FREDHOLM'S EQUATION AND FOURIER TRANSFORM ON DISCRETE ELECTROMAGNETIC SYSTEMS

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ABSTRACT
Fredholm's equation [1-4]and Fourier transform can be used to extend some important features of discrete acoustic systems to the electromagnetic domain that have an important role on obtaining great signals localization. Also the mathematical structure obtained of the related equations is very near to some results on nuclear scattering theory like the existence of resonances. Because of their practical importance on applications to communications and other fields, we give a mathematical support and explain actual experimental results like super-localization of microwave signals and suggest that we can improve the usual methods to avoid the leak of information in a broadcasting process. The fundamental procedure is the use of the Fourier transform of Fredholm's time dependent equation and some techniques of operator methods that resemble the equivalent ones on quantum mechanics. Finally we propose the use of the obtained equations for the Fourier's transform of the Green's function associated to discrete systems, to describe the propagation of the electromagnetic signals for restrictive but very important applications.

Keywords: *Fredholm's equation, Fourier transform, electromagnetic high definition.*

1. INTRODUCTION

Between the relevant techniques used on real systems to improve the signal localization we find Time Reversal [5-11] that nowadays is used on several applications in medical, engineering among other fields, and its power is based principally on the discrete systems property to avoid some kind of destructive interference while information is propagated. For example, we can localize an acoustic signal previously emitted from a source if we can record it and then send it in time reverse mode to reach the initial source. But if we try to reproduce the same localization on the electromagnetic field, we have theoretical and experimental difficulties. One of them is that electromagnetic field can't be emitted exactly in time reverse order because of the high speed of light (we can only take the conjugated of the complex frequency). Another problem is the electromagnetic field equations whose application requires very complicated calculations for continuous systems and, in addition, interference forbids an efficient localization. An alternative is an approximation for discrete systems based on acoustic techniques success.

In this work, we define a general discrete electromagnetic system and describe the signal propagation first by means of a time dependent Fredholm's equation, and then by taking the Fourier transform of them. We need to build the Green's function and set the condition that time reversal equations must be of the same kind as the forward ones. By exploiting the properties of Fredholm's equation, we find properties like the existence of resonances and recover the high localization for time reversed signals with the inherent very low leak of information. On sections 2 and 3 we introduce the inhomogeneous Fredholm´s equation and write the forward vector-matrix equation formalism. From section 4 to 5 we set the time reversal conditions with details and present the time reversal vector matrix equation for a best transmission, and then we give an academic example. On sections 6 and 7 we build the homogeneous vector matrix equation for resonances and show how to obtain them with a second example. Also, in chapter 6 we show the orthogonality relation for resonances.

2. THE ASSOCIATED INHOMOGENEOUS FREDHOLM'S EQUATION

We first assume that due to the linearity of the wave equation, we can relate the values of the electromagnetic field $E^m(\bar{r},t)$ at different times and places $\bar{r},t$ and $\bar{r}',t'$ by means of an integral equation. Because we are interested on discrete systems we define $E^m(\bar{r}_j,t)=E_j^m(t)$, that is, the signal measured at $\bar{r}_j$, so we can write

$$E_j^m(t)=E_j^{m(\circ)}(t)+\sum_{n=1}^{3}\sum_{k\neq j}\int_{-\infty}^{\infty}G^{m,n(\circ)}(\bar{r}_j,t;\bar{r}_k,t')A_k^{m,n}E_k^n(t')dt' \qquad (1)$$

Here $G^{m,n(\circ)}\left(\overline{r}_j,t;\overline{r}_k,t'\right)$ is the free Green's function and the complex dispersion coefficients are $A_k^{m,n}$ which contains the complete non-linear interaction. By interchanging the double sum with the integral in equation (1) we obtain (which implies a column vector form for $E_j^m(t)$) :

$$E^m{}_j(t) = E^m{}_j{}^{(\circ)}(t) + \int_{-\infty}^{\infty} K_{j,k}^{m,n(\circ)}(t,t')E_k^n(t')dt' \tag{2}$$

This is an inhomogeneous Fredholm's integral equation (IFE) but not as defined in scalar conventional form. Also, we have used summation convention over k, and defined the kernel:

TT

$$K_{j,k}^{m,n(\circ)}(t,t') = G^{m,n(\circ)}(\overline{r_j},t;\overline{r_k},t')A_k^{m,n} \tag{3}$$

The signal $E_k^n(t')$ can be written in terms of the non-null function $S_k^n(t')$ defined by

$$E_k^n(t') = \begin{Bmatrix} 0\ if\ t' \in (-\infty,0) \cup (T,\infty) \\ S_k^n(t')\ if\ t' \in [0,T] \end{Bmatrix} \tag{4}$$

For convenience, we return to equation (1) which can be written as

$$S^m{}_j(t) = S^m{}_j{}^{(\circ)}(t) + \sum_{n=1}^{3}\sum_{k\neq j} \int_0^T G^{m,n(\circ)}(\overline{r_j},t;\overline{r_k},t')A_k^{m,n}S_k^n(t')dt' \tag{5}$$

On the other hand, we can express the Green's function in terms of its Fourier transform associated with frequency ω

$$G^{m,n(\circ)}(\overline{r_j},t;\overline{r_k},t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_\omega^{m,n(\circ)}(\overline{r_j},\overline{r_k})e^{i\omega(t-t')}d\omega \tag{6}$$

So that equation (5) becomes

$$S^m{}_j(t) = S^m{}_j{}^{(\circ)}(t) + \frac{1}{2\pi}\sum_{n=1}^{3}\sum_{k\neq j} A_k^{m,n}\int_{-\infty}^{\infty} e^{i\omega t}G_\omega^{m,n(\circ)}(\overline{r_j},\overline{r_k})\,f_k^n(\omega)d\omega \tag{7}$$

