FUNDAMENTAL PROBLEMS OF AN INFINITE PLATE WITH A HOLE CONFORMALLY MAPPED INSIDE THE UNIT CIRCLE

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ABSTRACT
Complex variable methods are used to derive the Goursat functions for the first and second fundamental problems of an infinite plate with a curvilinear hole C. The hole is mapped in the domain inside a unit circle by means of the rational mapping function. Many special cases are discussed and established of these functions .Also, many applications and examples are considered. Moreover the components of stress , in each application , are computed.

Keywords: *Complex variable method ,an infinite plate , curvilinear hole , conformal mapping , Goursat functions,*

1. INTRODUCTION AND BASIC EQUATIONS

Problems dealing with isotropic homogeneous perforated infinite plate have been investigated by many authors , see [1-9] . Some of them [2-4] used Laurant's theorem to express each complex potential as a power series in order to obtain the first fundamental problem . Others , [5-9]used complex variable method to obtain the solution of the first and second fundamental problem in the closed form of Goursat functions .

Fundamental problems in the plane theory of elasticity are equivalent to finding two analytic functions $\phi_1(z)$ and $\psi_1(z)$ of one complex argument $z = x + iy$.

These functions satisfy the boundary conditions

$$k\phi_1(t) - t\overline{\phi_1'}(t) - \overline{\psi_1}(t) = f(t) \tag{1.1}$$

where t denotes the affix of a point on the boundary. In the first fundamental problem $k = -1$, $f(t)$ is a given function of stresses. While in the second fundamental problem $k = \chi = \frac{(\lambda + 3\mu)}{\lambda + \mu}$, $\lambda = \frac{E}{(1-2v)(1+v)}$ and $f = 2\mu\ g$ is a given function of the displacement , λ and μ are called the Lame constants, E is called Young's constant and v is called Poisson's ratio.

Let , the complex potentials $\phi_1(t)$ and $\psi_1(t)$ take the forms , see [1]

$$\phi_1(\zeta) = -\frac{X + iY}{2\pi(1+\chi)}\ln\zeta + c\Gamma\zeta + \phi(\zeta)\ , \tag{1.2}$$

$$\psi_1(\zeta) = \chi\frac{(X - iY)}{2\pi(1+\chi)}\ln\zeta + c\Gamma^*\zeta + \psi(\zeta)\ , \tag{1.3}$$

where X, Y are the components of the resultant vector of all external forces acting on the boundary, and Γ , Γ^* are constants . Generally the two complex functions , Goursat functions, $\phi(\zeta)$ and $\psi(\zeta)$ are single valued analytic functions within the region outside the unit circle γ and $\phi(\infty) = 0$.

All the authors [1,5-9] used the conformal mapping $z = cw(\zeta)$, $c > 0$, $w'(\zeta) \neq 0, \infty$ to conform the curvilinear hole of an infinite elastic plate onto the domain outside the unit circle γ .

It is known that ,the components of stresses are given by , see [1]

$$\sigma_{xx} + \sigma_{yy} = 4\,\mathrm{Re}\{\phi'(z)\}\ , \tag{1.4}$$

$$\sigma_{yy} - \sigma_{xx} + i\sigma_{xy} = 2\left\{\ \bar{z}\ \phi''(z) + \psi'(z)\right\} \tag{1.5}$$

Hence , we have

$$\sigma_{yy} = \mathrm{Re}\left\{\ 2\ \phi'(z) + M(z,\bar{z})\right\}\quad , \quad M(z,\bar{z}) = \bar{z}\ \phi''(z) + \psi'(z) \tag{1.6}$$

$$\sigma_{xx} = \mathrm{Re}\left\{ 2\ \phi'(z) - M(z,\bar{z}) \right\} \quad , \quad M(z,\bar{z}) = \bar{z}\ \phi''(z) + \psi'(z), \tag{1.7}$$

and

$$\sigma_{xy} = 2\,\mathrm{Im}\left\{ \bar{z}\ \phi''(z) + \psi'(z) \right\} = 2\,\mathrm{Im}\{M(z,\bar{z})\}. \tag{1.8}$$

