FIXED POINT THEOREM IN MENGER SPACE FOR SEMI-COMPATIBLE MAPPINGS

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ABSTRACT
In this paper, the concept of semi-compatibility and weak compatibility in Menger space has been applied to prove a common fixed point theorem for six self maps.

Keywords : *Probabilistic metric space, Menger space, common fixed point, compatible maps, semi-compatible maps, weak compatibility.*
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1. INTRODUCTION

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [4]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [8] studied this concept and gave some fundamental results on this space. Sehgal and Bharucha-Reid [9] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed-point theory in Menger space.
Recently, Jungck and Rhoades [3] termed a pair of self maps to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. Sessa [10] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weak commuting maps in metric spaces. Jungck [2] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [5].
Cho, Sharma and Sahu [1] introduced the concept of semi-compatibility in a d-complete topological space. Popa [7] proved interesting fixed point results using implicit real functions and semi-compatibility in d-complete topological space. In the sequel, Pathak and Verma [6] proved a common fixed point theorem in Menger space using compatibility and weak compatibility.
In this paper a fixed point theorem for six self maps has been proved using the concept of semi-compatible maps and weak compatible maps.

2. PRELIMINARIES

Definition 2.1. A mapping $\mathcal{F} : R \longrightarrow R^+$ is called a distribution if it is non-decreasing left continuous with

$$\inf \{ \mathcal{F}(t) \mid t \in R \} = 0 \quad \text{and} \quad \sup \{ \mathcal{F}(t) \mid t \in R\} = 1.$$

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}.$$

Definition 2.2. A triangular norm * (shortly t-norm) is a binary operation on the unit interval [0, 1] such that for all a, b, c, d $\in$ [0, 1] the following conditions are satisfied :

(a) $a * 1 = a$;
(b) $a * b = b * a$;
(c) $a * b \le c * d$ whenever $a \le c$ and $b \le d$;
(d) $a * (b * c) = (a * b) * c$.

Examples of t-norms are $a * b = \max\{a + b - 1, 0\}$ and $a * b = \min\{a, b\}$.

Definition 2.3. [8] A probabilistic metric space (PM-space) is an ordered pair $(X, \mathcal{F})$ consisting of a non empty set X and a function $\mathcal{F} : X \times X \to L$, where L is the collection of all distribution functions and the value of $\mathcal{F}$ at $(u, v) \in X \times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ assumed to satisfy the following conditions:

(PM-1) $F_{u,v}(x) = 1$, for all $x > 0$, if and only if $u = v$;

(PM-2) $F_{u,v}(0) = 0$;
(PM-3) $F_{u,v} = F_{v,u}$;
(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$,
for all $u,v,w \in X$ and $x, y > 0$.

Definition 2.4. [8] A Menger space is a triplet $(X, \mathcal{F}, t)$ where $(X, \mathcal{F})$ is a PM-space and $*$ is a t-norm such that the inequality
(PM-5) $F_{u,w}(x + y) \geq F_{u,v}(x) * F_{v,w}(y)$, for all $u, v, w \in X$, $x, y \geq 0$.

Proposition 2.1. [9] Let (X, d) be a metric space. Then the metric d induces a distribution function F defined by $F_{x,y}\{t\} = H(t - d(x,y))$ for all $x, y \in X$ and $t > 0$. If t-norm $*$ is $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ then $(X, \mathcal{F}, *)$ is a Menger space. Further, $(X, \mathcal{F}, *)$ is a complete Menger space if (X, d) is complete.

Definition 2.5. [5] Let $(X, \mathcal{F}, *)$ be a Menger space and $*$ be a continuous t-norm.
(a) A sequence $\{x_n\}$ in X is said to be converge to a point x in S (written $x_n \to x$) iff for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists an integer $n_0=n_0(\varepsilon, \lambda)$ such that $F_{x_n, x}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$.
(b) A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n, x_{n+p}}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$ and $p > 0$.
(c) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Remark 2.1. If $*$ is a continuous t-norm, it follows from (PM-4) that the limit of sequence in Menger space is uniquely determined.

