MAXIMUM LIKELIHOOD PARAMETER ESTIMATORS FOR THE TWO POPULATIONS GEV DISTRIBUTION

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ABSTRACT

The method of maximum likelihood for estimating the parameters of the two populations general extreme value (TPGEV) probability distribution function for the maxima is presented for the case of flood frequency analysis. The proposed methodology is compared with widely used models, namely: two component extreme value (TCEV), general extreme value (GEV) and Gumbel distributions. The TPGEV distribution behaved well for those selected sets of data in Northwestern Mexico and the results of this distribution proved to be better than the TCEV model, when there are two populations present in the flood sample of data. The paper contains several numerical examples of the application of the proposed methodology.

Keywords: *probability, flood frequency analysis, maximum likelihood, parameter estimation, mixed distributions*

1. INTRODUCTION

The method of maximum likelihood has been acknowledged as one of the best methods for parameter estimation of probability distribution functions. The properties of its estimators like the invariance property, Mood et al [1], and the asymptotically unbiasedness, sufficiency, for a particular class of probability distribution functions, consistency and efficiency, (Haan [2]) and the remarkable suitability when being applied to cumbersome mathematical expressions in its likelihood functions under some strict regularity conditions, have gained it the well-known condition of prime choice for solving problems of parameter estimation of probability distribution functions.

The use of the general extreme value (GEV) distribution function (Jenkinson [3], [4]) for flood frequency analysis is widespread enough and it is now a feasible option, by the practicing engineers, for application in the flood frequency analysis process (NERC [5]; Prescott and Walden [6]; and Hosking [7]).The use of a mixture of probability distributions functions for modeling samples of data coming from two populations have been proposed long time ago (Mood et al [1]). In the particular case of extreme value distributions, several options have been proposed so far, the TCEV distribution (Gumbel [8]; Todorovic and Rousselle [9]; Canfield [10]; and Rossi et al [11]) the mixed Gumbel distribution, (Gonzalez-Villarreal [12]; Raynal-Villasenor [13]; and Raynal-Villasenor and Guevara-Miranda [14]), and the mixed general extreme value distribution (Raynal-Villasenor and Santillan-Hernandez [15]; and Gutierrez-Ojeda and Raynal-Villasenor [16]).

It is the purpose of this paper to expand the knowledge of the two populations general extreme value distribution in the case of the maxima, by providing the procedure of application of the method of maximum likelihood for estimating its parameters.

2. THE GENERAL EXTREME VALUE DISTRIBUTION FOR THE MAXIMA

The probability distribution function of the GEV distribution for the maxima is, NERC [5]:

$$F(x) = \exp\left\{-\left[1-\frac{(x-\lambda)}{\alpha}\beta\right]^{1/\beta}\right\} \tag{1}$$

where α, β, and λ are the scale, shape and location parameters.

The probability density function is given by, NERC [5]:

$$f(x) = \frac{1}{\alpha}\exp\left\{-\left[1-\frac{(x-\lambda)}{\alpha}\beta\right]^{1/\beta}\right\}\left[1-\frac{(x-\lambda)}{\alpha}\beta\right]^{1/\beta-1} \tag{2}$$

For extreme value type I (Gumbel) distribution ($\beta \to 0$; $\gamma = 1.1396$):

$$-\infty < x < \infty \tag{3}$$

For extreme value type II (Frechet) distribution ($\beta < 0$; $\gamma > 1.1396$) :

$$\lambda + \alpha / \beta \leq x < \infty \tag{4}$$

For extreme value type III (Weibull) distribution ($\beta > 0$; $\gamma < 1.1396$):

$$-\infty < x \leq \lambda + \alpha / \beta \tag{5}$$

where γ is the coefficient of skewness.