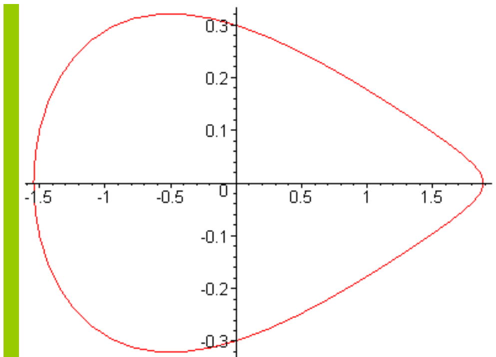
In this paper , we consider an infinite elastic plate with a curvilinear hole C where the origin lies inside the hole. This hole is conformally mapped in the domain inside a unit circle γ by means of the rational mapping function,

$$z = cw(\zeta) = c\,\frac{\zeta^{-1} + m\zeta}{1 - n\zeta} \quad (c > 0, |n| < 1\ , |\zeta| < 1)\ , \tag{1.9}$$

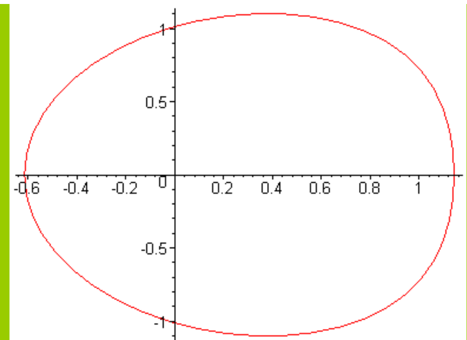
where , m, n are real parameters , subjected to the condition $w'(\zeta)$ does not vanish or become infinite inside the unit circle $\gamma\ , |\zeta| < 1$.

The solution inside the circle gives good information a bout the corresponding solution in the original problem since a conformal mapping is locally 1-1.

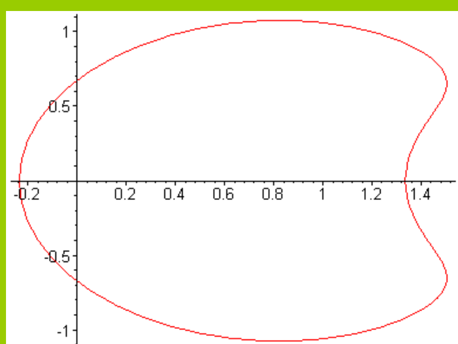
The following graphs clear up the shape of the rational mapping , of Eq.(1.9), in the domain inside a unit circle γ .