Definition 2.6. [11] Self mappings A and S of a Menger space $(X, \mathcal{F}, t)$ are said to be weak compatible if they commute at their coincidence points i.e. $Ax = Sx$ for $x \in X$ implies $ASx = SAx$.

Definition 2.7. [5] Self mappings A and S of a Menger space $(X, \mathcal{F}, t)$ are said to be compatible if $F_{ASx_n, SAx_n}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$ for some u in X, as $n \to \infty$.

Definition 2.8. Self mappings A and S of a Menger space $(X, \mathcal{F}, t)$ are said to be semi-compatible if $F_{ASx_n, Su}(x) \to 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to u$, for some u in X, as $n \to \infty$.
Now, we give an example of pair of self maps (S, T) which is semi-compatible but not compatible. Further we observe here that the pair (T, S) is not semi-compatible though (S, T) is semi-compatible.

Example 2.1. Let (X, d) be a metric space where X = [0, 1] and $(X, \mathcal{F}, t)$ be the induced Menger space with $F_{p,q}(\varepsilon) = H(\varepsilon - d(p, q))$, $\forall\, p, q \in X$ and $\forall\, \varepsilon > 0$. Define self maps S and T as follows :

$$Sx = \begin{cases} x & \text{if} \quad 0 \leq x < \frac{1}{2} \\ 1 & \text{if} \quad \frac{1}{2} \leq x \leq 1 \end{cases} \quad \text{and} \quad Tx = \begin{cases} 1 - x & \text{if} \quad 0 \leq x < \frac{1}{2} \\ 1 & \text{if} \quad \frac{1}{2} \leq x \leq 1 \end{cases}.$$

Take $x_n = \frac{1}{2} - \frac{1}{n}$. Now,

$$F_{Sx_n,1/2}(\varepsilon) = H(\varepsilon - (1/n)).$$

Therefore, $\lim_{n\to\infty} F_{Sx_n,1/2}(\varepsilon) = H(\varepsilon) = 1$.

Hence, $Sx_n \to 1/2$ as $n \to \infty$.

Similarly, $Tx_n \to 1/2$ as $n \to \infty$.

Also

$$F_{STx_n,TSx_n}(\varepsilon) = H\left(\varepsilon - \left(\frac{1}{2} - \frac{1}{n}\right)\right) \neq 1, \quad \forall\ \varepsilon > 0.$$

Hence, the pair (S, T) is not compatible.

Again, $\lim\limits_{n\to\infty} F_{STx_n,Tx}(\varepsilon) = \lim\limits_{n\to\infty} F_{STx_n,1}(\varepsilon) = H(\varepsilon - |1-1|) = 1\ \forall\ \varepsilon > 0.$

Thus, (S, T) is semi-compatible.

Now, $\lim\limits_{n\to\infty} F_{TSx_n,Sx}(\varepsilon) \neq 1,\ \forall\ \varepsilon > 0.$

Thus, (T, S) is not semi-compatible.

Remark 2.2. In view of above example, it follows that the concept of semi-compatibility is more general than that of compatibility.

Lemma 2.1. [11] Let $\{x_n\}$ be a sequence in a Menger space $(X, \mathcal{F}, *)$ with continuous t-norm * and $t * t \geq t$. If there exists a constant $k \in (0, 1)$ such that

$$F_{x_n, x_{n+1}}(kt) \geq F_{x_{n-1}, x_n}(t)$$

for all $t > 0$ and $n = 1, 2, 3, \ldots$, then $\{x_n\}$ is a Cauchy sequence in X.

