = G_m(k).
\end{aligned} \tag{13}$$

$$X_1(0) = d_1, \quad X_2(0) = d_2, \cdots, X_m(0) = d_m. \tag{14}$$

Therefore, according to DTM the N −term approximations for (3) can be expressed as

$$\begin{aligned}
&\varphi_{1,n}(t) = x_1(t) = \sum_{k=1}^{N} X_1(k) t^{k\alpha}, \\
&\varphi_{2,n}(t) = x_2(t) = \sum_{k=1}^{N} X_2(k) t^{k\alpha}, \\
&\vdots \\
&\varphi_{m,n}(t) = x_m(t) = \sum_{k=1}^{N} X_m(k) t^{k\alpha},
\end{aligned} \tag{15}$$

4. MULTI-STEP FRACTIONAL DIFFERENTIAL TRANSFORM METHOD

The approximate solutions (3)-(4) are generally not valid for large t. A simple way of ensuring validity of the approximations for large t is to treat (13)-(14) as an algorithm for approximating the solutions of (3)-(4) in a sequence of intervals choosing the initial approximations as

$$\begin{aligned}
&x_{1,0}(t) = x_1(t^*) = d_1^*, \\
&x_{2,0}(t) = x_2(t^*) = d_2^*, \\
&\vdots \\
&x_{m,0}(t) = x_m(t^*) = d_m^*.
\end{aligned} \tag{16}$$

In order to carry out the iterations in every subinterval $[0,t_1),[t_1,t_2),[t_2,t_3),\ldots,[t_{j-1},t)$ of equal length h, we would need to know the values of the following[28],

$$x_{1,0}^*(t) = x_1(t^*), \; x_{2,0}^*(t) = x_2(t^*), \; \ldots, \quad x_{m,0}^*(t) = x_m(t^*). \tag{17}$$

But, in general, we do not have these information at our clearance except at the initial point $t^* = t_0$. A simple way for obtaining the necessary values could be by means of the previous n-term approximations $\varphi_{1,n}, \varphi_{2,n}, \ldots, \varphi_{m,n}$ of the preceding subinterval, i.e.,

$$x_{1,0}^* \cong \varphi_{1,n}\left(t^*\right), x_{2,0}^* \cong \varphi_{2,n}\left(t^*\right), \ldots, x_{m,0}^* \cong \varphi_{m,n}\left(t^*\right). \tag{18}$$

5. EQUILIBRIUM POINTS AND THEIR ASYMPTOTIC STABILITY

To evaluate the equilibrium points of (2), let

$$\begin{cases} D^{q_1} x = 0, \\ D^{q_2} y = 0, \\ D^{q_3} z = 0. \end{cases} \tag{19}$$

Then $E_0\left(x_0, y_0, z_0\right) = E_0\left(\frac{1-\beta}{d\left(1-\beta-\delta\right)}, \frac{1-\beta}{\left(1-\beta-\delta\right)}, \frac{\delta}{\left(1-\beta-\delta\right)}\right)$ is the equilibrium point. The jacobian matrix $J(E_0)$ for system (2) evaluated at the E is given by

$$J(x,y,z) = \begin{pmatrix} b\left(1-\beta\right)z - b\delta y & -b\delta x & b\left(1-\beta\right)x \\ 1+\left(\delta-1\right)y+\beta z & \left(\delta-1\right)x & \beta x \\ -ad & a & 0 \end{pmatrix}, \tag{20}$$

$$J(E_0) = \begin{pmatrix} 0 & \frac{-b\delta\left(1-\beta\right)}{d\left(1-\beta-\delta\right)} & \frac{b\left(1-\beta\right)^2}{d\left(1-\beta-\delta\right)} \\ \frac{2\left(1-\beta-\delta+\delta\beta\right)}{\left(1-\sigma-\delta\right)} & \frac{\left(\delta-1\right)\left(1-\beta\right)}{d\left(1-\beta-\delta\right)} & \frac{\sigma\left(1-\beta\right)}{d\left(1-\beta-\delta\right)} \\ -ad & a & 0 \end{pmatrix}, \tag{21}$$

Denote

$$Q = \frac{1-\beta}{1-\beta-\delta}. \tag{22}$$

Then the characteristic equation of the linearized system is

$$P\left(\lambda\right) = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3, \tag{23}$$

where

$$b_1 = \frac{\left(1-\delta\right)Q}{d}, \quad b_2 = \frac{aQ\left(bd(1-\beta)-\beta\right)}{d}, \quad b_3 = \frac{ab(1-\beta)Q}{d}. \tag{24}$$

Proposition 5.1. $E_0\left(\frac{Q}{d}, Q, \frac{\delta Q}{1-\beta}\right)$ is asymptotically stable if all of the eigenvalues λ of $J(E_0)$ satisfy $|\arg(\lambda)| > \frac{q\pi}{2}$.