3. THE TWO POPULATIONS GENERAL EXTREME VALUE DISTRIBUTION

Based in the general form for two populations probability distributions functions (Mood et al [1]):

$$F(x)_{mix} = (1-p)F(x;\underline{\theta_1}) + pF(x;\underline{\theta_2}) \tag{6}$$

where p is the proportion of the second population in the mixture. The TPGEV distribution can be constructed as:

$$F(x)_{mix} = (1-p)\exp\left\{-\left[1-\frac{(x-\lambda_1)\beta_1}{\alpha_1}\right]^{1/\beta_1}\right\} + p\exp\left\{-\left[1-\frac{(x-\lambda_2)\beta_2}{\alpha_2}\right]^{1/\beta_2}\right\} \tag{7}$$

and the corresponding probability density function is:

$$f(x)_{mix} = \frac{(1-p)}{\alpha_1}\left[1-\left(\frac{(x-\lambda_1)}{\alpha_1}\right)\beta_1\right]^{1/\beta_1-1} \exp\left\{-\left[1-\frac{(x-\lambda_1)\beta_1}{\alpha_1}\right]^{1/\beta_1}\right\}$$

$$+\frac{p}{\alpha_2}\left[1-\left(\frac{(x-\lambda_2)}{\alpha_2}\right)\beta_2\right]^{1/\beta_2-1} \exp\left\{-\left[1-\frac{(x-\lambda_2)\beta_2}{\alpha_2}\right]^{1/\beta_2}\right\} \tag{8}$$

4. THE METHOD OF MAXIMUM LIKELIHOOD

The method of maximum likelihood have been defined and applied to several probability distribution functions with defined probability density functions (pdf) (NERC [5]). Such method has suitable characteristics like the invariance property (Mood et al [1]), and the asymptotically unbiasedness, sufficiency, consistency and efficiency (Haan [2]) in large sample estimation and applicability in estimating the parameters of complex probability density functions.

The likelihood function of N independent random variables is defined to be the joint probability density function of N random variables and is viewed as a function of the parameters. If $X_1, ..., X_N$ is a random sample of a univariate probability density function, the corresponding likelihood function for the observed $X_1, ..., X_N$ sample is (Mood et al [1]):

$$L(\underline{x},\underline{\theta}) = \prod_{i=1}^{N} f(x_i) \tag{9}$$

where θ denotes the parameter set and *f(.)* is the probability density function.

The logarithmic version of eq. (8) is:

$$Ln[L(\underline{x},\underline{\theta})] = \sum_{i=1}^{N} Ln[f(x_i)] \tag{10}$$

and will be used instead of the former equation because it is easier to handle in the computational procedure described in the next section.

The set of parameters that maximize equation (9), if they exists, will be the maximum likelihood estimators for the parameters of the probability distribution function.

5. MAXIMUM LIKELIHOOD PARAMETER ESTIMATORS FOR THE TWO POPULATIONS GEV DISTRIBUTION FOR THE MAXIMA

Based in the principles contained in the previous section, the log-likelihood function for the TPGEV distribution for the maxima is:

$$Ln[L(x_i;\lambda_1,\alpha_1,\beta_1,\lambda_2,\alpha_2,\beta_2, p)] = \sum_{i=1}^{N} Ln\left\{\frac{(1-p)}{\alpha_1}\left(1-\frac{(x_i-\lambda_1)}{\alpha_1}\beta_1\right)^{1/\beta_1-1}\right.$$

$$\exp\left\{-\left[1-\frac{(x_i-\lambda_1)}{\alpha_1}\beta_1\right]^{1/\beta_1-1}\right\}+\frac{p}{\alpha_2}\left[1-\frac{(x_i-\lambda_2)}{\alpha_2}\beta_2\right]^{1/\beta_2-1}\exp\left\{-\left[1-\frac{(x_i-\lambda_2)}{\alpha_2}\beta_2\right]^{1/\beta_2}\right\}\Bigg\} \tag{11}$$

and the corresponding first order partial derivatives of such function with respect to each of the parameters are:

$$\frac{\partial Ln(L)}{\partial \lambda_j}=\frac{C_j}{\alpha_j^{\,2}}\sum_{i=1}^{N}\left\{\frac{-F(x_i;\underline{\theta_j})\left(1-\frac{(x_i-\lambda_j)}{\alpha_j}\beta_j\right)^{2/\beta_j-2}+F(x_i;\underline{\theta_j})\left(1-\frac{(x_i-\lambda_j)}{\alpha_j}\beta_j\right)^{1/\beta_j-2}}{DEN}\right\} \tag{12}$$

j = 1, 2

$$\frac{\partial Ln(L)}{\partial \alpha_j}=\frac{C_j}{\alpha_j^{\,2}}\sum_{i=1}^{N}\Bigg\{-F(x_i;\underline{\theta_j})\left(1-\frac{(x_i-\lambda_j)}{\alpha_j}\beta_j\right)^{1/\beta_j-1}$$

$$-f(x_i;\underline{\theta_j})(x_i-\lambda_j)\left((1-\beta_j)/\left(1-\frac{(x_i-\lambda_j)}{\alpha_j}\beta_j\right)+\left(1-\frac{(x_i-\lambda_j)}{\alpha_j}\beta_j\right)^{1/\beta_j-1}\right)\Bigg\}\Big/ DEN \tag{13}$$

j = 1, 2

$$\frac{\partial Ln(L)}{\partial \beta_j}=\frac{C_j}{\alpha_j}\sum_{i=1}^{N}\left\{\frac{\exp\left[-\left(1-\frac{(x_i-\lambda_j)}{\alpha_j}\beta_j\right)^{1/\beta_j}\right]\left[-\frac{1}{\beta_j^{\,2}}Ln\left(1-\frac{(x_i-\lambda_j)}{\alpha_j}\beta_j\right)+\frac{(1-\beta_j)(x_i-\lambda_j)}{\beta_j\left(1-\frac{(x_i-\lambda_j)}{\alpha_j}\beta_j\right)}\right]}{DEN}\right\} \tag{14}$$

j = 1, 2

$$\frac{\partial Ln(L)}{\partial p}=\sum_{i=1}^{N}\left\{\frac{-f(x;\underline{\theta_1})+f(x;\underline{\theta_2})}{DEN}\right\} \tag{15}$$

where:

$$DEN = f(x)_{mix} \tag{16}$$

$$F(x;\underline{\theta_1}) = \exp\left\{-\left[1-\frac{(x-\lambda_1)}{\alpha_1}\beta_1\right]^{1/\beta_1}\right\} \tag{17}$$

$$F(x;\underline{\theta_2}) = \exp\left\{-\left[1-\frac{(x-\lambda_2)}{\alpha_2}\beta_2\right]^{1/\beta_2}\right\} \tag{18}$$

$$C_1 = 1\text{-}p\ ;\ C_2 = p \tag{19}$$

The exact solution provided by the system of equations (12)-(14) is not known for the case of the of TPGEV distribution, so the maximum likelihood estimators of the parameters of the TPGEV distribution may be obtained by either solving numerically, e.g. by the method of Newton, the system of non-linear equations (equations (12)-(14)), or by a direct maximization of the log-likelihood function, equation (11), by a non-linear optimization procedure, e.g. the multivariable constrained Rosenbrock method (Kuester and Mize [17]). In this study the former option was the choice for estimating the parameters of the TPGEV distribution by the method of maximum likelihood.

6. THE TCEV DISTRIBUTION

The Two Component Extreme Value probability distribution has been defined (Rossi et al [11]) as:

$$F(x) = \exp\left[-\Lambda_1 \exp\left(-\frac{x}{\theta_1}\right) - \Lambda_2 \exp\left(-\frac{x}{\theta_2}\right)\right] \tag{20}$$

where Λ_i is the location parameter and θ_i is the shape parameter of the TCEV distribution.
The maximum likelihood parameters of the TCEV distribution are obtained by an iterative scheme using the following equations (Rossi et al [11]):

