ANALYSIS AND APPLICATIONS OF LAPLACE /FOURIER TRANSFORMATIONS IN ELECTRIC CIRCUIT

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ABSTRACT
Solution of second order linear and ordinary, differential equations can be obtained by Laplace transformation, which converts differential equation into an algebraic equation with incorporation of the boundary conditions from the beginning. Also non-periodic function can be represented as an integral over a continuous range of frequencies. The concept of Laplace Transformation and Fourier Transformation play a vital role in diverse areas of science and technology such as electric analysis, communication engineering, control engineering, linear system, analysis, statistics, optics, quantum physics, solution of partial differential operation, etc. In solving problems relating to these fields, one usually encounters problems on time invariants, differential equations, time and frequency domains for non – periodic wave forms. This paper provides the reader with a solid foundation in the fundamentals of Laplace transform and Fourier transform; and gain an understanding of some of the very important and basic applications of these fundamentals to electric circuits and signal design and solution to related problems.

Keywords: *Laplace functions, Dirac delta function, Inverse Laplace, linearity.*

1. INTRODUCTION

Basically in communication, there is need to provide a link between time domain and the frequency domain for non-periodic waveforms. The solution of physical problems has been a challenge to the scientist and engineers alike. The analysis of electronic circuits and solution of linear differential equations is simplified by use of Laplace transform. The Laplace transform provides a method of analyzing a linear system using algebraic methods. The basic process of analyzing a system using Laplace transform involves conversion of the system transfer function or differential equation into *s* – domain, using *s* – domain to convert input functions, finding an output function by algebraically combing the input and transfer functions, using partial functions to reduce the output function to simpler components and conversion of output equation back to time domain [7,8]. The Laplace transform is denoted by in [8] $L\ \{f\ (t)\}$, has it function f (*t*), with *t* ($t \geq 0$) that transforms it to a function f (*s*) with a real argument s.

In engineering and science, the Laplace transform is used for solving problems of time invariant systems such as electrical circuits, harmonics, oscillations, mechanical system, control theory and optical devices. The Laplace in its analysis transforms the time domain in which outputs and inputs are function of time to the frequency domain (the inputs and output function of complex angular frequency in radians per unit time). Based on specifications, Laplace transform simplifies the process of analyzing the behaviour of a dynamic or synthesizing a new system. Fourier transform is used for decomposing signals into its constituent frequencies and its oscillatory functions. It represent signal in frequency domain and transforms one complex value function of a real variable into another.

Fourier transform entails representation of a non-periodic function not as a sum but as an integral over a continuous range of frequencies. [7]. This is done by converting infinite Fourier series in terms of series and cosines into a double infinite series involving complex exponentials. On other hands, Fourier transform basically involve frequency domain representation of non-periodic function, in which such representation is valid over the entire time domain and accomplished by means of Fourier integral[1].

2. LAPLACE TRANSFORMATION

In analyzing a system, one usually recounts *Time – Invariant, Linear Differential Equations* of second or high orders. Generally, it is difficult to obtain solutions of these equations in closed form via the solution methods in ordinary differential equations. One way to circumvent this problem is to apply LAPLACE TRANSFORMATION. Laplace transforms (converts) a differential equation into an algebraic equation in terms of the transform function of the unknown quantity intended.

The Laplace transform technique is based on the transformation expressed by

$$F(s)L\{f\ (t)\} = \int_{t=0}^{\infty} e^{-st} f(t)dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where:

f (*s*) indicates the Laplace transform of the function f (*t*) on condition that

f (*t*) = 0

t < 0

s = Complex variable known as *Laplace Variable*

L = *Laplace transform operator*.

For successful application of Laplace technique, it is imperative to include the transform integral based on the following conditions.

C_1 ==> f (*t*) is a piecewise continuous on every finite (time) interval of (0,∞).

C_2 ==> The magnitude of f (*t*) is bounded by an exponential function

$$| f(t)| \le Me^{at} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

For some real constant *a* and *M* and all $t \in (0, \infty)$

If the condition (C_1 and (C_2) exist for f (*t*), then its Laplace transform, f (*s*) exist for all Re (*s*) > *a*

Proof:

Whereby C_1 ensures that the integrand function, e^{-st} *(f (t),* is integrable on every finite interval.

