FIXED POINT THEOREM FOR SETVALUED MAPPINGS SATISFYING AN IMPLICIT RELATION IN INTUITIONISTIC FUZZY METRIC SPACE

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ABSTRACT
In this paper, we give a common fixed point theorem for setvalued mappings with intuitionistic fuzzy metric space.

Key words:- *Intuitionistic fuzzy metric space, Implicit relation, compatible mapping and noncompatible mapping*.
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1. INTRODUCTION

Throughout this paper, (X, M, N, *, ◊) stands for a intuitionistic fuzzy metric space with the induced metric M, N whereas CB(X) denotes the family of all nonempty closed bounded subsets of X. Let

$$H_M(A, B, t) \geq \min\{\inf_{x\in A}\sup_{y\in B} M(x, y, t), \inf_{y\in B}\sup_{x\in A} M(x, y, t)\}$$

$$H_N(A,B,t) \leq \max\{\sup_{x\in A}\inf_{y\in B} N(x, y, t), \sup_{y\in B}\inf_{x\in A} N(x, y, t)\}$$

where A, B $\in$ CB(X).

It is well known that if (X, M, N, *, ◊) is a complete intuitionistic fuzzy metric space, then so is the intuitionistic fuzzy metric space (CB(X), H_M, H_N, *, ◊).In 1999 Popa proved theorem for weakly compatible non-continuous mappings using implicit relation. It was extended by Imdad ([2], 2002) using coincidence commuting property. The main object of this paper is to obtain some common fixed point theorems in intuitionistic fuzzy metric space using "Implict Relaion". Our result differs from all above authors in the following ways:
(1) We have taken three self maps.
(2) Weak** commuting property is used.
(3) Relaxing the continuity requirement completely.

1.1.1 Definition . The mappings f : X → X and S : X →CB(X) are compatible if fSx $\in$ CB(X) for all x $\in$ X and

$$\lim_{n\to\infty} M(Sf x_n, f Sx_n, t) = 1, \quad \lim_{n\to\infty} N(S f x_n, f Sx_n, t) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = A \in CB(X)$ and $\lim_{n\to\infty} f x_n = t \in A$.

Now, we consider the following conditions.

1.1.1 Condition . The mappings f : X →X and S : X →CB(X) are said to be satisfy Condition 1.1.1 iff fSx $\in$ CB(X) for all x $\in$ X and

$$\lim_{n\to\infty} M(Sfx_n, fSx_n, t) \geq \lim_{n\to\infty} M(Sfx_n, Sx_n, \frac{t}{2}) * M(Sx_n, fSx_n, \frac{t}{2})$$

$$\lim_{n\to\infty} N(Sfx_n, fSx_n, t) \leq \lim_{n\to\infty} N(S f x_n, Sx_n, \frac{t}{2}) \Diamond N(Sx_n, fSx_n, \frac{t}{2})$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = A \in CB(X)$ and $\lim_{n\to\infty} f x_n = t \in A$.

1.1.2 Condition . The mappings $f : X \to X$ and $S : X \to CB(X)$ are said to be satisfy Condition 2 iff for all $x \in X$ and

$$\lim_{n\to\infty} M(ffx_n, f x_n, t) \geq \lim_{n\to\infty} M(Sf x_n, Sx_n, t)$$

$$\lim_{n\to\infty} N(f f x_n, f x_n, t) \leq \lim_{n\to\infty} M(Sf x_n, Sx_n, t)$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = A \in CB(X) \text{ and } f x_n = t \in A.$$

1.1 Lemma . Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space and for all $x, y \in X$, $t > 0$ and if for a number $k \in (0, 1)$,

$$M(x, y, kt) \geq M(x, y, t) \text{ and } N(x, y, kt) \leq N(x, y, t) \text{ then } x = y.$$

Implicit relation

Implicit relations on intuitionistic fuzzy metric spaces

Let F be the set of all continuous functions $F(R^+)^6 \to R^+$ which is a non-decreasing and non-increasing in the argument satisfying the following conditions:

F_1 : $F(t_1, \ldots, t_6)$ is non decreasing in $t_2, \ldots, t_6$.