Denote

$$D(P) = \begin{bmatrix} 1 & b_1 & b_2 & b_3 & 0 \\ 0 & 1 & b_1 & b_2 & b_3 \\ 3 & 2b_1 & b_2 & 0 & 0 \\ 0 & 3 & 2b_1 & b_2 & 0 \\ 0 & 0 & 3 & 2b_1 & b_2 \end{bmatrix} \tag{25}$$

$$= 18b_1b_2b_3 + \left(b_1b_2\right)^2 - 4b_3b_1^3 - 4b_2^3 - 27a_3^2.$$

Using the results of Ref.[51-55], we have

Proposition 5.2.(i) if the discriminant of $P(\lambda)$, $D(P)$ is positive, then E_0 is asymptotically stable if Routh-Hurwitz conditions are satisfied, i.e

$$b_1 > 0, b_3 > 0, b_1 b_2 > b_3 \quad \text{if } D(P) > 0.$$

Table 1. As values of $a = 7, b = 0.4, d = 1.17$ are kept constant, other changes[24]

δ	β	b_1	b_2	b_3	$b_1 b_2 - b_3$	$D(p)$	$\min_i \lvert\arg(\lambda_i)\rvert$	Eigenvalues
0.65	0.278	2.999	3.593	17.326	-6.547	-6684	0.939	(-3.42, 0.21-2.23i, 0.21+2.23i)
0.4	0.05	0.885	4.077	3.926	-0.314	-430	0.99	(-0.94, 0.031-2.03i, 0.031+2.03i)
0.65	-1	0.443	17.15**9**	7.09	0.5140	-20543	1	(-0.41, 0.014-4.14i, 0.014+4.14i)
0.7	-1	0.394	17.819	7.363	-0.334	-23119	0.99	(-0.413, 0.009-4.22i, 0.009+4.22i,)

6. NUMERICAL RESULTS

We will apply classic DTM and the MsDTM to nonlinear chaotic fractional order ordinary differential Equation systems as a distributed of a self-developing market economy

By applying DTM to Eq.(2)

$$X(k+1) = \frac{\Gamma(q_1 k+1)}{\Gamma(q_1(k+1)+1)}\left\{b(1-\beta)\sum_{k_1=0}^{k} X(k_1)Z(k-k_1) - b\delta\sum_{k_1=0}^{k} X(k_1)Y(k-k_1)\right\},$$

$$Y(k+1) = \frac{\Gamma(q_2 k+1)}{\Gamma(q_2(k+1)+1)}\left\{X(k) - (1-\delta)\sum_{k_1=0}^{k} X(k_1)Y(k-k_1) + \beta\sum_{k_1=0}^{k} X(k_1)Z(k-k_1)\right\},$$

$$Z(k+1) = \frac{\Gamma(q_3 k+1)}{\Gamma(q_3(k+1)+1)}\left\{aY(k) - adX(k)\right\}, \tag{26}$$

with initial state

$X(0) = 1.31, Y(0) = 1.53, Z(0) = 0.5$ throughout the paper.

By applying the multi-step DTM to Eq.(2), $X_i(n), Y_i(n), Z_i(n)$ satisfy the following recurrence relations for $n = 1, 2 \ldots N-1$.

$$X_i(k+1) = \frac{\Gamma(q_1 k+1)}{\Gamma(q_1(k+1)+1)}\left\{b(1-\beta)\sum_{k_1=0}^{k} X_i(k_1)Z_i(k-k_1) - b\delta\sum_{k_1=0}^{k} X_i(k_1)Y_i(k-k_1)\right\},$$

$$Y_i(k+1) = \frac{\Gamma(q_2 k+1)}{\Gamma(q_2(k+1)+1)}\left\{X_i(k) - (1-\delta)\sum_{k_1=0}^{k} X_i(k_1)Y_i(k-k_1) + \beta\sum_{k_1=0}^{k} X_i(k_1)Z_i(k-k_1)\right\},$$

$$Z_i(k+1) = \frac{\Gamma(q_3 k+1)}{\Gamma(q_3(k+1)+1)}\left\{aY_i(k) - adX_i(k)\right\}, \tag{27}$$

$$X_0(0) = 1.31, Y_0(0) = 1.53, Z_0(0) = 0.5.$$

$$X_i(0) = X_{i-1}(t_i), Y_i(0) = Y_{i-1}(t_i), Z_i(0) = Z_{i-1}(t_i), i = 1, 2, \ldots, K \le M$$

We assume the following parameters of system (2): $a = 7, b = 0.4, d = 1.17$ and initial conditions $x_0 = 1.31$, $y_0 = 1.53$ and $z_0 = 0.5$.

6.1. Equal size order case $(q_1 = q_2 = q_3 = q)$

The equilibrium points of system (2) and the eigenvalues of its corresponding Jacobian matrix are given in Table 1. From Table 1, it can be seen that E_0 are saddle points for different values of δ and β. Figs. 1–3 indicate the approximate solutions obtained using the MSDTM and the classical Runge–Kutta method of $x(t)$, $y(t)$ and $z(t)$

when α is one. From the graphical results in Figs. 1–3, it can be seen that the results obtained using the multi-step differential transform method match the results of the classical Runge–Kutta method very good, which implies that the presented method can predict the behavior of these variables accurately for the region under consideration.

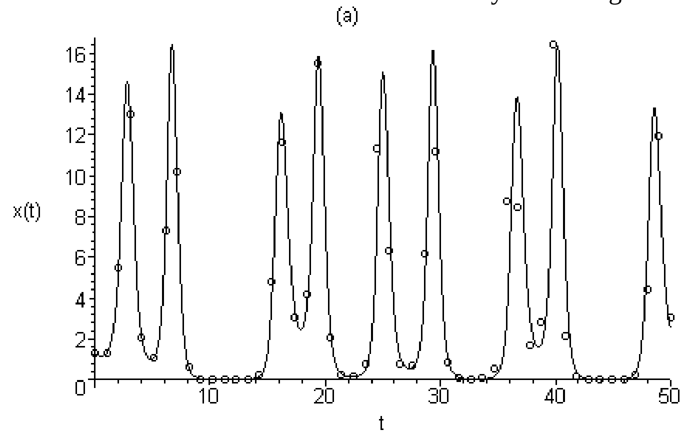

Figure 1. comparison of $x(t)$ *; solid line, 10 term-MSDTM* $(dt = 0.001)$ *; circle, Runge–Kutta method* $(dt = 0.01)$ *for* $q = 1$.