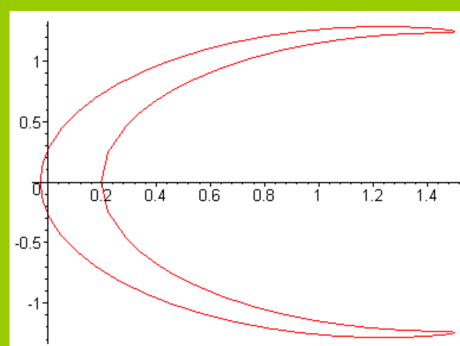
Fig(1): n=0.1 , m=0.7



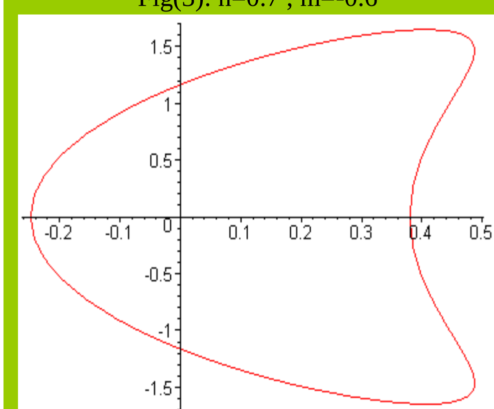
Fig(2): n=0.3 , m=-0.2



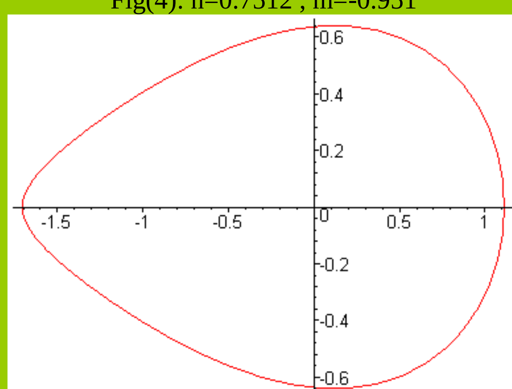
Fig(3): n=0.7 , m=-0.6



Fig(4): n=0.7512 , m=-0.951



Fig(5): n=0.21 , m=-0.7



Fig(6): n=-0.21 , m=0.347

2. GOURSAT FUNCTIONS

In order to obtain the Goursat functions we write

$$\frac{w(\zeta)}{\overline{w'(\zeta)}} = \alpha(\zeta) + \overline{\beta(\zeta)} \ , \tag{2.1}$$

$$\alpha(\zeta) = \frac{h}{(1-n\zeta)} \, , \; h = \frac{n^3(n^2+m)(1-n^2)^2}{[mn^2+2n^2-1]} \tag{2.2}$$

and $\overline{\beta(\zeta)}$ is a regular function for $|\zeta| < 1.$

Substituting from (1.3) and (1.4) into (1.1) , with the aid of (2.1), we get

$$k\phi(\sigma) - \alpha(\sigma)\ \overline{\phi'(\sigma)} - \overline{\psi_*(\sigma)} = G(\sigma) \ , \tag{2.3}$$

where

$$\psi_*(\sigma) = \psi(\sigma) + \beta(\sigma)\ \phi'(\sigma) \, , \tag{2.4}$$

$$G(\sigma) = F(\sigma) - ck\Gamma\sigma + \frac{c\overline{\Gamma^*}}{\sigma} + N(\sigma)\ \alpha(\sigma) + N(\sigma)\ \overline{\beta(\sigma)} \ ,$$

$$N(\sigma) = [c\overline{\Gamma} - \frac{\sigma(X-iY)}{2\pi(1+\chi)}] \quad , \quad F(\sigma) = f(t) \ . \tag{2.5}$$

The function $F(\sigma)$ with it's derivatives must satisfy the HÖlder condition.

Now ,in order to determine $\phi(\sigma)$,we multiply both sides of Eq.(2.3) by $\frac{1}{2\pi i(\sigma-\zeta)}$, where ζ is any point in the interior of γ , and integrating over the circle ,to obtain

$$k\phi(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\alpha(\sigma)\overline{\phi'(\sigma)}}{\sigma-\zeta} d\sigma = \frac{ck\Gamma}{(1-n\zeta)} + \frac{hn}{1-n\zeta} N(n^{-1}) - A(\zeta) \, , \tag{2.6}$$

where

$$A(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{F(\sigma)}{(\sigma-\zeta)} d\sigma \ . \tag{2.7}$$

The integral term of (2.6) can be written in the form

$$\frac{1}{2\pi i}\int_\gamma \frac{\alpha(\sigma)\overline{\phi'(\sigma)}}{\sigma-\zeta} d\sigma = \frac{ch\ bn}{1-n\zeta} \ , \tag{2.8}$$

where b is a complex constant , will be determined .