Where

$$f_k^m(\omega) = \int_0^T e^{-i\omega t'}S_k^m(t')dt' \tag{8}$$

That is, $f_k^m(\omega)$ is, the Fourier transform of $S_k^m(t)$

We also have

$$S_j^m(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}f_j^m(\omega)d\omega \tag{9}$$

Substituting in equation (7) and performing some algebra (described as the same of the backward in time case in section 4) we obtain

$$f_j^m(\omega) = f_j^{m(\circ)}(\omega) + \sum_{n=1}^{3}\sum_{k\neq j} A_k^{m,n}G_\omega^{m,n(\circ)}(\overline{r_j},\overline{r_k})\,f_k^n(\omega)d\omega \tag{10}$$

Or, in vector form,

$$\overline{f}^{m(\circ)}(\omega) = \left[\mathbf{1} - \mathbf{K}^{(\circ)}(\omega)\right]_n^m \overline{f}^n(\omega) \tag{11}$$

(Einstein summation convention was used here)
Where

$$\mathbf{K}_{j,k,n}^{m(\circ)}(\omega) = \begin{Bmatrix} 0 \ \ if \ \ j = \mathrm{k} \\ A_k^{m,n} G_\omega^{m,n(\circ)}(\overline{r_j}, \overline{r_k})\ if \ \ j \neq \mathrm{k} \end{Bmatrix} \tag{12}$$

3. DESCRIPTION IN FORWARD TIME DIRECTION

Equation (11) can be inverted formally as

$$\overline{f}^n(\omega) = [\left[\mathbf{1} - \mathbf{K}^{(\circ)}(\omega)\right]^{-1}]_m^n \overline{f}^{m(\circ)}(\omega) \tag{13}$$

By means of the development of this equation and by obtaining the Fourier transform of the complete Green's function $G_\omega^{m,n(\circ)}(r_j, r_k)$, the result is [1, 2]

$$\overline{f}^n(\omega) = \left[\mathbf{1} + \mathbf{K}(\omega)\right]_m^n \overline{f}^{m(\circ)}(\omega) \tag{14}$$

Here we have defined

$$\mathbf{K}_{i,j,m}^n(\omega) = \begin{Bmatrix} 0 \ \ if \ i = j \\ A_j^{n,m} G_\omega^{n,m}(\overline{r_i}, \overline{r_j})\ if \ \ i \neq j \end{Bmatrix} \tag{15}$$

Equation (14) allows the knowledge of experimental data on the components of $\mathbf{K}_{i,j,m}^n(\omega)$, because we have the original signals $f_k^{m(\circ)}(\omega)$ and then we can measure final signals $f_j^n(\omega)$.

4. MAKING THE RIGHT TIME INVERSION

The condition we must set to guarantee the minimum leak of information in time reverse process is to maintain the same kind of integral equation. That is, we must describe the time reversal with another inhomogeneous integral equation. This implies that we need to put explicitly the counterpart of the source term in the forward description. The equivalent time reversed equation of (1) is then (but now with a row column vector form for $E_s^n(T-t)$):

$$E_s^n(T-t) = E_s^{n(\circ)}\left(T-t\right) + \sum_{m=1}^{3} \sum_j \int_{-\infty}^{\infty} A_s^{m,n*} G^{m,n(\circ)*}\left(\overline{r}_j, T-t'; \overline{r}_s, t\right) E_j^m\left(T-t'\right) dt' \tag{16}$$

In this equation we may clearly distinguish the sink contribution, which allows overcoming the diffraction limit; specifically, this is represented by the term

$$E_s^{n(\circ)}(T-t) \tag{17}$$

This term converts the integral equation into an inhomogeneous equation, creating an acute localization by means of cancellation of those terms in the homogeneous equation which prevents its occurrence. The presence of a random distributed set of scatters in the nearby field (fig. 1) can represent or substitute the effect of this term because it transforms the evanescent waves into traveling waves in the time reverse process (On the Lerosey et al. experiment [10] the process and the device are called time-reversal mirror or TRM). So, we can consider the scatters in the near field as a source of travelling waves but time reversed (in practice a sink). In order to apply the Fourier transform to a well behaved function, we return to equation (16), and we can see that it can be written in terms of the non-null function $S_j^m(t)$ related to the signal $E_k^n(t')$ by the following definition

$$E_k^n(t') = \begin{Bmatrix} 0 \ if \ \ t' \in \ (-\infty, 0) \cup (T, \infty) \\ S_k^n(t') \ if \ \ t' \in \ [0, T] \end{Bmatrix} \tag{18}$$

So by substituting this relation we arrive to the expression

$$S_s^n(T-t) = S_s^{n(\circ)}(T-t)$$

$$+\sum_j \int_{-\infty}^{\infty} A_s^{m,n*} G^{m,n(\circ)*}(\overline{r_j}, T-t'; \overline{r_s}, t) \sum_{m=1}^{3} S_j^m(T-t')dt' \qquad (19)$$

We can express equation (19) in terms of the Fourier transform $G_{-\omega}^{n,m(\circ)*}(\overline{r_j}, \overline{r_s})$ defined as

$$G^{n,m(\circ)*}(\overline{r_j}, T-t'; \overline{r_s}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{-\omega}^{n,m(\circ)*}(\overline{r_j}, \overline{r_s}) e^{i\omega(T-t'-t)} d\omega \qquad (20)$$

That is,

$$S_s^n(T-t) = S_s^{n(\circ)}(T-t)$$

$$+\sum_{m=1}^{3}\sum_j \int_0^T \frac{1}{2\pi} A_s^{m,n*} \int_{-\infty}^{\infty} G_{-\omega}^{m,n(\circ)*}(\overline{r_j}, \overline{r_s}) e^{i\omega(T-t'-t)} d\omega S_j^m(T-t')dt' \qquad (21)$$