F_a : For $u, v \geq 0$, $F(u, v, u, v, 1) \geq 0$ implies that $u \geq v$.

F_b : $F(u, 1, 1, u, 1) \geq 0$ or $F(u, u, 1, 1, u) \geq 0$ or $F(u, 1, u, 1, u) \geq 0$ implies $u \leq 1$.

F_1 : $F(t_1, \ldots, t_6)$ is non -increasing in $t_2, \ldots, t_6$.

For $u, v \geq 0$, $F(u, v, u, v, 0) \leq 0$ implies that $u \leq v$.

$F(u, 0, 0, u, 0) \leq 0$ or $F(u, u, 0, 0, u) \leq 0$ or $F(u, 0, u, 0, u) \leq 0$ implies that $u \leq 1$.

1.1 Example .: Define $F(t_1, t_2, t_3, t_4, t_5) = 13t_1 - 11t_2 - 3t_3 - 5t_4 + t_5 - 1$,

then from (F a) $F(u, v, v, u, 1) \in 0$ or $F(u, v, u, v, 1) \geq 0$, $\Rightarrow u \geq v$.

and from (F b) $F(u, u, 1, 1, u) \in 0$, or $F(u, 1, 1, u, 1) \geq 0$, $\Rightarrow u \leq 1$.

1.2 Example .:

1. $F((t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6)$
2. $F((t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - \min\{t_i, t_j : i, j \in \{2, 3, 4, 5, 6\}$
3. $F((t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - \min\{t_i, t_j, t_k : i, j, k \in \{2, 3, 4, 5, 6\}$

1.3 Example .:

1. $F((t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\{t_2, t_3, t_4, t_5, t_6)$
2. $F((t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - \max\{t_i, t_j : i, j \in \{2, 3, 4, 5, 6\}$
3. $F((t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - \max\{t_i, t_j, t_k : i, j, k \in \{2, 3, 4, 5, 6\}$

2. MAIN RESULT

2.1 Theorem .: Let (X, M, N, *, ◊) be a complete intuitionistic fuzzy metric space. Consider $f, g: X \to X$ and $S, T : X \to CB(X)$,Where CB(X) the family of all closed bounded subsets of X be a continuous mapping such that f and S as well as g and T satisfy Condition 1 and Condition 2 Assume $T(X) \subseteq f(X)$, $S(X) \subseteq g(X)$ and for all $x, y \subseteq X$

$$F(M(Sx, Ty, kt), M(fx, gy, t), M(fx, Sx, t), M(gy, Ty, 2t), M(fx, Ty, 2t), M(gy, Sx, t)) \geq 0, \quad (2.1)$$

$$F(N(Sx, Ty, kt), N(fx, gy, t), N(fx, Sx, t), N(gy, Ty, 2t), N(fx, Ty, 2t), N(gy, Sx, t)) \leq 0, \quad (2.2)$$

where $F \in \mathcal{F}$. Then f, g, S and T have a common fixed point.

Proof. Let x_0 be arbitrary point in X. We shall construct two sequences $\{x_n\}$ and $\{y_n\}$ of elements in X and sequence $\{A_n\}$ of elements in CB(X).Since $S(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $y_1 = g x_1 \in Sx_0$. Then there exists an element $y_2 = fx_2 \in T x_1 = A_1$, because $T(X) \subseteq f(X)$, such that

$$M(y_1, y_2, kt)=M(gx_1, fx_2, kt) \geq M(Sx_0, Tx_1, kt)$$
$$\geq M(y_0, y_1, t) \text{ and}$$
$$N(y_1, y_2, kt) = N(gx_1, fx_2, kt) \leq N(Sx_0, Tx_1, kt)$$
$$\leq N(y_0, y_1, t)$$