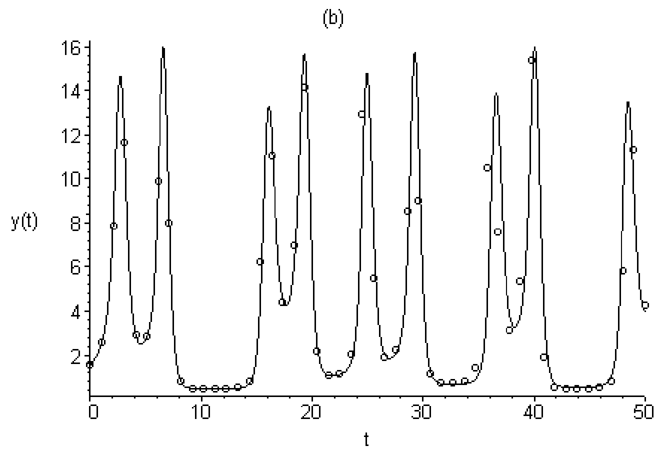

Figure 2. comparison of $y(t)$ *; solid line, 10 term-MSDTM* $(dt = 0.001)$ *; circle, Runge–Kutta method* $(dt = 0.01)$ *for* $q = 1$.

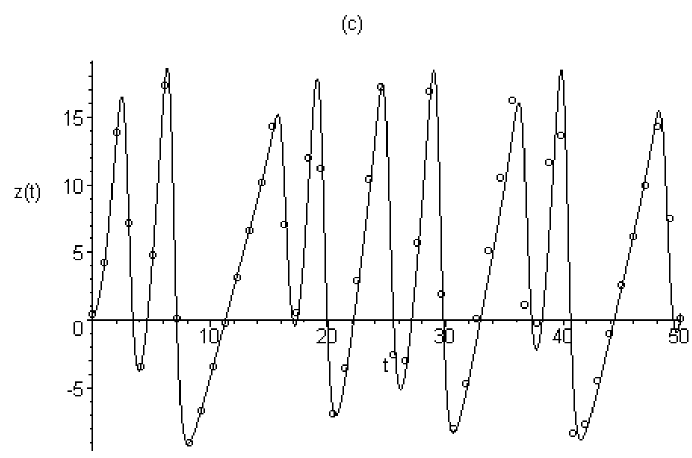

Figure 3. comparison of $z(t)$ *; solid line, 10 term-MSDTM* $(dt = 0.001)$ *; circle, Runge–Kutta method* $(dt = 0.01)$ *for* $q = 1$.