This equation can be written as

$$S_s^n(T-t) = S_s^{n(\circ)}(T-t)$$

$$+\sum_{m=1}^{3}\sum_j \int_{-\infty}^{\infty} \frac{1}{2\pi} A_s^{m,n*} G_{-\omega}^{m,n(\circ)*}(\overline{r_j}, \overline{r_s}) e^{i\omega(T-t)} \left[\int_0^T e^{-i\omega t'} S_j^m(T-t')dt'\right] d\omega \qquad (22)$$

Now we write the last equation in terms of the Fourier transform of the electromagnetic field

$$S_j^m(T-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(T-t)} f_j^m(-\omega) d\omega \qquad (23)$$

And find the following equation

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(T-t)} f_s^n(-\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(T-t)} f_s^{n(\circ)}(-\omega) d\omega$$

$$+\sum_{m=1}^{3}\sum_j \int_{-\infty}^{\infty} \frac{1}{2\pi} A_s^{m,n*} G_{-\omega}^{m,n(\circ)*}(\overline{r_j}, \overline{r_s}) e^{i\omega(T-t)} \left[\int_0^T e^{-i\omega t'} S_j^m(T-t')dt'\right] d\omega \qquad (24)$$

Because we have that

$$\int_0^T e^{-i\omega t'} S_j^m(T-t')dt' = \int_{-\infty}^{\infty} e^{-i\omega t'} S_j^m(T-t')dt' = e^{i\omega T} f_j^m(-\omega) \qquad (25)$$

We can write equation (24) as

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(T-t)} f_s^n(-\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(T-t)} f_s^{n(\circ)}(-\omega) d\omega$$

$$+\sum_{m=1}^{3}\sum_j \frac{1}{2\pi} \int_{-\infty}^{\infty} A_s^{m,n*} G_{-\omega}^{m,n(\circ)*}(\overline{r_j}, \overline{r_s}) e^{i\omega(T-t)} e^{i\omega T} f_j^m(-\omega) d\omega \qquad (26)$$

Equation (26) can be written

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(T-t)} f_s^{\,n}(-\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(T-t)} f_s^{\,n(\circ)}(-\omega)d\omega$$
$$+\sum_{m=1}^{3}\sum_{j} \frac{1}{2\pi}\int_{-\infty}^{\infty} f_j^{\,m}(-\omega)A_s^{m,n*}G_{-\omega}^{m,n(\circ)*}(\overline{r_j},\overline{r_s})e^{i\omega(T-t)}e^{i\omega T}d\omega \tag{27}$$

And by defining the variable

$$u = t - T \tag{28}$$

We can write equation (27) as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u} f_s^{\,n}(-\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u} f_s^{\,n(\circ)}(-\omega)d\omega$$
$$+\sum_{m=1}^{3}\sum_{j} \frac{1}{2\pi}\int_{-\infty}^{\infty} f_j^{\,m}(-\omega)A_s^{m,n*}G_{-\omega}^{m,n(\circ)*}(\overline{r_j},\overline{r_s})e^{-i\omega u}e^{i\omega T}d\omega \tag{29}$$

The exponential function on the second term of the right side of equation (29) can be separated so we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u} f_s^{\,n}(-\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u} f_s^{\,n(\circ)}(-\omega)d\omega$$
$$+\sum_{m=1}^{3}\sum_{j} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u} f_j^{\,m}(-\omega)A_s^{m,n*}G_{-\omega}^{m,n(\circ)*}(\overline{r_j},\overline{r_s})e^{i\omega T}d\omega \tag{30}$$

To avoid the negative sign in the argument of the functions that appears in equation (30) we can change it without problem and obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega u} f_s^{\,n}(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega u} f_s^{\,n(\circ)}(\omega)d\omega$$

$$+\sum_{m=1}^{3}\sum_{j} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega u} f_j^{\,m}(\omega)A_s^{m,n*}G_{\omega}^{m,n(\circ)*}(\overline{r_j},\overline{r_s})e^{-i\omega T}d\omega \tag{31}$$

Then we have the expression

$$f_j^{\,n}(\omega) = f_j^{\,n(\circ)}(\omega) + \sum_{m=1}^{3}\sum_{k} f_k^{\,m}(\omega)A_j^{m,n*}G_{\omega}^{m,n(\circ)*}(\overline{r_k},\overline{r_j})e^{-i\omega T} \tag{32}$$

This equation in vector form (as we said above, we have row vectors instead of column vectors) looks like

$$\overline{\mathrm{f}}^{\,n(\circ)}(\omega) = \overline{\mathrm{f}}^{\,n}(\omega) - \overline{\mathrm{f}}^{\,m}(\omega)\mathbf{R}^{m,n(\circ)*}(\omega) \tag{33}$$

Where

$$\mathbf{R}^{m,n(\circ)*}(\omega)_{s,j} = \begin{cases} 0 & if \quad j = s \\ A_s^{m,n*}G_{\omega}^{m,n(\circ)*}(\bar{r}_j,\bar{r}_s)e^{-i\omega T} & if \quad j \neq s \end{cases} \tag{34}$$

And also

$$\left[\overline{\mathrm{f}}^{\,n}(\omega)_{\{\text{row vector}\}}\right]^{\tau} = \overline{f}^{\,n}(\omega)_{\{\text{column vector}\}} \tag{35}$$