3. MAIN RESULT

Theorem 3.1. Let A, B, S, T, L and M be self maps of a complete Menger space $(X, \mathcal{F}, *)$ with $t * t \geq t$ satisfying :

(3.1.1) $L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$;

(3.1.2) AB = BA, ST = TS, LB = BL, MT = TM;

(3.1.3) either L or AB is continuous;

(3.1.4) (L, AB) is semi-compatible and (M, ST) is weak compatible;

(3.1.5) there exists a constant $k \in (0, 1)$ such that

$$F^2_{Lx,My}(kt)*[F_{ABx,Lx}(kt).F_{STy,My}(kt)] \geq [pF_{ABx, Lx}(t) + qF_{ABx, STy}(t)].F_{ABx, My}(2kt)$$

for all $x, y \in X$ and $t > 0$ where $0 < p, q < 1$ such that $p + q = 1$.

Then A, B, S, T, L and M have a unique common fixed point in X.

Proof. Suppose $x_0 \in X$. From condition (3.1.1) $\exists\ x_1, x_2 \in X$ such that

$$Lx_0 = STx_1 \quad \text{and} \quad Mx_1 = ABx_2.$$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Lx_{2n} = STx_{2n+1} \quad \text{and} \quad y_{2n+1} = Mx_{2n+1} = ABx_{2n+2} \text{ for } n = 0, 1, 2, \ldots .$$

Step 1. Taking $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1.5), we have

$$F^2_{Lx_{2n},Mx_{2n+1}}(kt)*[F_{ABx_{2n},Lx_{2n}}(kt).F_{STx_{2n+1}, Mx_{2n+1}}(kt)]$$
$$\geq [pF_{ABx_{2n}, Lx_{2n}}(t) + qF_{ABx_{2n}, STx_{2n+1}}(t)].F_{ABx_{2n}, Mx_{2n+1}}(2kt)$$

$$F^2_{y_{2n}, y_{2n+1}}(kt)*[F_{y_{2n-1},y_{2n}}(kt).F_{y_{2n}, y_{2n+1}}(kt)]$$
$$\geq [pF_{y_{2n}, y_{2n-1}}(t) + qF_{y_{2n-1}, y_{2n}}(t)].F_{y_{2n}, y_{2n+1}}(2kt)$$

$$F_{y_{2n}, y_{2n+1}}(kt)[F_{y_{2n-1},y_{2n}}(kt) * F_{y_{2n}, y_{2n+1}}(kt)]$$
$$\geq (p + q)F_{y_{2n}, y_{2n-1}}(t).F_{y_{2n-1}, y_{2n+1}}(2kt)$$

$$F_{y_{2n},y_{2n+1}}(kt)F_{y_{2n-1}, y_{2n+1}}(2kt) \geq Fy_{2n-1, y_{2n}}(t)Fy_{2n-1, y_{2n+1}}(2kt).$$

Hence, we have

$$F_{y_{2n}, y_{2n+1}}(kt) \geq F_{y_{2n-1}, y_{2n}}(t).$$

Similarly, we also have

$$F_{y_{2n+1}, y_{2n+2}}(kt) \geq F_{y_{2n}, y_{2n+1}}(t).$$

In general, for all n even or odd, we have

$$F_{y_n, y_{n+1}}(kt) \geq F_{y_{n-1}, y_n}(t)$$

for $k \in (0, 1)$ and all $t > 0$. Thus, by lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X. Since $(X, \mathcal{F}, *)$ is complete, it converges to a point z in X. Also its subsequences converge as follows :

$$\{Lx_{2n}\} \to z,\ \{ABx_{2n}\} \to z,\ \{Mx_{2n+1}\} \to z \text{ and } \{STx_{2n+1}\} \to z. \tag{3.1.6}$$

Case I. Suppose AB is continuous.

As AB is continuous and (L, AB) is semi-compatible, we get

$$LABx_{2n+2} \to Lz \quad \text{and} \quad LABx_{2n+2} \to ABz. \tag{3.1.7}$$

Since the limit in Menger space is unique, we get

$$Lz = ABz. \tag{3.1.8}$$

Step 2. By taking $x = ABx_{2n}$ and $y = x_{2n+1}$ in (3.1.5), we have

$$F^2_{LABx_{2n},Mx_{2n+1}}(kt)*[F_{ABABx_{2n},LABx_{2n}}(kt).F_{STx_{2n+1},Mx_{2n+1}}(kt)]$$
$$\geq [pF_{ABABx_{2n},LABx_{2n}}(t) + qF_{ABABx_{2n},STx_{2n+1}}(t)].F_{ABABx_{2n},Mx_{2n+1}}(2kt).$$