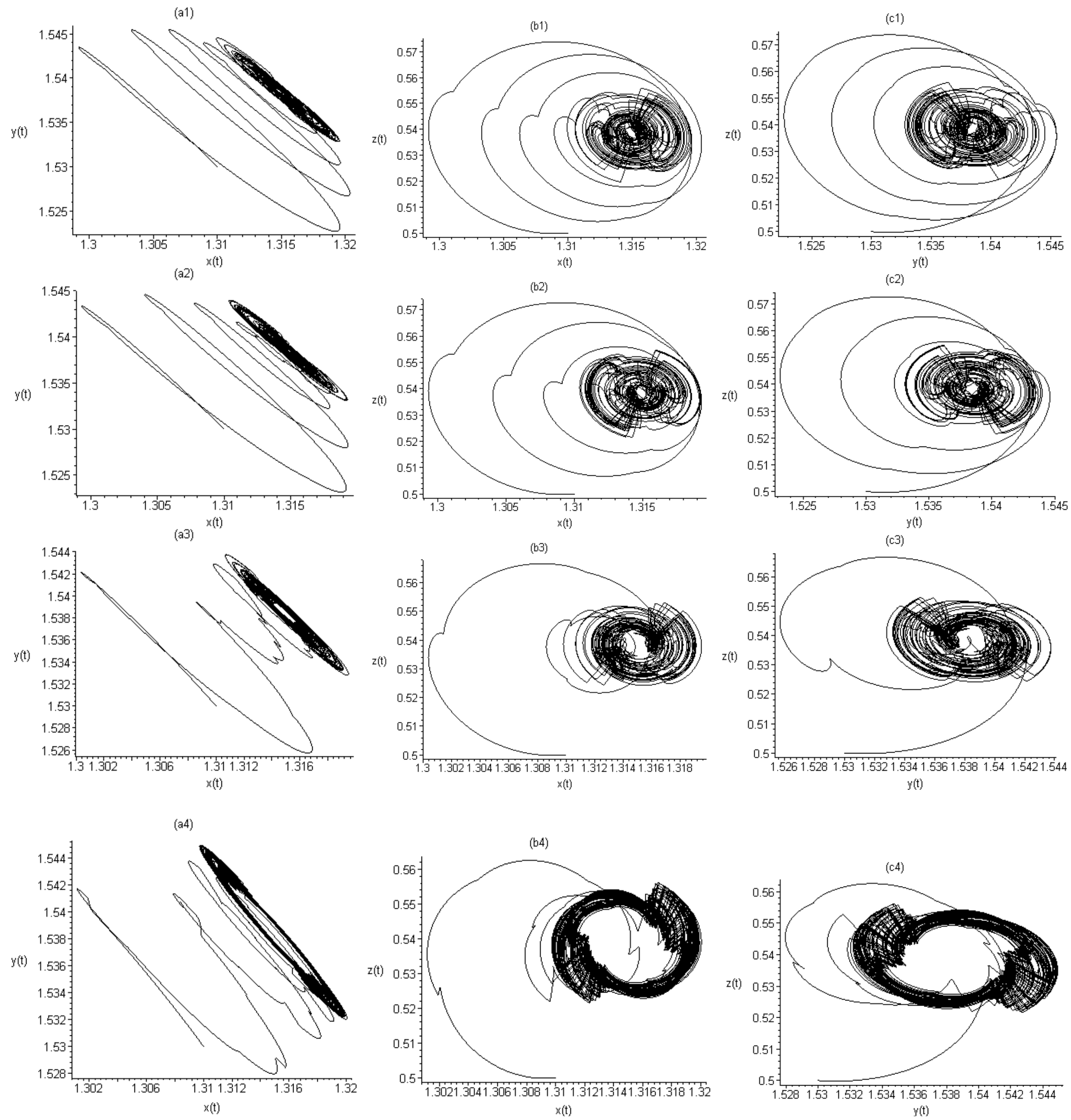

Fig.4. Chaotic attractors obtained 5 term-MsDTMwith $a=7, b=0.4, d=1.17, \beta=-1, \delta=0.7$ *,* $dt=0.01$ *(a1)-(b1)-(c1) for* $q=1$ *(a2)-(b2)-(c2) for* $q=0.99$ *(a3)-(b3)-(c3) for* $q=0.9$ *and (a4)-(b4)-(c4) for* $q=0.7$ *with time span* $[0,100]$.

Fix $(q_1 = q_2 = q_3 = q)$, and let q vary. The system (2) is calculated numerically against $q \in [0.7, 1]$, while the incremental value of q is 0.01. Fig. 5 showed the phase diagram with $q = 1, 0.99, 0.9, 0.7$, respectively. In addition, $x-y, x-z$ and $y-z$ phase diagrams are shown in Fig. 5, respectively.

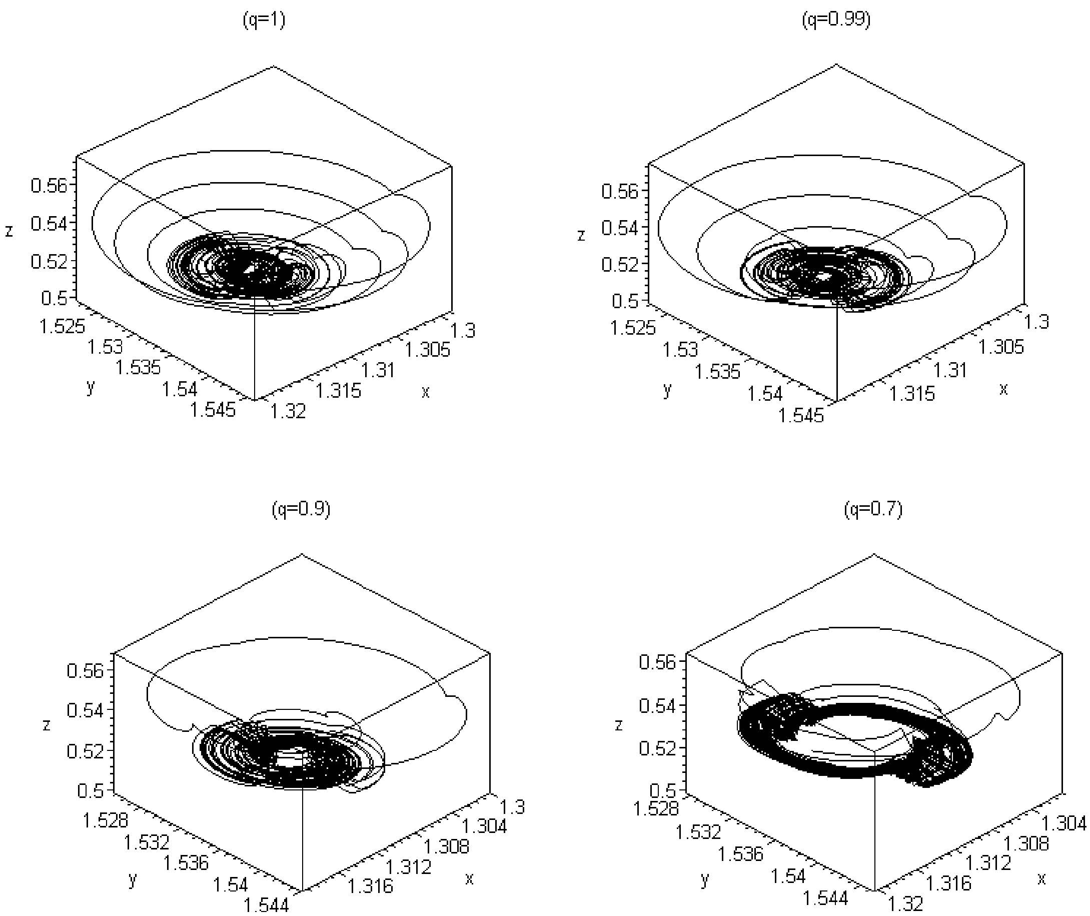

Figure 5. Phase portrait of a fractional-order distributed of a self-developing market economy for different q values and $a = 7, b = 0.4, d = 1.17, \beta = -1, \delta = 0.7$, $dt = 0.01$ *with time span* $[0,100]$.