$$\Lambda_j = \frac{\Lambda_j\left[\sum_{i=1}^{N}\frac{\exp\left(-\frac{x_i}{\theta_1}\right)}{\psi(x_i)}\right]}{\theta_j\sum_{i=1}^{N}\exp\left(\frac{x_i}{\theta_j}\right)} \quad ;\ j = 1, 2 \tag{21}$$

$$\theta_j = \frac{\dfrac{\sum_{i=1}^{N} x_i \exp\left(-\frac{x_i}{\theta_j}\right)}{\psi(x_i)}}{\sum_{i=1}^{N} x_i \exp\left(-\frac{x_i}{\theta_j}\right) + \sum_{i=1}^{N}\frac{\exp\left(-\frac{x_i}{\theta_j}\right)}{\psi(x_i)}} \quad ;\ j = 1, 2 \tag{22}$$

where $\psi(.)$ is the digamma function with argument (.).

7. RESULTS AND DISCUSSION

As examples of application, the annual flood discharges of several gauging stations, located in the states of Sinaloa and Chihuahua, in Northwestern Mexico, were processed and the sample maximum likelihood estimators of the parameters of the TPGEV distribution were computed. Those gauging stations are located in an area that every year is affected by tropical cyclones, during summer and fall, and cold fronts, during winter, causing the presence of at least two populations in the samples of flood data. The years of record, computed sample mean, standard deviation and coefficient of skewness of the samples of flood data for the selected gauging stations are shown in table 1.

Table 1. Statistical characteristics of flood data of the selected gauging stations

		Statistical Characterisics		
Gauging Station	Years of Record	Mean	Standard Deviation	Coefficient of Skewness
El Oregano	50	153.73	97.30	0.65
Santa Cruz	52	753.91	726.74	2.54
Huites	51	2498.96	2221.35	2.07
El Zopilote	56	163.37	169.15	2.08
Jaina	52	696.48	692.05	3.27
Ixpalino	41	864.85	808.08	2.76
Acatitan	41	400.52	468.40	2.64
San Bernardo	35	741.58	550.21	2.14
Choix	40	160.68	112.74	2.54
Tezocoma	34	87.79	125.05	3.84

The one population general extreme value and Gumbel distributions computed parameters, were obtained through the application of user-friendly computer package FLODRO 4.0 (Raynal-Villasenor [18]) for the selected gauging stations, and they are shown in tables 2 and 3.
The TPGEV and TCEV distribution computed parameters for such gauging stations were evaluated by using computer code FLODRO 4.0 (Raynal-Villasenor [18]) and the results are contained in tables 4 and 5.
In order to compare the results provided by the TPGEV distribution with those produced by other widely applied models, such the one population general extreme value (GEV), Gumbel (G) and Two Component Extreme Value (TCEV) distributions, in table 6 a compilation is presented of the design values for several return periods and their standard errors of fitting, EE, produced by the methods mentioned above and the one proposed in the paper. The EE is defined as (Kite [19]):

$$EE = \left[\frac{\sum_{i=1}^{N} (x_i - y_i)^2}{(N - m_j)} \right]^{1/2} \tag{23}$$

where x_i are the historical values of the sample, y_i are the values produced by the distribution function corresponding to the same return periods of the historical values, N is the sample size, and m_j is the number of parameters of the distribution function.

Table 2. One population GEV and Gumbel (EV-I) distributions parameters for the selected gauging stations

	Gumbel Parameters		GEV Parameters		
Gauging Station	λ	α	λ	α	β
El Oregano	108.84	95.98	109.43	77.05	0.016
Santa Cruz	567.28	426.88	406.89	349.47	-0.320
Huites	1650.26	1212.08	1402.62	874.92	-0.455
El Zopilote	97.05	100.98	78.34	80.35	-0.372
Jaina	451.58	361.29	386.64	284.05	-0.357
Ixpalino	572.42	436.54	510.07	370.85	-0.275
Acatitan	241.58	241.34	194.24	181.28	-0.407
San Bernardo	527.17	325.54	476.43	272.37	-0.305
Choix	109.28	73.92	104.97	71.14	-0.107
Tezocoma	48.74	54.37	36.31	36.87	-0.482