Applying C_2, its magnitude is given as:

$$| e^{-st} f(t)| = e^{-st} \, | f(t)| \le e^{-st} Me^{at} Me^{(a-s)t} \ldots\ldots\ldots\ldots (3)$$

$$==> | F(s)| = | \int_{0-}^{\infty} f(t) e^{-st} dt | \le \int_{0-}^{\infty} Me^{(a-s)t} dt = \frac{M}{a-s} e^{(a-s)t} \Big|_{t=0}^{X} = \frac{M}{s-a} \ldots\ldots\ldots (4)$$

where the assumption has been used for integral evaluation

3. APPLICATIONS OF LAPLACE TRANSFORM

The Laplace transform technique is applicable in many fields of science and technology such as:

* Control Engineering
* Communication
* Signal Analysis and Design
* System Analysis
* Solving Differential Equations

4. DERIVATIVES OF BASIC LAPLACE TRANSFORM SPECIAL FUNCTIONS FOR SYSTEM ANALYSIS AND ELECTRIC SIGNAL DESIGN

Let us consider several special functions that are frequently encountered in system analysis and the derivation of their Laplace transforms. They include:

1. **Exponential Functions:** If we consider the graphical representation of the Laplace transform of exponential function in fig. 1.0.In[2,5], the Laplace transform is expressed as:

$$\mathcal{L}\{f(t)\} = \int_{0-}^{\infty} e^{-at} e^{-st} dt = \int_{0-}^{0+} e^{-at} e \; dt + \int_{0+}^{\infty} e^{-at} e^{-st} dt$$

$$= \frac{1}{-(s+a)} e^{-(st-a)t} \Big|_{t=0}^{\infty} = \frac{1}{s+a} \quad \infty \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

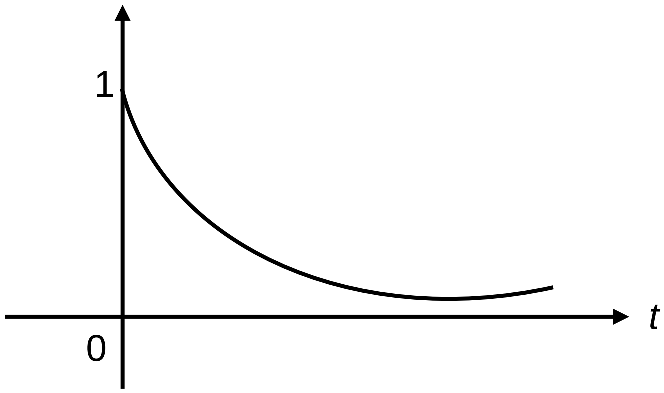


Fig 1.0

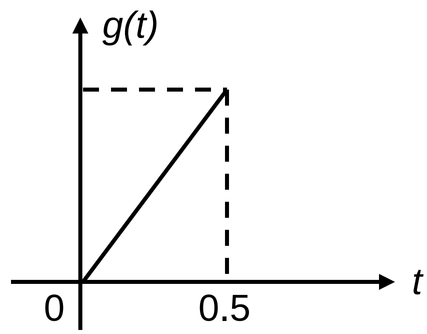


Fig 2.0

If we analyze the fig. 2.0 geometrically using Laplace transformation method. The analytical description of the signal is

$$gt = \begin{cases} 2t \text{ if } 0 < 0.5 \\ 0 \text{ otherwise} \end{cases}$$

Applying Laplace transform and integral by parts [4], gives

$$G(s) = \int_{0-}^{0.5} 2te^{-st}\,dt = \left(2t\,\frac{e^{-st}}{-s}\right)_{t=0}^{0.5} \int_{0-}^{0.5} 2\left(\frac{e^{-st}}{-s}\right) at$$

$$= \frac{e^{-0.5\,s}}{S^2}\left(2 + x\right) + \frac{2}{S^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

2. **<u>Unit Step Function (U_S(t) :</u>** The Unit – step function (fig. 3.0) can be said to be the force of magnitude 1 which is applied to a system at time t≥ 0 $^+$. So, it is defined in [7] as:

$$\mathcal{L}\,(U_S(t)) = U_S(s) = \int_0^{\infty} U_S(t)\,e^{-st}\,dt$$

$$= \int_0^{\infty} U_S(t)\,e^{-st}\,dt + \int_0^{\infty} U_S(t)\,e^{-st}\,dt = \int_{0-}^{\infty} e^{-st}$$