Substituting from (2.8) into (2.6) , we have

$$k\phi(\zeta) = A(\zeta) - \frac{ck\Gamma}{(1-n\zeta)} + \frac{hn}{1-n\zeta}(cb + N(n^{-1})) \ . \tag{2.9}$$

Differentiating (2.9) with respect to ζ , and using the result of $\overline{\phi'(\sigma)}$ in (2.8) , we obtain

$$cb = \frac{k\ E - \upsilon\ h\ \overline{E}}{(k^2 - \upsilon^2 h^2)} \tag{2.10}$$

where

$$E=\overline{A'(n^{-1})}-\frac{nck\Gamma}{(1-n^2)^2}+\upsilon\ h\ \overline{N(n^{-1})}\ ,\quad \upsilon=\frac{n^2}{(1-n^2)^2}\ . \tag{2.11}$$

Hence, $\phi(\zeta)$ is completely determined .
The second function of Goursat can be obtained in the form

$$\psi(\zeta)=-B(\zeta)-\frac{c\Gamma^*}{(1-n\zeta)}-\frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta)+\frac{h\zeta}{(\zeta-n)}\phi_*(n)-B\ , \tag{2.12}$$

where

$$\phi_*(\zeta)=\phi'(\zeta)+\overline{N(\zeta)}\ , \tag{2.13a}$$

and

$$B(\zeta)=\frac{1}{2\pi i}\int_\gamma\frac{\overline{F(\sigma)}}{(\sigma-\zeta)}d\sigma\ ,\quad B=\frac{1}{2\pi i}\int_\gamma\frac{\overline{F(\sigma)}}{\sigma}d\sigma\ . \tag{2.13b}$$

Eqs.(2.9) and (2.12) represent the Goursat functions.

3. SPECIAL CASES

Now , we are in a position to consider several cases :
(i) let $m=0\ ,n\neq 0$, we get the following conformal mapping , see Figs.(7-8)

$$z=cw(\zeta)=\frac{c}{\zeta-n\zeta^2}\ ,$$

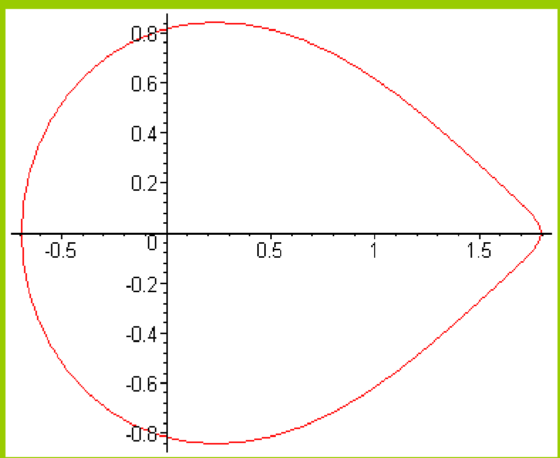
The two functions of Goursat of Eqs.(2.9) and (2.12), in this case, become

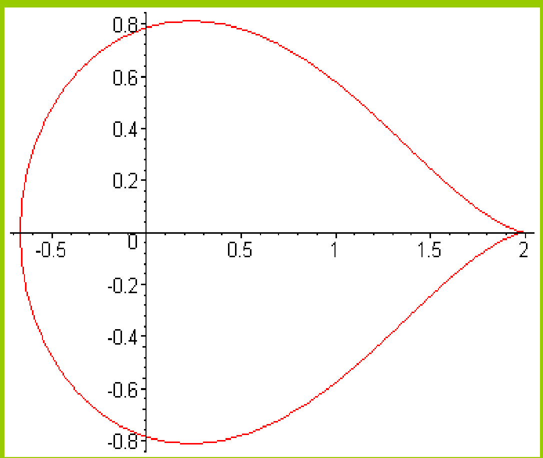
$$k\phi(\zeta)=A(\zeta)-\frac{ck\Gamma}{(1-n\zeta)}+\frac{n^6(1-n^2)^2}{(2n^2-1)(1-n\zeta)}[N(n^{-1})+\frac{kE-q\overline{E}}{k^2-q^2}]\ , \tag{3.1}$$

$$\psi(\zeta)=-B(\zeta)-\frac{c\Gamma^*}{(1-n\zeta)}-\frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta)+\frac{n^5(1-n^2)^2\zeta}{(2n^2-1)(\zeta-n)}\phi_*(n)-B, \tag{3.2}$$

where

$$E=\overline{A'(n^{-1})}-\frac{nck\Gamma}{(1-n^2)}+q\overline{N(n^{-1})}\ ,\quad q=\frac{n^7}{(2n^2-1)}.$$