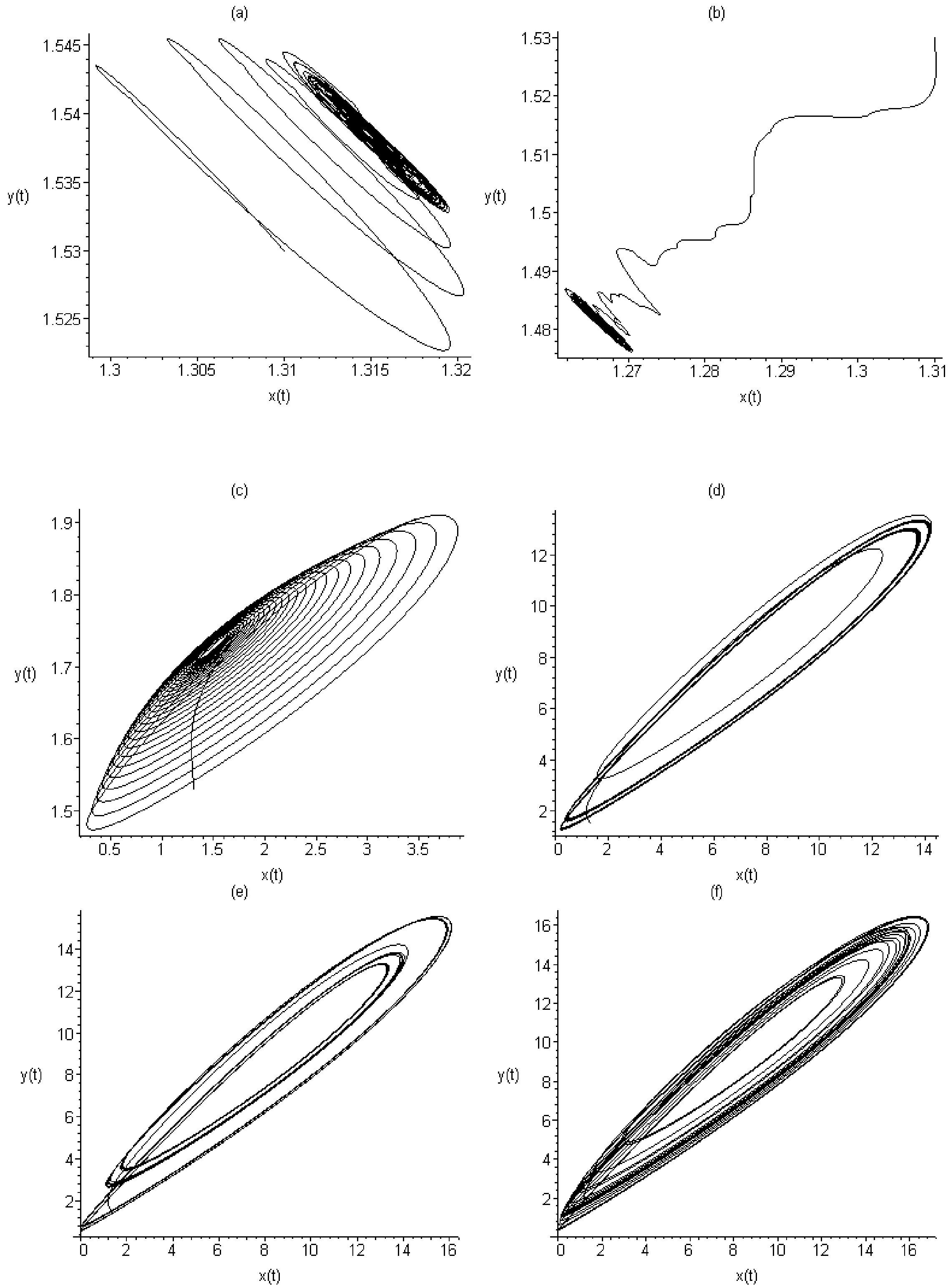

Figure 6. phase diagram of system (2) with fractional order $q=1$ *and (a)* $\beta=-1, \delta=0.7$ *, (b)* $\beta=-1, \delta=0.65$ *,(c)* $\beta=0.05, \delta=0.4$ *, (d)* $\beta=0.266, \delta=0.65$ *(e)* $\beta=0.276, \delta=0.65$ *and(f)* $\beta=0.28, \delta=0.65$ *with time span* $[0,100]$.

According to the fixed fractional order $q = 1$, phase diagram of system (2) given for values of the different β and δ in Figure 6. Also, the economy is destroyed due to a global crisis in the event of $\beta = 0.284, \delta < 0.65$.