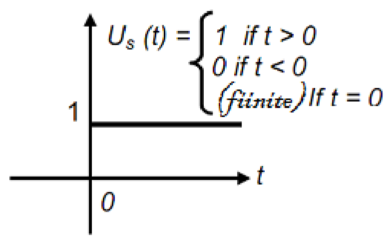


Fig. 3.0

$$= \frac{1}{s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

OR Unit step function U (x) is defined as:

$$U(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

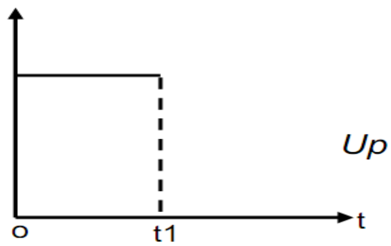
From the above definition, for any number C,

$$U(x - c) = \begin{cases} 0 & x < c \\ 1 & x \geq 0 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

3. **Unit – Pulse Function (*Up (t)*)** is defined in [6,8] as:

$$\mathcal{L}\,(Up(t)) = Up(s)$$

$$= \int_0^{t_1} e^{-st}\,dt = \left(\frac{e^{-st}}{-s}\right)_{t=0}^{t1} = \frac{1}{s}\left(1 - e^{-st1}\right) \quad \text{................................(10)}$$



$$Up\ (t) = \begin{cases} 1 \text{ if } 0 < t < t_1 \\ 0 \text{ if } t < 0 \\ \text{and } t > t_1 \end{cases}$$

Fig. 4.0

If we consider the above diagram of a unit - pulse function (fig. 4.0). $t_1 \longrightarrow \infty$. Assume the limit of the transform of the unit – pulse is given by:

$$\lim_{t \to \infty} = \left(\frac{1}{s}\,(1 - e^{-st})\right) = \frac{1}{s}$$

Generally, unit – pulse function is described by:

$$Up\ (t) = \begin{pmatrix} 1/t,\ if\ o < t < t_1 \\ 0\ if\ t < 0\ and\ t > t_1 \end{pmatrix}$$

Appling equation (1) to the above, the Laplace transformation gives:

$$\mathcal{L}\ Up\ (t) = \frac{1}{st_1}\ (1 - e^{-st}) \quad \text{………………………………………….. ..(11)}$$

4. **Unit Sinusoidal Function:** Is expressed as:

$$f(t) = \begin{cases} \sin \omega t \text{ if } t \geq 0 \\ 0 \quad \text{if } t < 0 \end{cases}$$

$$\Longrightarrow f(s) = \mathcal{L}\ \{\cos \omega t\} \quad \text{……………………………………….… (12)}$$

$$= \quad e^{-st}\ \cos \omega t\ dt$$

Integration by parts gives

$$f(s) = \frac{S}{S^2 + \omega^2} \quad \text{……………………………………………………………. (13)}$$

5. **Dirac Delta (Unit Impulse) Function:** This is denoted by *δ(t)*. It is physically realized as a force, with large magnitude and applied for a very short period as indicated in fig. 5.0.

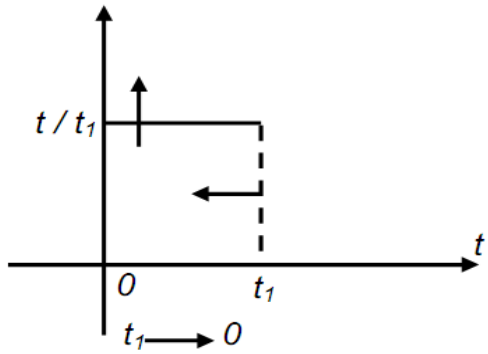


Fig. 5.0

The Laplace transform of a Unit – Impulse is expressed as:

$$\mathcal{L}\{\delta(t)\} = \Delta s = \lim_{t_1\to 0} \left\{\frac{1}{st_1}(1 - e^{-st})\right\} = \lim_{t_1\to 0}\left\{\frac{se^{-st_1}}{s}\right\} = 1 \quad \text{.........} (14)$$

Generally, impulse function is given by:

A $\delta(t)$

Where:

A denotes area

$\delta(t-\tau)$ = Unit impulse applied at the time $t = \tau$. This function is generally applicable in a filtering system.