Fig(7): m=0 , n=0.4432 Fig(8): m=0 , n=0.49876

(ii) For n=0 , $0 \le m \le 1$, we get the rational mapping function $z = c(1 + m\zeta^2)/\zeta$.
Then (2.9) and (2.12) become

$$k\phi(\zeta) = A(\zeta) - ck\Gamma \ , \tag{3.3}$$

$$\psi(\zeta) = -B(\zeta) - \frac{c\Gamma^*}{(1-n\zeta)} - \frac{(\zeta^2 + m)}{(m\zeta - 1)}\phi_*(\zeta) - B , \tag{3.4}$$

(iii) let $m = n = 0$, we get the mapping function $z = c\zeta$ .Then (2.9) and (2.12) yield

$$k\phi(\zeta) = A(\zeta) - ck\Gamma \ , \tag{3.5}$$

$$\psi(\zeta) = -B(\zeta) - c\Gamma^* - \frac{1}{\zeta}\phi_*(\zeta) - B . \tag{3.6}$$

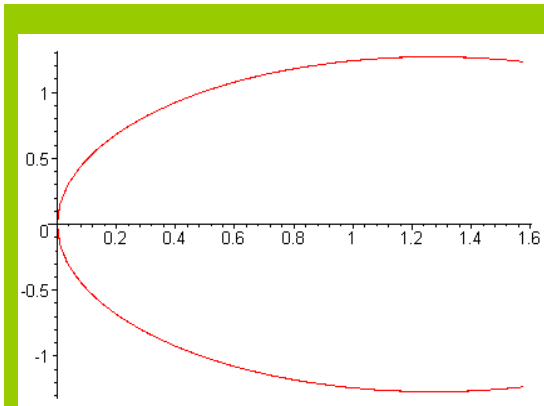
(iv) For m=-1 the boundary C degenerates into a circular cut , for values of m near -1 the edge of the hole resembles the shape of a crescent , see Fig (10-11).
Then ,in this case, (2.9) and (2.12) give

$$k\phi(\zeta) = A(\zeta) - \frac{ck\Gamma}{(1-n\zeta)} + \frac{n^4(1-n^2)^2}{1-n\zeta}\left[N(n^{-1}) + \frac{kE + n^5\overline{E}}{k^2 - n^{10}} \right] , \tag{3.7}$$

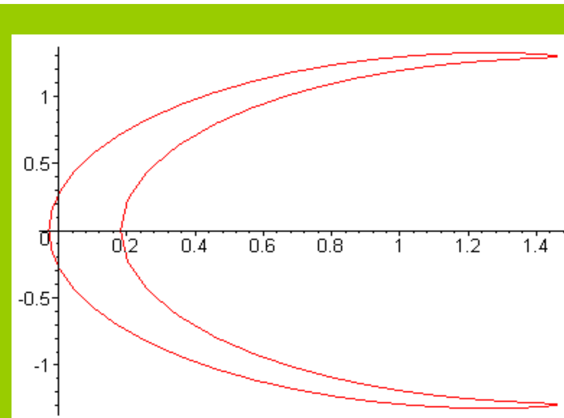
$$\psi(\zeta) = -B(\zeta) - \frac{c\Gamma^*}{(1-n\zeta)} - \frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta) + \frac{n^3(1-n^2)^2\zeta}{\zeta - n}\phi_*(n) - B , \tag{3.8}$$

where

$$E = \overline{A'(n^{-1})} - \frac{cnk\Gamma}{(1-n^2)^2} + n^5\overline{N(n^{-1})} .$$