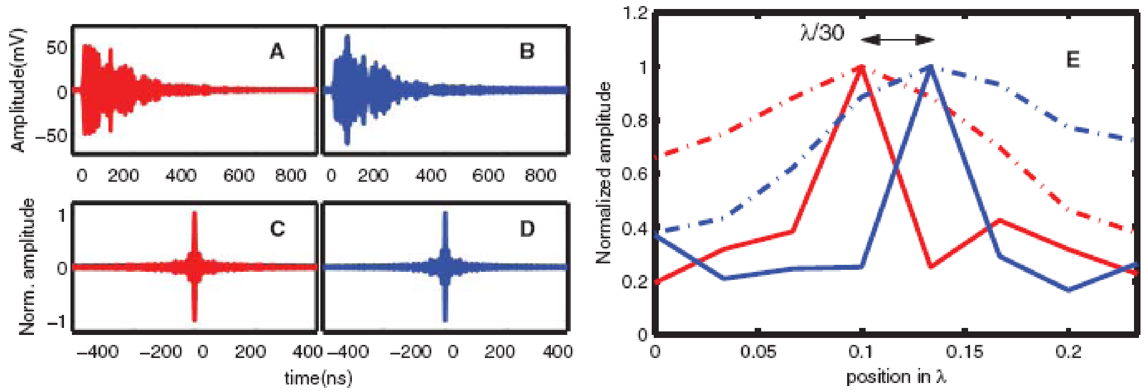

Fig.1. on the Lerosey et al. experiment [10]: Focusing beyond the diffraction limit. (A and B) show the signal received at one antenna of the TRM when a 10-ns pulse is sent from antennas 3 and 4, respectively, of the micro structured array. The signals in (A) and (B) look considerably different, although antennas 3 and 4 are only λ/30 apart. (C and D) show the time compression obtained at antennas 3 and 4, respectively, when the eight signals coming from antennas 3 and 4 are time-reversed and sent back from the TRM. (E) In full line are shown the focusing spots obtained around antennas 3 and 4. Their typical width is λ/30. Thus, antennas 3 and 4 can be addressed independently. The focal spots obtained when there are no copper wires are shown for comparison (dashed-dotted line). All maxima have been normalized by scaling factors in the ratios: 1 (red and blue dashed-dotted lines), 2.2 (red full line), 2.5 (blue full line).

Factorizing,

$$\bar{\mathrm{f}}^{n(\circ)}(\omega)=\bar{\mathrm{f}}^{m}(\omega)\left[\mathbf{1}-\mathbf{R}^{(\circ)*}(\omega)\right]_{n}^{m} \tag{36}$$

It is possible to invert formally equation (36)

$$\bar{\mathrm{f}}^{n}(\omega)=\bar{\mathrm{f}}^{m(\circ)}(\omega)[\left[\mathbf{1}-\mathbf{R}^{(\circ)*}(\omega)\right]^{-1}]_{n}^{m} \tag{37}$$

We can also make a formal span of the right side of the last equation as

$$\begin{aligned}\bar{\mathrm{f}}^{n}(\omega)=\sum_{m=1}^{3}\ \{&\bar{\mathrm{f}}^{m(\circ)}(\omega)\\ &+\bar{\mathrm{f}}^{m(\circ)}(\omega)\mathbf{R}^{m,n(\circ)*}(\omega)+\bar{\mathrm{f}}^{m(\circ)}(\omega)\left[\mathbf{R}^{m,n(\circ)*}(\omega)\right]^{2}\\ &+\bar{\mathrm{f}}^{m(\circ)}(\omega)\left[\mathbf{R}^{m,n(\circ)*}(\omega)\right]^{3}+\cdots\}\end{aligned} \tag{38}$$

From this equation we can see that the k-th component is then

$$\begin{aligned}f_{k}^{n}(\omega)=\sum_{m=1}^{3}\ \{&\delta_{m}^{n}f_{k}^{m(\circ)}(\omega)\\ &+\sum_{t}\ f_{t}^{m(\circ)}(\omega)A_{k}^{m,n*}G_{\omega}^{m,n(\circ)*}\left(\bar{r}_{t},\bar{r}_{k}\right)e^{-i\omega T}\\ &+\sum_{t}\ \sum_{l}\ f_{t}^{m(\circ)}(\omega)A_{k}^{m,n*}G_{\omega}^{m,n(\circ)*}\left(\bar{r}_{l},\bar{r}_{k}\right)e^{-i\omega T}A_{l}^{m,n*}G_{\omega}^{m,n(\circ)*}\left(\bar{r}_{t},\bar{r}_{l}\right)e^{-i\omega T}+\cdots\}\end{aligned} \tag{39}$$

In equation (39) we have used the Kroneckerdelta δ_{m}^{n} .
Now we substitute in equation (32) this last expression for $f_{k}^{n}(\omega)$:

$$f_j^n(\omega) = \sum_{m=1}^{3} [\delta_m^n f_j^{m(\circ)}(\omega)$$
$$+\sum_k A_k^{m,n*} G_\omega^{m,n(\circ)*}\left(\bar{r}_k, \bar{r}_j\right) e^{-i\omega T} \{ f_k^{m(\circ)}(\omega)$$
$$+\sum_t A_k^{m,n*} G_\omega^{m,n(\circ)*}\left(\bar{r}_t, \bar{r}_k\right) e^{-i\omega T} f_t^{m(\circ)}(\omega)$$
$$+\sum_t \sum_l A_k^{m,n*} G_\omega^{m,n(\circ)*}\left(\bar{r}_l, \bar{r}_k\right) e^{-i\omega T} A_l^{m,n*} G_\omega^{m,n(\circ)*}\left(\bar{r}_t, \bar{r}_l\right) e^{-i\omega T} f_t^{m(\circ)}(\omega) + \cdots \}] \qquad (40)$$

The parentheses can be cancelled, and we obtain

$$f_j^n(\omega) = \sum_{m=1}^{3} [\delta_m^n f_j^{m(\circ)}(\omega)$$
$$+\sum_k A_j^{m,n*} G_\omega^{m,n(\circ)*}\left(\bar{r}_k, \bar{r}_j\right) e^{-i\omega T} f_k^{m(\circ)}(\omega)$$
$$+\sum_k \sum_t A_k^{m,n*} G_\omega^{m,n(\circ)*}\left(\bar{r}_j, \bar{r}_k\right) e^{-i\omega T} A_t^{m,n*} G_\omega^{m,n(\circ)*}\left(\bar{r}_k, \bar{r}_t\right) e^{-i\omega T} f_t^{m(\circ)}(\omega) + \cdots] \qquad (41)$$