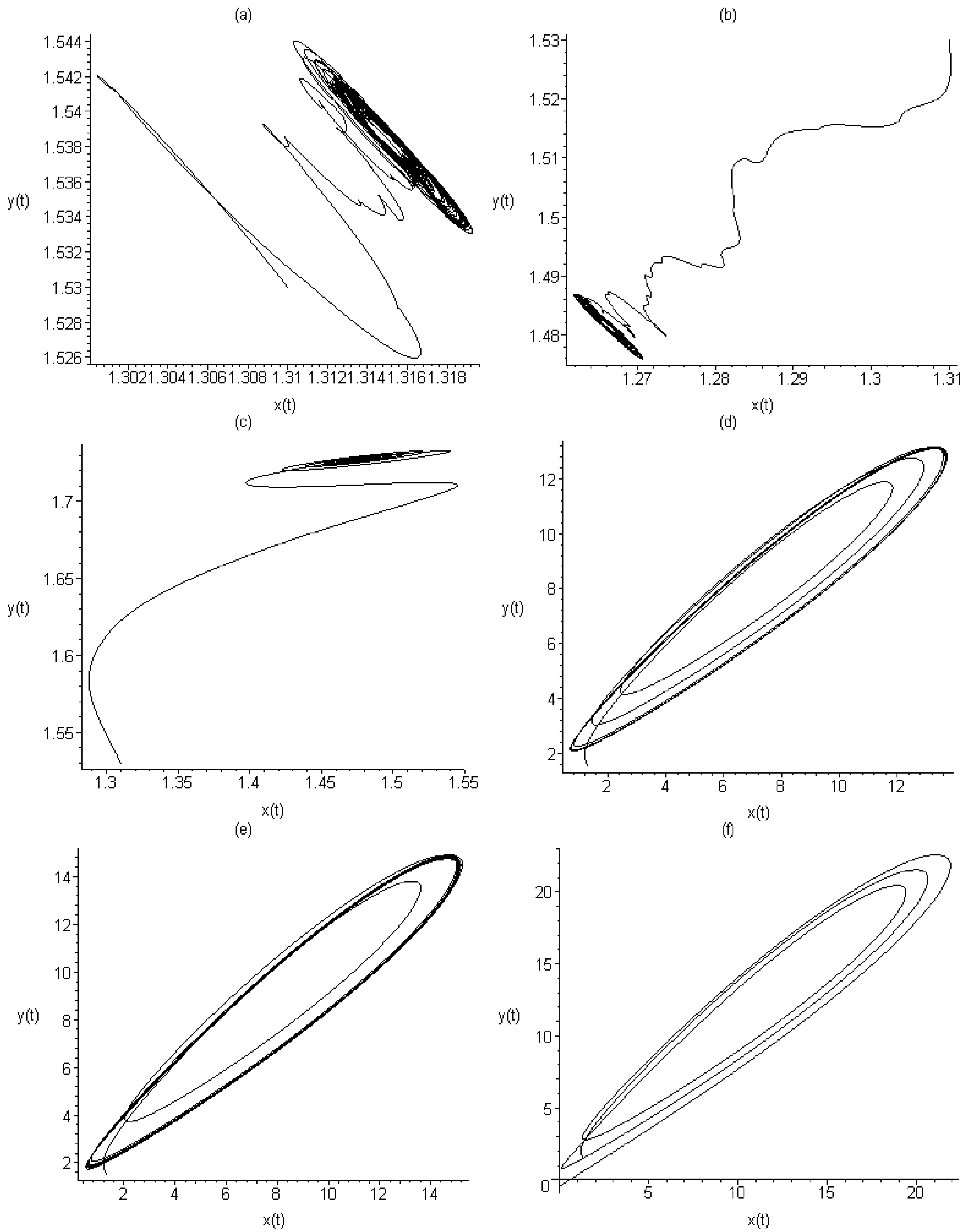

Figure 7. phase diagram of system (2) with fractional order $q = 0.89$ *and (a)* $\beta = -1, \delta = 0.7$ *, (b)* $\beta = -1, \delta = 0.65$ *,(c)* $\beta = 0.05, \delta = 0.4$ *, (d)* $\beta = 0.266, \delta = 0.65$ *(e)* $\beta = 0.276, \delta = 0.65$ *and(f)* $\beta = 0.3, \delta = 0.65$ *with time span* $[0,100]$.

According to the fixed fractional order $q = 0.89$, phase diagram of system (2) given for values of the different β and δ in Figure 7.

6.2. Unequal size order case
In the section, as examples we examine the following situations.

Case I: when $q_2 = q_3 = 1,$ and let q_1 reduce to less than 1

$q = (0.99,1,1)$ and $v_1 = 99, u_1 = 100, v_2 = u_2 = v_3 = u_3 = 1$, we get $m = 100$, $a = 7, b = 0.4, d = 1.17, \beta = -1, \delta = 0.7$ and $\det\left(diag\left(\left[\lambda^{Mq_1}, \lambda^{Mq_2}, \cdots, \lambda^{Mq_n}\right]\right) - J\left(E_0\right)\right) = 0$ becomes,

By Lemma 3.2, Eq.(12) is simplified as

$$\lambda^{299} + .394477317300000018\lambda^{199} + 9.204470737\lambda^{99} + 7.363576585 + 8.615384611\lambda^{100} = 0. \quad (28)$$

Solving the above equation we have

$$\frac{\pi}{2M} - \min_i \left|\arg\left(\lambda_i\right)\right| = 0.01570796327 > 0. \quad (29)$$

Thus, for the given derivative orders $q = (0.99,1,1)$, system (2) satisfy the necessary condition to exhibit chaos. Numerical results in Fig. 8 confirms this conclusion.