Fig(10) : n=0.787 , m= -1



Fig(11) : n=0.73 , m= -0.951

(v) For m= $-n^2$,we get the function $z = c\dfrac{(1+n\zeta)}{\zeta}$ .Then (2.9) and (2.12) become

$$k\phi(\zeta) = A(\zeta) - \frac{ck\Gamma}{1-n\zeta} \ , \tag{3.9}$$

$$\psi(\zeta) = -B(\zeta) - \frac{c\Gamma^*}{1-n\zeta} + \zeta^2 (n+\zeta)\phi_*(\zeta) - B. \tag{3.10}$$

4. APPLICATIONS

Now , we can discuss some applications , and use program Maple 7 to compute the stress components

(i) For $k=-1,\ \Gamma = \dfrac{p}{4}\ ,\Gamma^* = -\dfrac{1}{2} p\, e^{-2i\theta}\ and\ X = Y = f = 0,$ we have the case of an infinite plate stretched at infinity by the application of a uniform tensile stress of intensity p , making an angle θ with the x-axis . The plate weakened by the curvilinear hole C which is free from stresses , see Fig. (12), (n:=0.25; m:=3;c:=2;p:=0.25;).

Then , the functions in (2.9) and (2.12) become

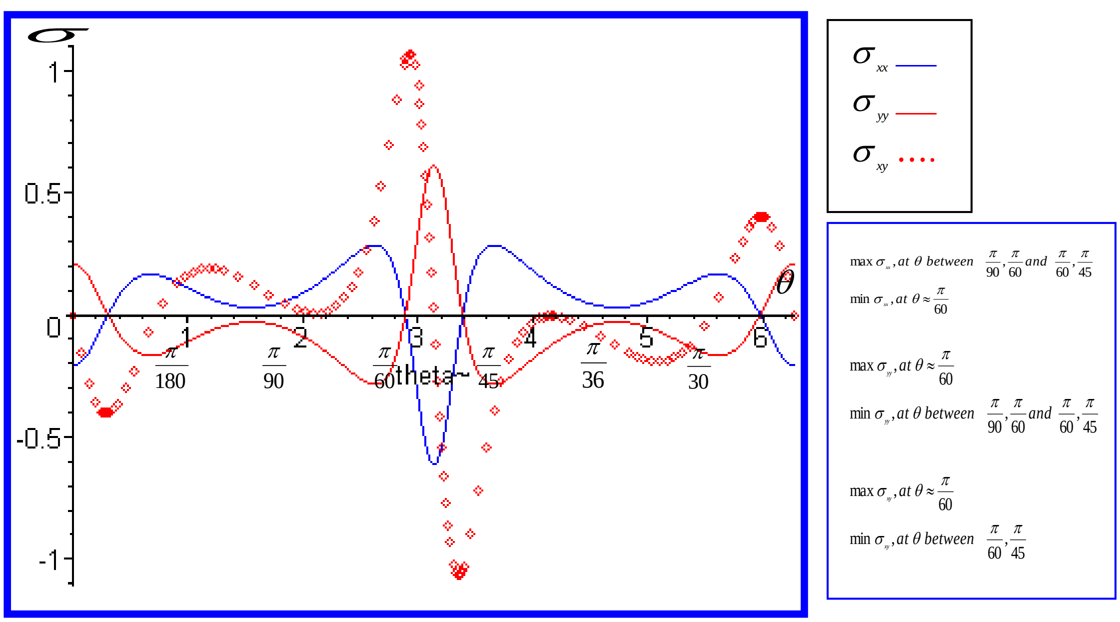
$$\phi(\zeta) = \frac{ch^2 n\upsilon p}{4(1-n\zeta)(1-\upsilon h)} - \frac{chnp}{4(1-n\zeta)}\left(1 - \frac{n}{(1-n^2)(1-\upsilon h)}\right), \tag{4.1}$$

$$\psi(\zeta) = -\frac{cp}{2(1-n\zeta)} e^{-2i\theta} - \frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta) + \frac{h\zeta}{(\zeta - n)}\phi_*(n)\ , \tag{4.2}$$

where

$$\phi_*(\zeta) = \phi'(\zeta) + \frac{cp}{4}.$$

The components of stresses for $0 \le \theta \le 2\pi$ are computed in Fig.(12), using Maple 7.