6. **<u>Unit – Ramp Function (Ur (t)):</u>** A unit ramp function is shown in fig. 6.0. It is given by the definition, [6,8]

$$\mathcal{L}\{U_r(t)\} = U_r(s)$$

$$= \int_0^\infty te^{-st}\,dt = \left\{t\left(\frac{e^{-st}}{-s}\right)\right\}_{t=0}^{\infty} - \int_0^\infty \left(\frac{e^{-st}}{-s}\right) dt = \frac{e^{-st}}{s^2}\Bigg|_{t=0}^{\infty} = \frac{1}{s^2} \text{..}(15)$$

$$U_r(t) = \begin{cases} t \text{ if } t > 0 \\ 0 \text{ if } t < 0 \end{cases}$$

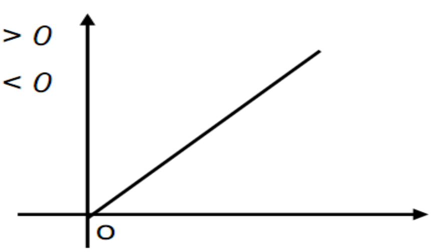


Fig. 6.0

5. INVERSE LAPLACE TRANSFORMATION

The inverse Laplace transformation is a process of obtaining time history, $f(t)$ from the Laplace transformation function f (s) when solving a differential equation via the Laplace transformation technique. The inverse Laplace is represented by:

$$f(t) = \mathcal{L}^{-1}\{f(s)\} \quad \text{...} (16)$$

The operation involved in the Laplace transform and its inverse is depicted pictorially as shown in the figure 7.0

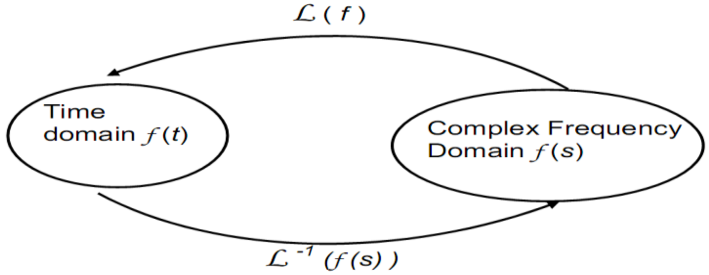


Fig. 7.0

6. LINEARITY OF LAPLACE AND INVERSE LAPLACE TRANSFORM:

linear Laplace transformation is obtained when a Laplace transform has two functions $f_1(t)$ and $f_2(t)$ and denoted by $f_1(s)$ and $f_2(s)$, α and β are constant scalar [5,7,8]. Then

$$L\{\alpha f_1(t) + \beta\}\{f_2(t)\} = \alpha L\{f_1(t)\} + \beta L\{f_2(t)\}$$
$$= \alpha L\{f_1(s)\} + \beta L\{f_2(s)\} \ldots\ldots (17)$$

The linearity of inverse Laplace operator L^{-1} can also be establish by taking the L^{-1} of both sides of eq. (1), we have

$$\alpha f_1(t) + \beta f_2(t) = L^{-1}\{\alpha f_1(s)\} + \beta f_2(s)\} \ldots\ldots (18)$$

7. SOLUTION OF INVERSE LAPLACE: CONVOLUTION METHOD:

In system analysis, a product of two transform functions, *G (s)* and *h (s)* is encountered, where their respective inverses, *g (t)* and *h* (*t*) are known. The convolution method is used to determine the inverse of the product f (*s*) = *G (s) H (s)*, directly from the knowledge of *g (t)* and *h* (*t*).

So, the convolution of g and *h* is expressed as

$$f(t) = \mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\{G(s)H(s)\} = \{g * h\}(t) = \{h * g\}(t) \ldots\ldots (19)$$

Let *g (t)* and *h* (*t*) satisfy condition (C_1) and (C_2).