From which we get a generalized Neumann series [12] for the discrete case of the Fourier transform of the integral equation solution (16),

$$f_j^n(\omega) = \sum_{m=1}^{3} [\delta_m^n f_j^{m(\circ)}(\omega)$$
$$+\sum_t A_j^{m,n*} f_t^{m(\circ)}(\omega) \{ G_\omega^{m,n(\circ)*}\left(\bar{r}_t, \bar{r}_j\right) e^{-i\omega T}$$
$$+\sum_k G_\omega^{m,n(\circ)*}\left(\bar{r}_t, \bar{r}_k\right) e^{-i\omega T} A_t^{m,n*} G_\omega^{m,n(\circ)*}\left(\bar{r}_k, \bar{r}_j\right) e^{-i\omega T}$$
$$\sum_{k,l} G_\omega^{m,n(\circ)*}\left(\bar{r}_t, \bar{r}_k\right) e^{-i\omega T} A_k^{m,n*} G_\omega^{m,n(\circ)*}\left(\bar{r}_k, \bar{r}_l\right) e^{-i\omega T} A_l^{m,n*} G_\omega^{m,n(\circ)*}\left(\bar{r}_l, \bar{r}_j\right) e^{-i\omega T} + \cdots \}] \qquad (42)$$

If the bracketed expression in equation (42) is convergent, then it must be equal to the Fourier transform of the complete time reversed Green function $\mathfrak{G}_\omega^{m,n*}(r_t, r_j)$, so that we may write

$$f_j^n(\omega) = \sum_{m=1}^{3} [\delta_m^n f_j^{m(\circ)}(\omega) + \sum_k A_j^{m,n*} \mathfrak{G}_\omega^{m,n*}\left(\bar{r}_k, \bar{r}_j\right) f_k^{m(\circ)}(\omega)] \qquad (43)$$

Equation (43) can be written in a compact vector form as:

$$\bar{\mathrm{f}}^m(\omega) = \bar{\mathrm{f}}^{n(\circ)}(\omega) \left[\mathbf{1} + \mathbf{R}^*(\omega) \right]_m^n \qquad (44)$$

In this equation, we define the kernel

$$\mathbf{R}^{m,n*}(\omega)_{j,k} = \begin{cases} 0 & if \quad j = k \\ A_j^{m,n*} \mathfrak{G}_\omega^{m,n*}(\bar{r}_k, \bar{r}_j) & if \quad j \neq k \end{cases} \qquad (45)$$

Where $\mathbf{R}_\omega^{n,m}(\vec{r}_t, \vec{r}_j)$ signifies that the Green's function is that obtained in a time reversed process.
Transposing equation (44) we get

$$\left\{\overline{\mathrm{f}}^{m}(\omega)\right\}^{\tau}=\left\{\overline{\mathrm{f}}^{n(\circ)}(\omega)\left[\mathbf{1}+\mathbf{R}^{*}(\omega)\right]_{m}^{n}\right\}^{\tau} \tag{46}$$

When we transpose the product of factors in the right hand member of this equation, we must permute the order:

$$\left\{\overline{\mathrm{f}}^{m}(\omega)\right\}^{\tau}=\left\{\left[\mathbf{1}+\mathbf{R}^{*}(\omega)\right]_{m}^{n}\right\}^{\tau}\left\{\overline{\mathrm{f}}^{n(\circ)}(\omega)\right\}^{\tau} \tag{47}$$

Equation (47) can be written without the explicit indication that we must transpose the row vectors. But after transposing we have column vectors. Then we finally obtain (for real interactions):

$$\overline{f}^{n}(\omega)=\left[\mathbf{1}+\mathbf{R}(\omega)\right]_{m}^{n}\overline{f}^{m(\circ)}(\omega) \tag{48}$$

Equations (45) and (48) resume the efforts to make a better envoy of information in a discrete electromagnetic system (for example the broadcast by an antenna arrangement in MIMO technology [13, 14]) by using time reversal. This equation contains all the necessary information for performing the image envoy (experimentally a resolution of $\lambda/30$ is observed on reference [10]).

For the sake of completeness, we will describe the Lerosey et al experiment (see reference [10] and figure 1) where the researchers put a set of aleatory distributed wires around eight antennas to push the evanescent waves to transform into travelling waves improving substantially the signal localization when they make a time reversal procedure. They put a crown of very thin wires in the electromagnetic near zone of every one of the eight antennas they employed to send a message (indeed a television color image). When they implement a broadcast in time reversed order, they obtain an amazing localization as we can see in the corresponding figure 1. The crown of aleatory distributed wires must be located in the near zone because it is impossible to control strictly the time reversal process as occur with acoustic signals. That is, when we record an acoustic signal we have the time to know with a high precision how is the specific shape of the signal at every time interval. With electromagnetic waves there is no enough time to analyze, record the shape, and then make a time reversal. But the trick is in the location of the "sink term" because when the electromagnetic waves start its departure from the antennas they preserve the carried information because their strike with the wires changes the exponentially decaying shape with a travelling shape. The effect is the same as inside of a metamaterial. But we have shown that the employment of this kind of negative index devices is equivalent to a time reversal process that is the existence of a sink. So we cannot in practice build a strictly named time reversed source (or simply sink), but their mathematical presence is in the time reversal matrix-vector equation and also the physical effect as we will show in a very simple academic example. It is not very difficult to conjecture that we are accomplished a mathematical base that we have lost when we carry out an inappropriate process (without a sink term), and we simply retrieve those stationary elements (lost members of the mathematical base) on the electromagnetic near zone and converting them on travelling waves. That is the central concept is the completeness of a mathematical vector space.