Fig(12):

(ii) For $k=-1,\ \Gamma = \Gamma^* = X = Y = 0\ and\ f = Pt$, where P is a real constant , see Fig. (13) , (n:=0.25; m:=3;c:=2;p:=0.25;).

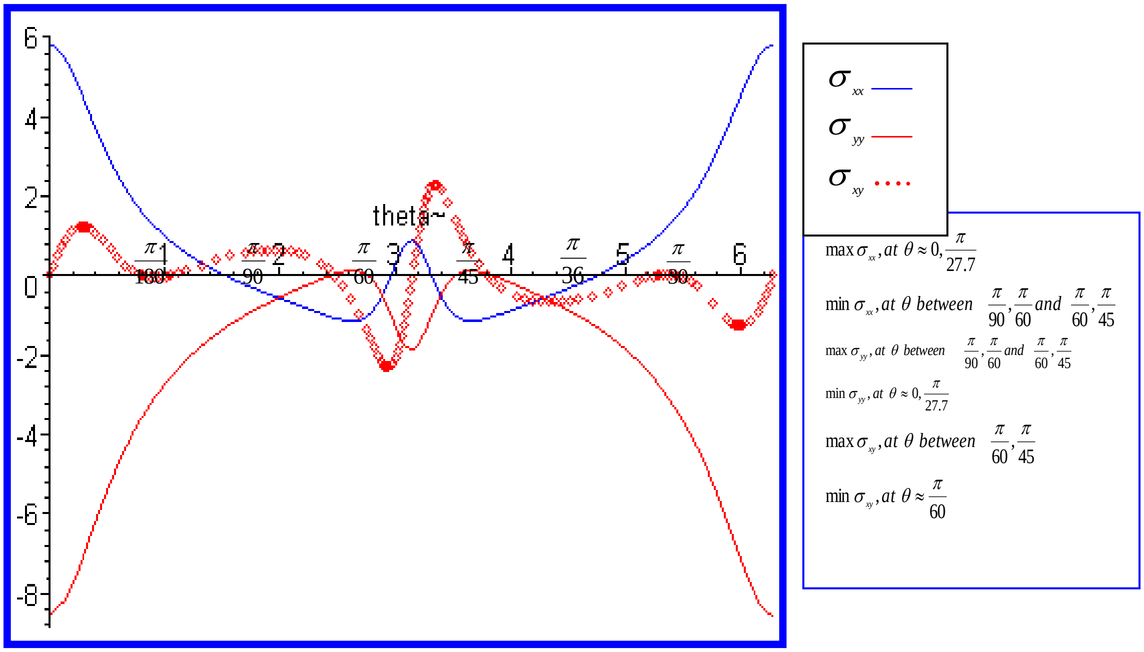
Then, the functions in (2.9) and (2.12) become

$$\phi(\zeta) = -\frac{cp(n^2+m)}{(1-n\zeta)}\left[1-\frac{hn^2}{(1-n^2)^2(1-h\upsilon)}\right], \tag{4.3}$$

and

$$\psi(\zeta) = \frac{-cp(1+mn^2)}{n(1-n^2)(1-n\zeta)} - \frac{cp(1+mn^2)}{(1-n^2)} - \frac{w(\zeta^{-1})}{w'(\zeta)}\phi'(\zeta) + \frac{h\zeta}{(\zeta-n)}\phi'(n). \tag{4.4}$$

Thus (4.3), (4.4) give the solution of the first fundamental problem when the edge of the hole is subject to a uniform pressure p .The component of stresses , in this case , are computed in Fig.(13).