Taking limit $n \to \infty$

$$F^2_{z,ABz}(kt)*[F_{ABz,ABz}(kt).F_{z,z}(kt)] \geq [pF_{ABz,ABz}(t) + qF_{z,ABz}(t)].F_{z,ABz}(2kt)$$
$$\geq [p + qF_{z,ABz}(t)]F_{z,ABz}(kt)]$$
$$F_{z,ABz}(kt) \geq p + qF_{z,ABz}(t)$$
$$\geq p + qF_{z,ABz}(kt)$$
$$F_{z,ABz}(kt) \geq \frac{p}{1-q} = 1$$

for $k \in (0, 1)$ and all $t > 0$. Thus, we have

$$z = ABz.$$

Step 3. By taking $x = z$ and $y = x_{2n+1}$ in (3.1.5), we have

$$F^2_{Lz,Mx_{2n+1}}(kt)*[F_{ABz,Lz}(kt).F_{STx_{2n+1},Mx_{2n+1}}(kt)]$$
$$\geq [pF_{ABz,Lz}(t) + qF_{ABz,STx_{2n+1}}(t)].F_{ABz,Mx_{2n+1}}(2kt)$$

Taking limit $n \to \infty$

$$F^2_{z,Lz}(kt)*[F_{z,Lz}(kt).F_{z,z}(kt)] \geq [pF_{z,Lz}(t) + qF_{z,z}(t)].F_{z,z}(2kt)$$

$$F^2_{z,Lz}(kt)*F_{z,Lz}(kt) \geq pF_{z,Lz}(t) + q.$$

Noting that $F^2_{z,Lz}(kt) \leq 1$ and using (c) in Definition 2.2, we have

$$F_{z,Lz}(kt) \geq pF_{z,Lz}(t) + q$$
$$\geq pF_{z,Lz}(kt) + q$$
$$F_{z,Lz}(kt) \geq \frac{q}{1-p} = 1$$

for $k \in (0, 1)$ and all $t > 0$. Thus, we have $z = Lz = ABz$.

Step 4. By taking $x = Bz$ and $y = x_{2n+1}$ in (3.1.5), we have

$$F^2_{LBz,Mx_{2n+1}}(kt)*[F_{ABBz,LBz}(kt).F_{STx_{2n+1},Mx_{2n+1}}(kt)]$$
$$\geq [pF_{ABBz,LBz}(t) + qF_{ABBz,STx_{2n+1}}(t)].F_{ABBz,Mx_{2n+1}}(2kt).$$

Since $AB = BA$ and $BL = LB$, we have

$L(Bz) = B(Lz) = Bz$ and

$AB(Bz) = B(ABz) = Bz$.

Taking limit $n \to \infty$, we have

$$F^2_{z,Bz}(kt)*[F_{Bz,Bz}(kt).F_{z,z}(kt)] \geq [pF_{Bz,Bz}(t) + qF_{z,Bz}(t)].F_{z,Bz}(2kt)$$

$$F^2_{z,Bz}(kt) \geq [p + qF_{z,Bz}(t)]F_{z,Bz}(2kt)$$

$$\geq [p + qF_{z,Bz}(t)]F_{z,Bz}(kt)$$

$$F_{z,Bz}(kt) \geq p + qF_{z,Bz}(t)$$

$$\geq p + qF_{z,Bz}(kt)$$

$$F_{z,Bz}(kt) \geq \frac{p}{1-q} = 1$$

for $k \in (0, 1)$ and all $t > 0$.
Thus, we have

$$z = Bz.$$

Since $z = ABz$, we also have

$$z = Az.$$

Therefore, $z = Az = Bz = Lz$.