The results of this study provide the arguments to establish the following points:

1) The TPGEV distribution function behaved very well in the selected gauging stations, just in two out of ten it cannot reach convergence. The TCEV failed to attain convergence in three samples of flood data. In the case of the TPGVE, the lack of convergence was not solved by changing the initial values in the optimization procedure, it seems that for a specific sample of flood data the procedure just will have a lack of convergence, so in those cases such model simply won´t work. The lack of convergence in the case of the TCEV it seems is associated by the estimation procedure itself, it won´t converge in many instances.
2) The TPGEV distribution function has the least standard error of fit (EE) in five gauging stations and was very close to the least value in four additional gauging stations. The GEV reached the least value of the EE in five of the gauging stations
3) None of the Gumbel (extreme value type I) nor the TCEV distributions were even close to any of the least values of the EE in the ten selected gauging stations
4) The TPGEV distribution function has the least standard error of fit (EE) in five gauging stations and was very close to the least value in three additional gauging stations. The GEV reached the least value of the EE in five of the gauging stations
5) With regard to the design values, for those gauging stations where the TPGEV distribution produced the best fit, the produced values were much higher than those for the GEV distribution
6) The computation of the parameters and design values for the TPGEV distribution were made possible by the use of a personal computer. It will be very difficult, if not impossible, to evaluate such parameters and design values with a portable calculator or some other computing device with less capacity than a personal computer. This is a drawback that the proposed method has and there is no way to overcome it, given the enormous number of calculations that the optimization code has to perform in order to obtain the maximum likelihood estimators of the parameters of the TPGVE distribution

8. CONCLUSIONS

The procedure of finding the estimators of the parameters of the TPGEV distribution for the maxima, using the method of maximum likelihood, has been presented.

The TPGEV distribution behaved well for those selected sets of flood data, just in two out of the ten cases considered for analysis, the TPGEV could not reach convergence in the estimation of parameters process. The lack of convergence was not solved by changing the initial values in the optimization procedure, it seems that for a specific sample of flood data the procedure just will have a lack of convergence, so in those cases such model simply won´t work. In these cases another model of mixed distributions should be used.

The TCEV had three failures in the estimation of the parameters process due to the lack of convergence. The lack of convergence in the case of the TCEV it seems is associated by the estimation procedure itself, it won´t converge in many instances.

In five cases the TPGEV distribution produced the least standard error of fit and in other three cases was very close to the GEV distribution which has the least standard error of fit in such samples of flood data.

It will be wise to consider the presence of two populations in the sample of flood data, in addition to the standard error of fit, to reach a decision on which model to use for flood frequency analysis.

Table 3. TPGEV distribution parameters for the selected gauging stations

Station	λ_1	α_1	β_1	λ_2	α_2	β_2	p
El Oregano	67.47	41.42	$9.6\ 10^{-6}$	195.54	67.31	-0.01	0.441
Santa Cruz	*	*	*	*	*	*	*
Huites	1194.05	506.37	0.068	3389.28	2034.51	-0.082	0.309
El Zopilote	*	*	*	*	*	*	*
Jaina	259.02	146.21	0.0131	632.19	395.34	-0.269	0.541
Ixpalino	445.49	190.14	1.020	583.68	444.93	-0.266	0.755
Acatitan	241.71	74.48	0.704	188.29	206.94	-0.597	0.731
San Bernardo	413.44	190.27	$5.4\ 10^{-4}$	699.78	413.00	-0.233	0.422
Choix	85.86	24.85	0.535	178.29	54.13	-0.347	0.483
Tezocoma	36.53	34.15	-0.3050	553.31	163.70	1.1448	0.033