Since $S(X) \subseteq g(X)$, we may choose $x_3 \in X$ such that $y_3 = g x_3 \in Sx_2 = A_2$

$$M(y_2, y_3, kt) \geq M(Tx_1, Sx_2, kt)$$
$$\geq M(y_1, y_2, t) \text{ and}$$
$$N(y_2, y_3, kt) \leq N(Tx_1, Sx_2, t)$$
$$\leq M(y_1, y_2, t)$$

and by induction we produce the sequences $\{x_n\}$, $\{y_n\}$ and $\{A_n\}$ such that

$$y_{2n+1} = g x_{2n+1} \in S_{2n} = A_{2n}, \quad (2.3)$$
$$y_{2n+2} = f x_{2n+2} \in T x_{2n+1} = A_{2n+1}, \quad (2.4)$$
$$M(y_{2n+1}, y_{2n}, kt) \geq kM(S_{2n}, T x_{2n-1}, kt)$$
$$\geq M(y_{2n}, y_{2n-1}, t) \quad (2.5)$$
$$N(y_{2n+1}, y_{2n}, kt) \leq N(S_{2n}, T x_{2n-1}, kt)$$
$$\leq N(y_{2n}, y_{2n-1}, t) \quad (2.6)$$
$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(S_{2n}, T x_{2n+1}, kt)$$
$$\geq M(y_{2n}, y_{2n+1}, t) \quad (2.7)$$
$$N(y_{2n+1}, y_{2n+2}, kt) \leq N(S_{2n}, T x_{2n+1}, kt)$$
$$\leq N(y_{2n}, y_{2n+1}, t) \quad (2.8)$$

for every $n \in N$. Letting $x = x_{2n}$, $y = x_{2n+1}$ in (2.1) and (2.2), we have successively

$$F\{H_M(Sx_{2n},T x_{2n+1}, kt), M(f x_{2n}, g x_{2n+1}, t), M(f x_{2n}, Sx_{2n}, t),$$
$$M(g x_{2n+1},T x_{2n+1}, 2t), M(f x_{2n},T x_{2n+1}, 2t) M(g x_{2n+1},Sx_{2n}, t)\} \geq 0$$
$$F\{H_N(Sx_{2n},T x_{2n+1}, kt), N(f x_{2n}, g x_{2n+1}, t), N(f x_{2n},Sx_{2n}, t),$$
$$N(g x_{2n+1},T x_{2n+1}, 2t), N(f x_{2n},T x_{2n+1}, 2t), N(g x_{2n+1},Sx_{2n}, t)\} \leq 0$$

and so

$$F\{H_M(Sx_{2n},T x_{2n+1}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1},y_{2n+2}, 2t),$$
$$M(y_{2n},y_{2n+2}, 2t), M(y_{2n+1},y_{2n+1}, t)\} \geq 0$$
$$F\{H_M(Sx_{2n},T x_{2n+1}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1},y_{2n+2}, 2t), \quad M(y_{2n},y_{2n+2}, 2t), 1\} \geq 0.$$
$$F\{M(y_{2n+1},y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n},y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t), 1\} \geq 0. \quad (2.9)$$
$$F\{H_N(Sx_{2n},T x_{2n+1}, kt), N(y_{2n}, y_{2n+1}, t), N(y_{2n}, y_{2n+1}, t), N(y_{2n+1},y_{2n+2}, 2t),$$
$$N(y_{2n},y_{2n+2}, 2t), N(y_{2n+1},y_{2n+1}, t)\} \leq 0$$
$$F\{H_N(Sx_{2n},T x_{2n+1}, kt), N(y_{2n}, y_{2n+1}, t), N(y_{2n}, y_{2n+1}, t), N(y_{2n+1},y_{2n+2}, 2t), \quad N(y_{2n},y_{2n+2}, t), 0\} \leq 0.$$
$$F\{N(y_{2n+1},y_{2n+2}, kt), N(y_{2n}, y_{2n+1}, t), N(y_{2n+1}, y_{2n+2}, t), N(y_{2n},y_{2n+1}, t), N(y_{2n+1}, y_{2n+2}, t), 0\} \leq 0. \quad (2.10)$$

Using Lemma 1.1 we have

$$M(y_{2n+2}, y_{2n+1}, kt) \geq M(y_{2n+1}, y_{2n}, t) \quad (2.11)$$

Similarly

$$N(y_{2n+2}, y_{2n+1}, kt) \leq N(y_{2n+1}, y_{2n}, t) \quad (2.12)$$

Thus for any n and t we have

$$M(y_{n+1}, y_n, kt) \geq M(y_n, y_{n-1}, t)$$
$$N(y_{n+1}, y_n, kt) \leq N(y_n, y_{n-1}, t)$$

We shall prove that {yn} is a Cauchy sequence

M(yn+1, yn, t) ≥ M(yn, yn-1, t/k) ≥ M(yn-1, yn-2, t/k²) ≥ - - - ≥ M(y1, yo, t/kn) →1 as n → ∞.