5. AN ACADEMIC EXAMPLE

Even we have not strictly proved that we are completing a mathematical base as we have stated on the last section, we can show how to use the new tools in practice. With the purpose of showing how to construct the Green's function in the vector-matrix case, we will assume that there is only one emitter and one transmitter. Then equation (48) takes the simple form [1]

$$f_{1}^{n}(\omega)=\mathbf{R}_{m}^{n}(\omega;\overline{r}_{1},\overline{r}_{0})\,f_{0}^{m}(\omega) \tag{49}$$

Before to write explicitly $f_{1}^{n}(\omega)$ and $f_{0}^{m}(\omega)$, we justify its expressions. From communication theory [15], we have the ensemble

$$f(a_i,t)=\sum_{n=-\infty}^{+\infty}a_n\left[\frac{\sin\pi\left[2Wt-n\right)\right]}{\pi(2Wt-n)}\right] \tag{50}$$

With the a_i normal and independent all with the same standard deviation $\sqrt{N}$. This ensemble is a representation of "white" noise, band limited to the band from 0 to W cycles per second and with average power N^2 . Our example shows how we can recover a monochromatic signal from a disperse one represented for one term of equation (50) by means of time reversal formalism. So we can take only one term of the sum in order to make simpler the following equations. We can also simplify the parameters and rename the function $f(a_i,t)$:

$$\frac{a_n}{W} = A\,\frac{n}{2W} = \Delta t \text{ and } f(a_i,t) = h_0^{1,2(\circ)}(t) \tag{51}$$

Let us introduce a simplified notation for the input and output signals and suppose they have some special shapes like, first the output or emitted ones:

$$f_0^{1(\circ)}(\omega) = X_1(\omega) = f_0^{2(\circ)}(\omega) = X_2(\omega) \text{ and } f_0^{3(\circ)}(\omega) = 0 \tag{52}$$

These functions are the Fourier transform of the corresponding following functions (retarded and central-peaked)

$$f_0^{1(\circ)}(\omega) = X_1(\omega) = f_0^{2(\circ)}(\omega) = X_2(\omega) \text{ and } f_0^{3(\circ)}(\omega) = 0 \tag{53}$$

($h_0^{1,2(\circ)}(t)$ Is a short notation for $h_0^{1(\circ)}(t)$ or $h_0^{1,2(\circ)}(t)$)

In this expression "a" is the dispersion, A is a normalization constant for the final signal and

$$\Delta t = \frac{\left|\bar{r}_1 - \bar{r}_0\right|}{c} \tag{54}$$

We have then the explicit behavior of their Fourier transforms (also $X_{1,2}(\omega)$ is a short notation for $X_1(\omega)$ or $X_2(\omega)$):

$$X_{1,2}(\omega) = \begin{cases} A e^{\frac{-i\omega(\bar{r}_1-\bar{r}_0)}{c}} & if \quad \omega \le a \\ 0 \quad if & \omega > a \end{cases} \tag{55}$$

Additionally, let us suppose that the third component of the input or arrived signal vanishes and

$$f_1^1(\omega) = Y_1(\omega) = f_1^2(\omega) = Y_2(\omega) \text{ and } f_1^3(\omega) = 0 \tag{56}$$

Is the Fourier transform of the limited range cosine function (monochromatic pulse of width d)

$$h_1^{1,2}(t) = p_d(t)\cos(\omega_0 t) \tag{57}$$

Where $p_d(t)$ represents the unit pulse function of width d, so that

$$Y_{1,2}(\omega) = \frac{\sin\left(\frac{1}{2}d(\omega-\omega_0)\right)}{\omega-\omega_0} + \frac{\sin\left(\frac{1}{2}d(\omega+\omega_0)\right)}{\omega+\omega_0} \tag{58}$$

(This is a pulse propagating toward the z direction, with polarization at 45^0 respects to x axis). Substituting (55) and (58) into equation (49) we can see that the Fourier transform of the Kernel function is then

$$\mathbf{R}(\omega;\bar{r}_1,\bar{r}_0) = \frac{1}{X_{1,2}(\omega)}\begin{bmatrix} b_{11}(\omega) & b_{12}(\omega) \\ b_{21}(\omega) & b_{22}(\omega)\end{bmatrix} = \begin{bmatrix} \mathbf{R}_1^1(\omega) & \mathbf{R}_2^1(\omega) \\ \mathbf{R}_1^2(\omega) & \mathbf{R}_2^2(\omega)\end{bmatrix} \tag{59}$$

Where the explicit components of the first matrix or ***b***-matrix are

$$b_{11}(\omega)=2\left[\frac{\sin\left(\frac{1}{2}d(\omega-\omega_0)\right)}{\omega-\omega_0}+\frac{\sin\left(\frac{1}{2}d(\omega+\omega_0)\right)}{\omega+\omega_0}\right]$$

$$-Ap_{2a}(\omega)\mathrm{e}^{\frac{-\mathrm{i}\omega(\bar{r}_1-\bar{r}_0)}{c}} \tag{60}$$

$$b_{12}(\omega)=Ap_{2a}(\omega)\mathrm{e}^{\frac{-\mathrm{i}\omega(\bar{r}_1-\bar{r}_0)}{c}}$$
$$-\left[\frac{\sin\left(\frac{1}{2}d(\omega-\omega_0)\right)}{\omega-\omega_0}+\frac{\sin\left(\frac{1}{2}d(\omega+\omega_0)\right)}{\omega+\omega_0}\right] \tag{61}$$

$$b_{21}(\omega)=3\left[\frac{\sin\left(\frac{1}{2}d(\omega-\omega_0)\right)}{\omega-\omega_0}+\frac{\sin\left(\frac{1}{2}d(\omega+\omega_0)\right)}{\omega+\omega_0}\right]$$

$$-2Ap_{2a}(\omega)\mathrm{e}^{\frac{-\mathrm{i}\omega(\bar{r}_1-\bar{r}_0)}{c}} \tag{62}$$

$$b_{22}(\omega)=2Ap_{2a}(\omega)\mathrm{e}^{\frac{-\mathrm{i}\omega(\bar{r}_1-\bar{r}_0)}{c}}$$
$$-2\left[\frac{\sin\left(\frac{1}{2}d(\omega-\omega_0)\right)}{\omega-\omega_0}+\frac{\sin\left(\frac{1}{2}d(\omega+\omega_0)\right)}{\omega+\omega_0}\right] \tag{63}$$