* No convergence was attained

Table 4. TCEV distribution parameters for the selected gauging stations

Station	Λ_1	θ_1	Λ_2	θ_2	p
El Oregano	6.75	11.26	3.32	80.47	0.330
Santa Cruz	4.20	62.01	2.47	506.96	0.369
Huites	*	*	*	*	*
El Zopilote	2.33	10.60	2.02	125.07	0.465
Jaina	5.12	115.18	1.69	490.28	0.248
Ixpalino	3.35	198.30	1.75	563.66	0.550
Acatitan	2.29	77.71	1.50	309.42	0.396
San Bernardo	*	*	*	*	*
Choix	*	*	*	*	*
Tezocoma	2.54	17.06	1.05	86.64	0.292

* No convergence was attained

Based in the results presented in the paper, the author recommend this procedure to be included in the standard methods for flood frequency analysis, as an additional model for the flood frequency analysis when there is the possibility that two populations are present in the samples of flood data.

Table 5. Comparison of Design Values (in m^3/s) and Standard Errors of Fitting (in m^3/s) Between Several Models for One and Two Populations Samples

Model	Q_5	Q_{10}	Q_{20}	Q_{50}	Q_{100}	EE
			El Oregano			
TCEV	217	278	336	411	467	17
TPGEV	234	290	341	406	455	**10**
GEV	225	283	338	410	464	14
Gumbel	223	280	335	405	458	14
			Sta. Cruz			
TCEV	1218	1599	1964	2436	2790	275
TPGEV	*	*	*	*	*	*
GEV	1102	1413	1714	2105	2398	**161**
Gumbel	1112	1430	1736	2131	2427	321
			Huites			
TCEV	*	*	*	*	*	*
TPGEV	3153	5557	8451	9641	9938	**260**
GEV	2724	3385	4020	4842	5457	541
Gumbel	3468	4377	5250	6380	7226	1046
			Zopilote			
TCEV	256	340	421	525	604	62
TPGEV	*	*	*	*	*	*
GEV	199	259	317	392	448	**30**
Gumbel	248	324	397	491	562	69
			Jaina			
TCEV	995	1361	1714	2170	2513	324
TPGEV	981	1416	1916	2714	2453	245
GVE	813	1026	1230	1425	1693	**224**
Gumbel	993	1264	1524	1861	2114	372
			Ixpalino			
TCEV	1183	1591	1992	2516	2909	397
TPGEV	1199	1722	2324	3290	4186	**267**
GEV	1066	1345	1612	1957	2216	285
Gumbel	1227	1555	1869	2276	2581	416
			Acatitan			
TCEV	592	823	1045	1334	1550	227
TPGEV	527	930	1525	2787	4314	**48**
GEV	466	602	732	902	1028	135
Gumbel	595	776	951	1176	1345	251

TCEV = Two Component Extreme Value Distribution
TPGEV = Two Populations General Extreme Value Distribution
* No convergence was attained in the estimation of parameters process
Bold numbers correspond to the distribution with best fit

Table 5. Comparison of Design Values (in m^3/s) and Standard Errors of Fitting (in m^3/s) Between Several Models for Two Populations Samples (cont'd)

Model	Q_5	Q_{10}	Q_{20}	Q_{50}	Q_{100}	EE
			San Bernardo			
TCEV	*	*	*	*	*	*
TPGEV	987	1358	1804	2514	3153	153
GEV	454	659	855	1109	1299	**145**
Gumbel	1015	1260	1494	1797	2025	241
			Choix			
TCEV	*	*	*	*	*	*
TPGEV	216	281	359	490	619	37
GEV	191	232	271	322	360	**34**
Gumbel	215	263	309	369	414	53
			Tezocoma			
TCEV	135	199	262	342	403	73
TPGEV	110	173	293	598	664	**38**
GEV	92	119	146	180	206	57
Gumbel	130	171	210	261	299	86

TCEV = Two Component Extreme Value Distribution

TPGEV = Two Populations General Extreme Value Distribution

* No convergence was attained in the estimation of parameters process

Bold numbers correspond to the distribution with best fit

9. ACKNOWLEDGEMENTS

The author wish to express their gratitude to the Universidad de las Americas, Puebla for the support provided in the realization of this paper.

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