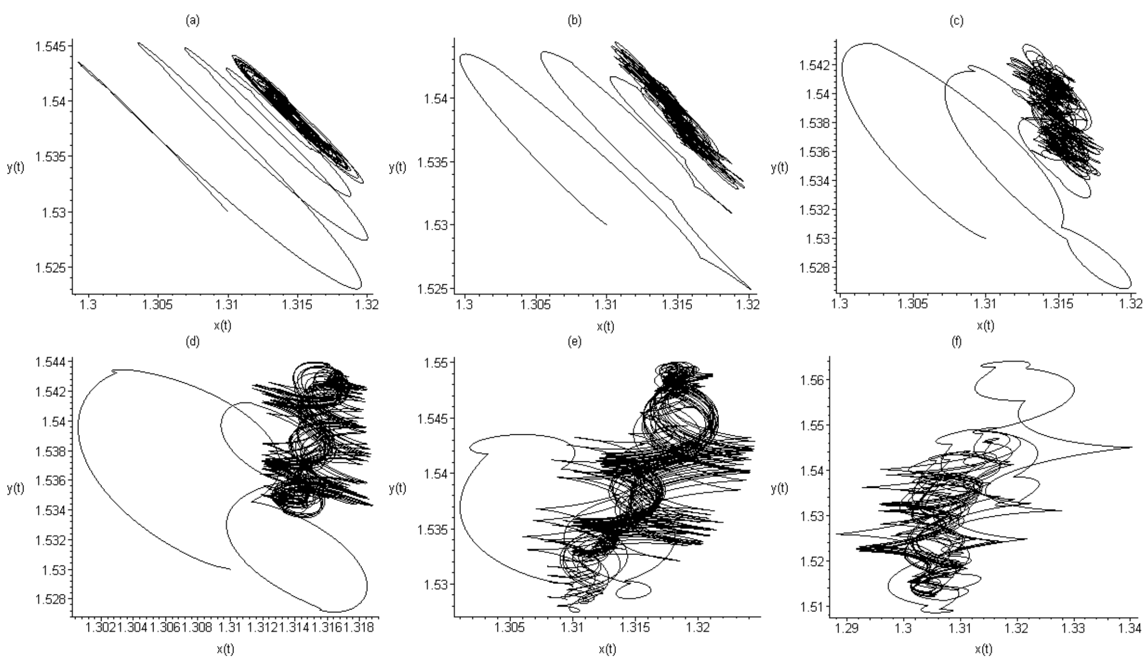

Figure 8. phase diagram of system (2) with $a = 7, b = 0.4, \beta = -1, \delta = 0.7$ *(a) fractional order* $q = (0.99,1,1)$ *(b)* $q = (0.89,1,1)$ *,(c)* $q = (0.79,1,1)$ *, (d)* $q = (0.69,1,1)$ *(e)* $q = (0.59,1,1)$ *and(f)* $q = (0.49,1,1)$ *with time span* $[0,100]$.

System (2) is obtained by numerical solution using MsDTMagainst $q_1 \in [0.49, 0.99]$, while the incrementalvalue of q_1 is 0.01. Numerical resultsindicate that system (2) will continue chaotic motion for $q_1 \in [0.49, 1]$.

Case II: when $q_1 = q_3 = 1,$ and let q_2 reduce to less than 1

$q = (1, 0.8, 1)$ and $v_1 = 1, u_1 = 1, v_2 = 80, u_2 = 100, v_3 = u_3 = 1$, we get $m = 100$, $a = 7, b = 0.4, d = 1.17, \beta = 0.266, \delta = 0.65$ and $\det\left(diag\left(\left[\lambda^{Mq_1}, \lambda^{Mq_2}, \cdots, \lambda^{Mq_n}\right]\right) - J(E_0)\right) = 0$ becomes,

By Lemma 3.2, Eq.(12) is simplified as

$$\lambda^{299} + .394477317300000018\lambda^{199} + 9.204470737\lambda^{99} + 7.363576585 + 8.615384611\lambda^{100} = 0. \quad (30)$$

Solving the above equation we have

$$\frac{\pi}{2M} - \min_i \left|\arg(\lambda_i)\right| = 0.01570796327 > 0. \quad (31)$$

Thus, for the given derivative orders $q = (0.99, 1, 1)$, system (2) satisfy the necessary condition to exhibit chaos. Numerical results in Fig. 9 confirm this conclusion.

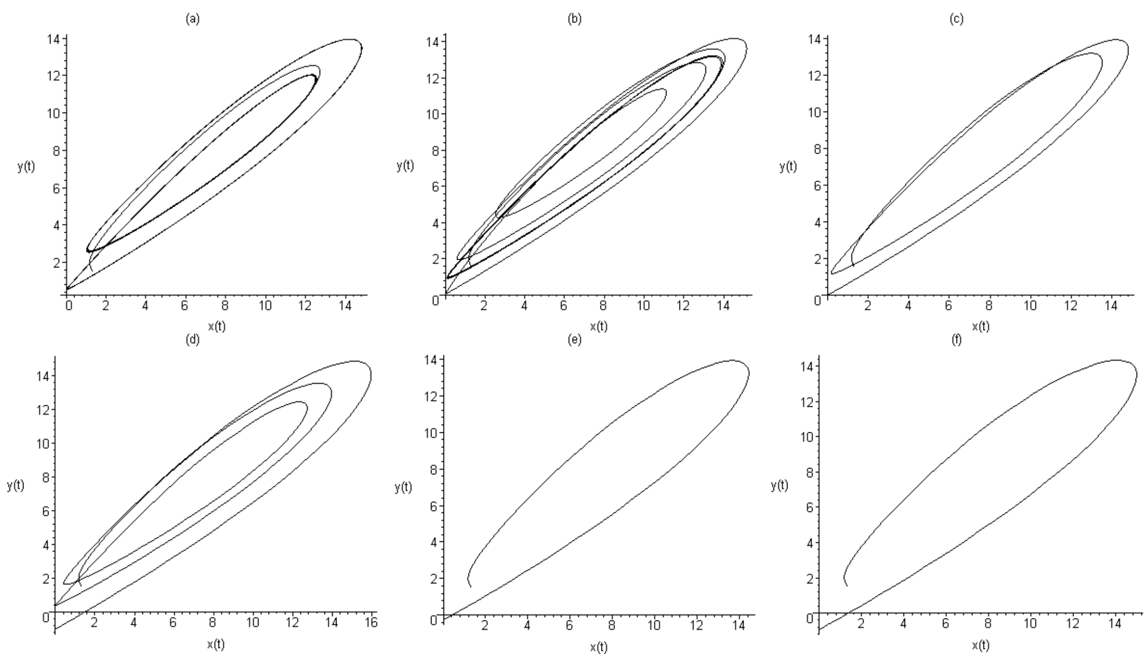

Figure 9.phase diagram of system (2) with $a = 7, b = 0.4, d = 1.17, \beta = 0.266, \delta = 0.65$ *(a) fractional order* $q = (1, 0.99, 1)$ *(b)* $q = (1, 0.98, 1)$ *,(c)* $q = (1, 0.97, 1)$ *, (d)* $q = (1, 0.96, 1)$ *(e)* $q = (1, 0.95, 1)$ *and(f)* $q = (1, 0.94, 1)$ *with time span* $[0, 100]$.