Let $G(s) = \mathcal{L}\{g(t)\}$, $H(s) = \mathcal{L}\{h(t)\}$, $f(s) = G(s)H(s)$.

$$\text{Then, } \mathcal{L}^{-1}\{f(s)\} = f(t) = (g * h)(t) = \int_0^t g(\tau)\, h(t-\tau)\, dr$$
$$= \int_0^t h(\tau)\, g(t-\tau)\, d\tau = (h * g)(t) \ldots\ldots (20)$$

8. FOURIER TRANSFORMATION

Fourier Transform was developed to provide a link between the time domain and frequency domain for non – periodic waveforms [1,6]. Fourier transform can be used in communications, linear system analysis, statistics, quantum physics, optics, solution of partial differential equations and antennas, etc.

Let us consider a pulse waveform shown fig. 8.0.

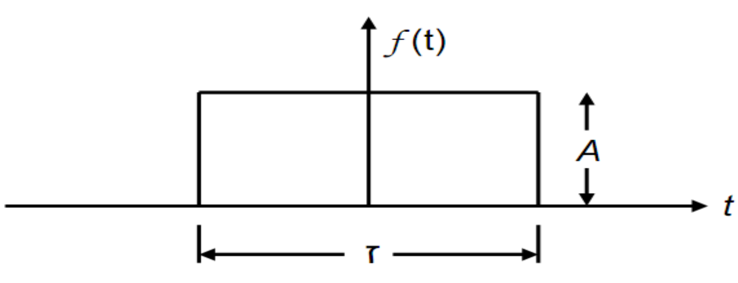


Fig 8.0

The Fourier Transform for this single pulse is:

$$F(\omega) = \frac{A\, r \sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}$$

Where:

F (ω)	=	Fourier transform.
A	=	Amplitude of the pulse in time domain.
ƭ	=	Pulse width in time domain.
ω	=	The radian frequency.

If we should analyze the frequency domain behaviour of the pulse as illustrated in fig 8.0. The Fourier transform, *f (ω)* is a continuous function whose shape is the sine function. The Fourier transform coefficients could be taken as a set of Fourier series whose period T, of the periodic function is near the infinity. In a rectangular waveform, the infinite period means a single pulse. The figure shows the behaviour of *f (ω)* is shown for negative frequencies. However, these frequencies are not physically realizable. They are by – product of mathematics used to device the Fourier transform.

9. PROPERTIES OF FOURIER TRANSFORM

The Fourier transform has the properties including [6,9,10]

(a) **Linearity:** For complex numbers of a and b. if *h* (x) = a f (x) + bg (x). Then $\hat{h}(\xi) = a\, f(\xi) + b\,.\,\hat{g}\,(\xi)$.

(b) **Translation:** For real number X_0, if h (x) = f (x – x_0), then h (ξ) = $e^{-2\pi i x 0\, \xi}\hat{f}\,(\xi)$

(c) **Modulation:** Real number ξ_0, if h (x) = f (ax), then $\hat{h}\,(\xi_1) = \frac{1}{(a)}\,\hat{f}\frac{(\xi)}{(a)}$

(d) **Scaling:** For non – zero real numbers a, if h (x) = f (ax), then $\hat{h}\,(\xi)\ \frac{1}{(a)}\,\hat{f}\frac{(\xi)}{(a)}$ where a = - 1 leads to time – reversal and states if h (x) = f (- x), then $\hat{h}\,(\xi) = \hat{f}\,(-\xi)$

(e) **Conjugation:** if h (x) = f (x), then then $\hat{h}\,(\xi) = \overline{\hat{f}}\,(-\xi)$. If f is real, $\hat{f}\,(-\xi) = \overline{\hat{f}}$. If f is purely imaginary, then $\hat{f}\,(-\xi) = \overline{-\hat{f}}\,(\xi)$

(f) **Duality:** If $h\,(x) = \hat{f}\,(x)$, then $\hat{h}\,(\xi) = f\,(-\xi)$

(g) **Convolution:** if h (x) = (f * g) (x) then $\hat{h}\,(\xi) = \hat{f}\,(\xi)\,.\,\hat{g}\,(\xi)$

10. CONCLUSION

The Fourier transform and Laplace transform are related. The Fourier transform resolves functions or signal into its mode of vibration whereas the Laplace transform resolves a function into its moments. Both are used for designing electrical circuits, solving differential and integral equations.

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