In terms of $X_{1,2}\ (\omega)$, $\mathbf{R}(\omega;\bar{r}_1,\bar{r}_0)$ and $Y_{1,2}\ (\omega)$ we can verify that equation(49)is developed like

$$\begin{bmatrix} Y_1\ (\omega) \\ Y_2\ (\omega) \end{bmatrix}=\begin{bmatrix} \mathbf{R}_1^1(\omega) & \mathbf{R}_2^1(\omega) \\ \mathbf{R}_1^2(\omega) & \mathbf{R}_2^2(\omega) \end{bmatrix}\begin{bmatrix} X_1\ (\omega) \\ X_2\ (\omega) \end{bmatrix} \tag{64}$$

From equation (45) we also have

$$\mathbf{R}(\omega;\bar{r}_1,\bar{r}_0)=\begin{bmatrix} \mathcal{G}_1^1(\omega) & \mathcal{G}_2^1(\omega) \\ \mathcal{G}_1^2(\omega) & \mathcal{G}_2^2(\omega) \end{bmatrix}\begin{bmatrix} A_1^1(\omega) & A_2^1(\omega) \\ A_1^2(\omega) & A_2^2(\omega) \end{bmatrix} \tag{65}$$

Because we can put an arbitrary interaction let us use the simplest one $\mathbf{A}=\mathbf{1}$. So we finally obtain

$$\mathcal{G}(\omega;\bar{r}_1,\bar{r}_0)=\mathbf{R}(\omega;\bar{r}_1,\bar{r}_0) \tag{66}$$

6. THE HOMOGENEOUS FREDHOLM'S EQUATION

After we obtain equation (14) we can use it as the starting point for analyzing discrete electromagnetic systems in a general sense. So we can explore the possible solutions when we suppose that no source and no sink exist. Then we obtain the equation [2]

$$\left[\mathbf{1}-\eta_e(\omega)\mathbf{K}^{(\circ)}(\omega)\right]_n^m \overline{w}_e^n(\omega)=0 \tag{67}$$

In this equation we introduce the Fredholm's eigenvalue η (ω) and the resonance solution w (ω).
And writing explicitly the kernel $\mathbf{K}^{(\circ)}(\omega)$ we can then write

$$\left[\mathbf{1}-\eta_e(\omega)\mathbf{G}^{(\circ)}(\omega)\mathbf{A}\right]_n^m \overline{w}_e^n(\omega)=0 \tag{68}$$

We also recall that Fredholm's determinant [16, 17]

$$\Delta\left(\eta_e(\omega)\right)=\prod_e \left(1-\frac{\eta(\omega)}{\lambda_e(\omega)}\right) \tag{69}$$

Must comply that

$$\Delta\left(\eta_e(\omega)\right)=0 \tag{70}$$

So we have that for each eigenvalue $\lambda_e(\omega)$ we have an Eigen function (or Eigen vector) $w_e^p(\omega)$ (remember that $p \equiv p,n$; that is, represent two indexes). Also we must have $\lambda_e(\omega)=1$. By developing expression (68)' we obtain, with $\mathbf{A}_e \equiv \eta_e(\omega)\mathbf{A}$

$$w_e^p(\omega)=\left[\mathbf{G}^{(\circ)}(\omega)\mathbf{A}_e\right]_s^p w_e^s(\omega) \tag{71}$$

From this equation and by using the properties of the integral operators we can obtain the orthogonality relation [2]

$$w_e^{p\dagger}(\omega)\mathbf{A}w_u^q(\omega)\left(\lambda_u-\lambda_e\right)=0 \tag{72}$$

The possible applications are on communications [10, 13, 14] because orthogonality guarantee less interference between the signals emitted by the antennas that support frequencies near the resonant ones but different on each device.

7. A SIMPLE APPLICATION FOR THE HOMOGENEOUS EQUATION

In this section we find the resonant frequencies for a specific problem, but we must remember that those resonant frequencies can be used only to associate an interval of frequencies of a real signal to a device that could be an antenna. The form of the kernel depends of the response of the media in some circumstances that can vary even from a different time interval. So we use an example that is very easy to work but that is not important how is the shape of the signal we used to get it. Now, we can find the resonant frequencies in this academic example. To this end we choose a convenient kernel $\mathbf{K}^{(\circ)}(\omega)$, for simplicity we do not take into accounts the three components of the electromagnetic field. Supposing we only has one, but we have two emitting antennas. A possible kernel is [2]:

$$\mathbf{K}^{(\circ)}(\omega)=\begin{pmatrix} \dfrac{\sin(\omega-\omega_0)d}{(\omega-\omega_0)d} & -i\dfrac{\cos(\omega-\omega_0)d}{(\omega-\omega_0)d} \\ i\dfrac{\cos(\omega-\omega_0)d}{(\omega-\omega_0)d} & \dfrac{\sin(\omega-\omega_0)d}{(\omega-\omega_0)d} \end{pmatrix} \tag{73}$$

At this point, it is important to remember that the equation we must solve is equation (67) that is

$$\left[\mathbf{1}-\eta_e(\omega)\mathbf{K}^{(\circ)}(\omega)\right]_n^m \overline{w}_e^n(\omega)=0$$

Where

$$\mathbf{K}_{i,j,m}^{n(\circ)}(\omega)=\left\{\begin{matrix} 0 \ \ if \ i=j \\ A_j^{n,m} G_\omega^{n,m(\circ)}(\overline{r}_i,\overline{r}_j) \ if \ i \neq j \end{matrix}\right\} \tag{74}$$