System (2) is obtained by numerical solution using MsDTMagainst $q_2 \in [0.94, 0.99]$, while the incremental value of q_2 is 0.01. Numerical results indicate that system (2) will continue periodic cyclefor $q_2 \in [0.96, 1]$, and it exhibits periodic motion in the other interval range. It is obvious in Fig. 9 (a)-(c) that system (2) displays cycle of period 3, a cycle of period 5 appears in Fig. 9(b), a cycle of period 2 appears in Fig. 9(c). But, this cycle is destroyed in Fig. 9(e)-(f).

Case III: when $q_1 = q_2 = 1,$ and let q_3 reduce to less than 1

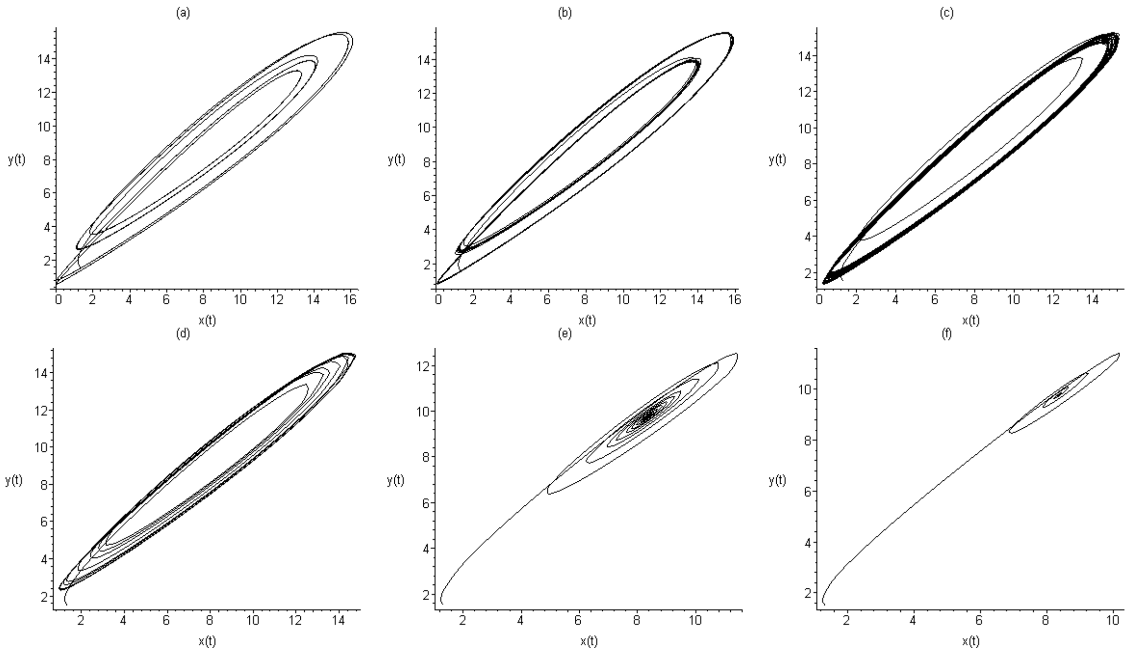

Figure 10. phase diagram of system (2) with $a = 7, b = 0.4, d = 1.17, \beta = 0.276, \delta = 0.65$ *(a) fractional order* $q = (1, 1, 0.99)$ *(b)* $q = (1, 1, 0.9)$ *,(c)* $q = (1, 1, 0.8)$ *, (d)* $q = (1, 1, 0.7)$ *(e)* $q = (1, 1, 0.6)$ *and(f)* $q = (1, 1, 0.5)$ *with time span* $[0,100]$.

System (2) is obtained by numerical solution using MsDTM against $q_3 \in [0.5, 0.99]$, while the incremental value of q_3 is 0.01. Numerical results indicate that system (2) will continue chaotic motion for $q_3 \in [0.7, 1]$, and it exhibits periodic motion in the other interval range. It is obvious in Fig. 10 (a) that system (2) displays cycle of period 5. It is explicit in Fig. 10 (b)-(c)-(d) that system (2) displays chaotic motion. This limit cycle contracts to a point and disappears through an inverse Hopf bifurcation in Fig. 10 (e)-(f).

7. CONCLUSION

In this paper, we deal with the fractional-order distributed of a self-developing market economy. We executed stability analysis on the fractional-order distributed of a self-developing market economy. Also, we carefully apply the multi-step DTM, a reliable modification of the DTM, that improves the convergenceof the series solution. The complex dynamics of the system are examined with the change of fractional order. The validity of the proposed method has beensuccessful by applying it for athe fractional-order distributed of a self-developing market economy . The method were used in a direct way without usinglinearization, perturbation or restrictive assumptions. It provides the solutions in terms of convergent series with easilycomputable components and the results have shown remarkable performance.

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