Step 5. Since $L(X) \subseteq ST(X)$ there exists $v \in X$ such that

$$z = Lz = STv.$$

By taking $x = x_{2n}$ and $y = v$ in (3.1.5), we get

$$F^2_{Lx_{2n},Mv}(kt)*[F_{ABx_{2n},Lx_{2n}}(kt).F_{STv,\,Mv}(kt)] \geq [pF_{ABx_{2n},\,Lx_{2n}}(t) + qF_{ABx_{2n},\,STv}(t)].F_{ABx_{2n},\,Mv}(2kt).$$

Taking limit as $n \to \infty$, we have

$$F^2_{z,Mv}(kt)*[F_{z,z}(kt).F_{z,\,Mv}(kt)] \geq [pF_{z,\,z}(t) + qF_{z,\,z}(t)].F_{z,\,Mv}(2kt)$$

$$F^2_{z,Mv}(kt)*F_{z,Mv}(kt) \geq (p + q)F_{z,\,Mv}(2kt).$$

Noting that $F^2_{z,\,Mv}(kt) \leq 1$ and using (c) in Definition 2.2, we have

$$F_{z,\,Mv}(kt) \geq F_{z,\,Mv}(2kt) \geq F_{z,\,Mv}(t).$$

Thus, we have

$$z = Mv \quad \text{and so} \quad z = Mv = STv.$$

Since (M, ST) is weakly compatible, we have

$$STMv = MSTv.$$

Thus, $STz = Mz$.

Step 6. By taking $x = x_{2n}$, $y = z$ in (3.1.5) and using Step 5, we have

$$F^2_{Lx_{2n},Mz}(kt)*[F_{ABx_{2n},Lx_{2n}}(kt).F_{STz,\,Mz}(kt)] \geq [pF_{ABx_{2n},\,Lx_{2n}}(t) + qF_{ABx_{2n},\,STz}(t)].F_{ABx_{2n},\,Mz}(2kt)$$

which implies that, as $n \to \infty$

$$F^2_{z,Mz}(kt)*[F_{z,z}(kt).F_{Mz,\,Mz}(kt)] \geq [pF_{z,\,z}(t) + qF_{z,\,Mz}(t)].F_{z,\,Mz}(2kt)$$

$$F^2_{z,Mz}(kt) \geq [p + qF_{z,\,Mz}(t)]F_{z,\,Mz}(2kt) \geq [p + qF_{z,\,Mz}(t)]F_{z,\,Mz}(kt)$$

$$F_{z,Mz}(kt) \geq p + qF_{z,\,Mz}(t) \geq p + qF_{z,\,Mz}(kt)$$

$$F_{z,Mz}(kt) \geq \frac{p}{1-q} = 1.$$

Thus, we have $z = Mz$ and therefore $z = Az = Bz = Lz = Mz = STz$.

Step 7. By taking $x = x_{2n}$, $y = Tz$ in (3.1.5), we have

$$F^2_{Lx_{2n},MTz}(kt)*[F_{ABx_{2n},Lx_{2n}}(kt).F_{STTz,\,MTz}(kt)] \geq [pF_{ABx_{2n},\,Lx_{2n}}(t) + qF_{ABx_{2n},\,STTz}(t)].F_{ABx_{2n},\,MTz}(2kt).$$

Since $MT = TM$ and $ST = TS$, we have

$$MTz = TMz = Tz \quad \text{and} \quad ST(Tz) = T(STz) = Tz.$$

Letting $n \to \infty$, we have

$$F^2_{z,\,Tz}(kt)*[F_{z,\,z}(kt).F_{Tz,\,Tz}(kt)] \geq [pF_{z,\,z}(t) + qF_{z,\,Tz}(t)].F_{z,\,Tz}(2kt)$$

$$F_{z,Tz}(kt) \geq p + qF_{z,\,Tz}(t) \geq p + qF_{z,\,Tz}(kt)$$

$$F_{z,Tz}(kt) \geq \frac{p}{1-q} = 1.$$

Thus, we have $z = Tz$. Since $Tz = STz$, we also have $z = Sz$.

Therefore, z = Az = Bz = Lz = Mz = Sz = Tz, that is, z is the common fixed point of the six maps.