The conditions for resonances are that Fredholm's determinant equals zero [16]:

$$\Delta\begin{pmatrix} \frac{\sin(\omega-\omega_0)d}{(\omega-\omega_0)d}-\lambda & -i\frac{\cos(\omega-\omega_0)d}{(\omega-\omega_0)d} \\ i\frac{\cos(\omega-\omega_0)d}{(\omega-\omega_0)d} & \frac{\sin(\omega-\omega_0)d}{(\omega-\omega_0)d}-\lambda \end{pmatrix}=0 \tag{75}$$

And also that

$$\lambda=1 \tag{76}$$

That is, the Fredholm's eigenvalue equals to one.
These two conditions give us the two resonant frequencies for the system constituted by these two antennas:

$$\omega_1=\frac{\pi}{4d}+\omega_0 \tag{77}$$

And

$$\omega_2=\frac{3\pi}{4d}+\omega_0 \tag{78}$$

The fact that we have founded two resonant frequencies on our example of the two antennas is not a general rule. The reason for this apparently one to one relation comes from the explicit form of the kernel $\mathbf{K}_{i,j,m}^{n(\circ)}(\omega)$, so we can have more or even less resonances than the number of antennas. Indeed it is possible that the conditions for the existence of a resonance, does not be accomplished by the kernel associated to a specific problem. Nevertheless we guess that the consideration of resonant frequencies can be a powerful tool for problems in which the implementation of other standard procedures that could be more expensive or considerably slower.

8. CONCLUDING REMARKS

All our results are original but some of them were stated without a rigorous proof in other papers [1, 2]. The very special view of the present work is the fact that our discrete electromagnetic system can be described by a Fredholm's equation (see equation 2) which has known properties that can be applied now on the electromagnetic domain. Because our results are essentially tools to improve the electromagnetic signal localization and avoid the loss of information, we suggest the use of the results to general communication systems [10, 13, 14]. In other works, as we have mentioned, the experimental results found by other authors when they improve the signal localization support the confidence that we have developed a useful tool.

Nevertheless, in the electromagnetic case in fact the sink term (eq. 17) does not have to show to perfection the behavior of a source operating inversely in time. Even there is an extremely restriction to put the sink inside the near field zone; indeed we note that our results (equations 14, 15, 45, 48, 67, 68 and 72) can be applied without demanding that the sink term behaves so rigorously as in our description. The formalism can be easily applied to discrete systems without having to resort to the use of meta-materials [20, 21, and 22]. For example, A. Grbic et al., [21] use a difficult to build device (left-handed transmission-line lens) that consist on a loaded grid of transmission lines in a two dimensional array to obtain a relatively poor signal localization (□ $\lambda/14$) for microwaves, these results are very far to those of G. Lerosey et al. [10], which use time reversal techniques and obtain a $\lambda/30$ signal localization. Equations 14, 15, 45, 48, 67, 68 and 72 signify the heart of the formalism and their usefully expression because we can take both, the outcome and income probe signal, built the Fourier transform of the complete Green function, and then take any arbitrary signal putting them into equation (48) and obtain the time reversed one. Although in an algebraic form we can observe how the signals used in the correspondent shown example (section 5) fulfill the condition that on time reversal the signal obtained at the final receptor returns to the form of a localized pulse. The possible application of this localization phenomenon comprises those fields which exhibit the same time symmetry, something no less useful for acoustic applications. In other work [3] we remark on the equivalence between the use of materials with a negative refraction index in optics [18-26] and the use of Time Reversal in acoustics, as a mean to overcome the diffraction limit, and to indicate that the expressions for the negative refraction index (optics) and the Green's function transform (acoustics) have similar properties, which represents a support to the development presented here, since it strongly suggests the parallelism in the behavior of signals in acoustics and optics in relation to the mechanism for the overcoming of the diffraction limit. As we said above, experimentally, G. Lerosey et al. [10] have obtained microwave focusing thanks to the recovery of evanescent waves using the equivalent of a sink term like the one we have integrated into the vector formalism. These sink equivalent consists in a random distribution of wires located at a distance from nearby fields.
By other side, we have roughly explored the possibility of occurrence of phenomena that have his nearly identical partner in quantum mechanical scattering. That is our Fredholm's problem brings us results very similar to the solutions of the Lippmann-Schwinger equation for non-local and non-separable potentials also when we involve the so called resonant states [5, 16, 17]. This kind of electromagnetic resonances could be related to phenomena like reverberance or a negative refraction index behavior when we consider space and emitter-receptors as a joint system. Essentially a resonance is a condition of the field that appears without the incidence of travelling waves. This is related to the evanescent waves, so if we excite these frequencies, we can recover the information contained in the evanescent field as was explained by G. Lerosey et al. [10]. In addition to our theoretical results, we propose a technological improvement to MIMO by using the resonance frequencies as the center of corresponding sets or packs of normal frequencies to the end of have a better signal reception. We can either add a step to MIMO systems or substitute completely the conventional switching with selecting the devices by finding the resonant frequencies. Even we do not give a real example for their application, we give an academic one (section 6) with the equations for orthogonality being also original. It is very important to note that we don't have necessary a set of resonances for every real condition and that if they exist, the real conditions may change abruptly and the resonances disappear in other moment, but we can use them to assign some intervals of frequencies to specific devices. In summary, we consider that our vector extension of the Time Reversal formalism may be useful in diverse discrete electromagnetic systems, including telecommunication systems [13, 14] and because of the strong subwavelength focusing ($\lambda/30$) also in nanotechnology. The final conclusion about this relatively new field is based on works like the accomplished by K. Volke et al. [27] about the optical angular momentum transfer to microparticles. On these works we can observe a remarkable need of an appropriate tool that allows very strong signal localization without assistance of left-hand materials.

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