Fig(13):

(iii) For $k=\chi,\ \Gamma=\Gamma^*=f=0$,see Fig.(14), n:=0.25;m:=3; c:=2;x:=0.25; X:=2; Y:=2;

Then, the functions in (2.9) and (2.10) become

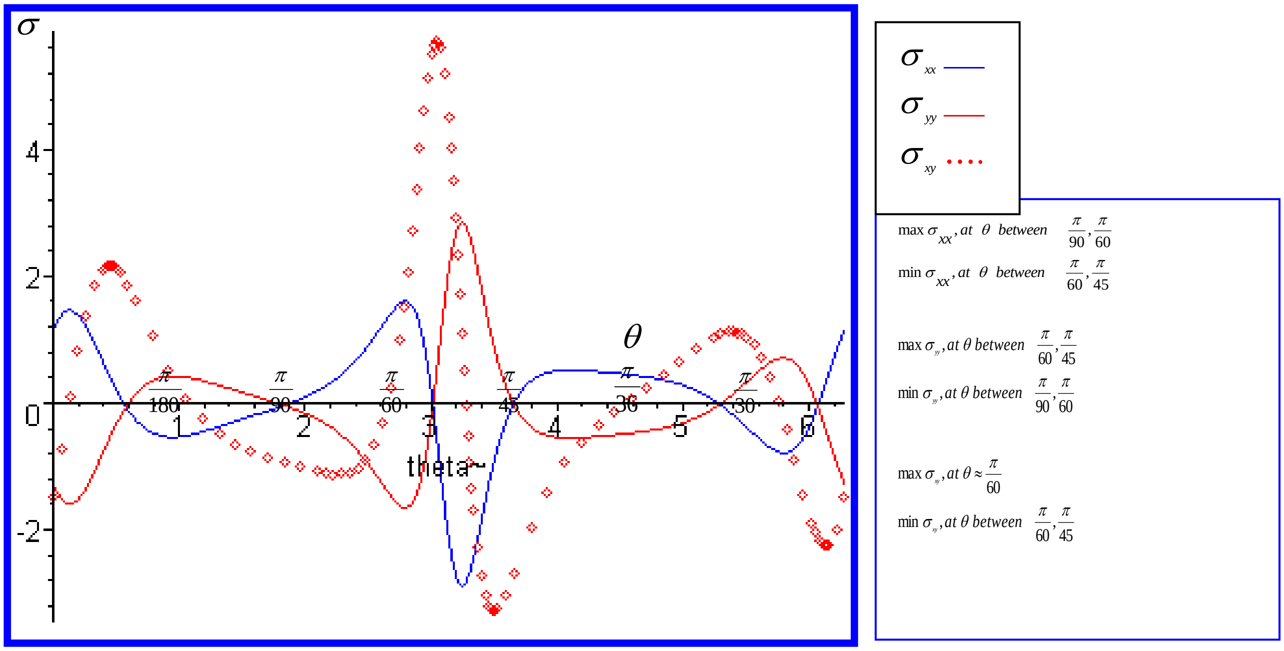
$$\chi\phi(\zeta) = \frac{hn}{2\pi(1-n\zeta)(1+\chi)}\left[\frac{h\upsilon\chi}{(\chi^2-h^2\upsilon^2)}(X+iY)-(X-iY)\left(1+\frac{h^2\upsilon^2}{c(\chi^2-h^2\upsilon^2)}\right)\right], \tag{4.5}$$

$$\psi(\zeta) = \frac{-w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta) + \frac{h\zeta}{\zeta-n}\phi_*(n), \tag{4.6}$$

where

$$\phi_*(\zeta) = \phi'(\zeta) - \frac{(X+iY)}{2\pi(1+\chi)\zeta}.$$

Therefore , we have the solution of the second fundamental problem of the case when a force (X,Y) acts on the centre of the curvilinear kernel . The components of stresses are computed , in this case , in Fig.(14).



Fig(14):

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