Case II. L is continuous.
Since L is continuous, $LLx_{2n} \to Lz$ and $L(AB)x_{2n} \to Lz$.
Since (L, AB) is semi-compatible, $L(AB)x_{2n} \to ABz$.

Step 8. By taking $x = Lx_{2n}$, $y = x_{2n+1}$ in (3.1.5), we have

$$F^2_{LLx_{2n},Mx_{2n+1}}(kt)*[F_{ABLx_{2n},LLx_{2n}}(kt).F_{STx_{2n+1},Mx_{2n+1}}(kt)]$$
$$\geq [pF_{ABLx_{2n},LLx_{2n}}(t) + qF_{ABLx_{2n},STx_{2n+1}}(t)].F_{ABLx_{2n},Mx_{2n+1}}(2kt)$$

Letting $n \to \infty$, we have

$$F^2_{z,Lz}(kt)*[F_{Lz,Lz}(kt).F_{z,z}(kt)] \geq [pF_{Lz,Lz}(t) + qF_{z,Lz}(t)].F_{z,Lz}(2kt)$$
$$F^2_{z,Lz}(kt) \geq [p + qF_{z,Lz}(t)]F_{z,Lz}(2kt)$$
$$\geq [p + qF_{z,Lz}(t)]F_{z,Lz}(kt),$$
$$F_{z,Lz}(kt) \geq p + qF_{z,Lz}(t)$$
$$\geq p + qF_{z,Lz}(kt),$$
$$F_{z,Lz}(kt) \geq \frac{p}{1-q} = 1.$$

Thus, we have z = Lz and using Steps 5-7, we have
z = Lz = Mz = Sz = Tz.

Step 9. Since L is continuous,
$LLx_{2n} \to Lz$ and $LABx_{2n} \to Lz$
Since (L, AB) is semi-compatible,
$L(AB)x_{2n} \to ABz$.
Since limit in Menger space is unique, so
Lz = ABz and using Step 4, we also have z = Bz.
Therefore, z = Az = Bz = Sz = Tz = Lz = Mz, that is, z is the common fixed point of the six maps in this case also.

Step 10. For uniqueness, let w ($w \neq z$) be another common fixed point of A, B, S, T, L and M.
Taking x = z, y = w in (3.1.5), we have

$$F^2_{Lz,Mw}(kt)*[F_{ABz,Lz}(kt).F_{STw,Mw}(kt)] \geq [pF_{ABz,Lz}(t) + qF_{ABz,STw}(t)].F_{ABz,Mw}(2kt)$$

which implies that

$$F^2_{z,w}(kt) \geq [p + qF_{z,w}(t)]F_{z,w}(2kt)$$
$$\geq [p + qF_{z,w}(t)]F_{z,w}(kt),$$
$$F_{z,w}(kt) \geq p + qF_{z,w}(t)$$
$$\geq p + qF_{z,w}(kt)$$
$$F_{z,w}(kt) \geq \frac{p}{1-q} = 1.$$

Thus, we have z = w.
This completes the proof of the theorem.
If we take $B = T = I_X$ (the identity map on X) in theorem 3.1, we have the following :

Corollary 3.2. Let A, S, L and M be self maps of a complete Menger space (X, $\mathcal{F}$, *) with $t * t \geq t$ satisfying :

(a) $L(X) \subseteq S(X)$, $M(X) \subseteq A(X)$;
(b) either L or A is continuous;
(c) (L, A) is semi-compatible and (M, S) is weak compatible;
(d) there exists a constant $k \in (0, 1)$ such that

$$F^2_{Lx,My}(kt)*[F_{Ax,Lx}(kt).F_{Sy,My}(kt)] \geq [pF_{Ax,Lx}(t) + qF_{Ax,Sy}(t)].F_{Ax,My}(2kt)$$

for all $x, y \in X$ and $t > 0$ where $0 < p, q < 1$ such that $p + q = 1$.
Then A, S, L and M have a unique common fixed point in X.

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