Thus the result holds for m=1.

$$M(y_{n+1}, y_n, t) \geq M(x_n, x_{n-1}, \frac{t}{q}) \geq M(x_{n-2}, x_{n-1}, \frac{t}{q^2})$$
$$\geq \cdots \geq M(x_1, x_2, \frac{t}{q^n}) \to 1$$

as $n \to \infty$

So, $M(x_n, x_{n+1}, t) \to 1$ as $n \to \infty$ for any $t > 0$.
By induction hypothesis suppose that the result holds for m = r.
Now $M(y_n, y_{n+r+1}, t) \geq M(y_n, y_{n+r}, t/2) * M(y_{n+1}, y_{n+r+1}, t/2) \to 1*1 = 1$
Thus the result holds for m=r+1

$$N(x_{n+1}, x_n, t) \leq N(x_n, x_{n-1}, \frac{t}{q}) \leq N(x_{n-1}, x_{n-2}, \frac{t}{q^2})$$

$$\leq \cdots \leq N(x_1, x_2, \frac{t}{q^n}) \to 0 \text{ as } n \to \infty.$$

So, $N(x_{n+1}, x_n, t) \to 0$ as $n \to \infty$. for any $t > 0$.
By induction hypothesis suppose that the result holds for m = r.
Now $N(y_n, y_{n+r+1}, t) \leq N(y_n, y_{n+r}, t/2) \lozenge N(y_{n+1}, y_{n+r+1}, t/2) \to 0\lozenge 0 = 0$
Thus the result holds for m=r+1

Hence $\{y_n\}$ is a Cauchy sequence in X which is complete. Therefore $\{y_n\}$ converges to $z \in X$ such that $y_n \to z$. Therefore, $g\, x_{2n+1} \to z$ and $f\, x_{2n} \to z$. Also from (2.11) and (2.12) and the fact that $\{y_n\}$ is Cauchy sequence it follows that $\{A_k\}$ is Cauchy sequence in the complete metric space $(CB(X), H_M, H_N)$. Thus $A_k \to A \in CB(X)$. This implies $Tx_{2n+1} \to A$ and $Sx_{2n} \to A$ and therefore $z \in A$, because

$$M(z, A, t) = \lim_{n\to\infty} M(y_n, A, t) \geq \lim_{n\to\infty} H_M(A_{n-1}, A_n, t) = 1,$$

$$N(z, A, t) = \lim_{n\to\infty} N(y_n, A, t) \leq \lim_{n\to\infty} H_N(A_{n-1}, A_n, t) = 0.$$

Since A is closed, A and f and S are satisfying Condition 1.1.1 and Condition 1.1.2 implies that

$$\lim_{n\to\infty} H_M(fSx_{2n}, S\,f\,x_{2n}) \geq \lim_{n\to\infty} H_M(Sx_{2n}, S\,f\,x_{2n}, \frac{t}{2}) * \lim_{n\to\infty} H_M(Sx_{2n}, f\,Sx_{2n}, \frac{t}{2})$$

$$\lim_{n\to\infty} H_N(f\,Sx_{2n}, S\,f\,x_{2n}) \leq \lim_{n\to\infty} H_N(Sx_{2n}, S\,f\,x_{2n}, \frac{t}{2}) \lozenge \lim_{n\to\infty} H_N(Sx_{2n}, f\,Sx_{2n}, \frac{t}{2})$$

and

$$\lim_{n\to\infty} M(f\,f\,x_{2n}, f\,x_{2n}, t) \geq H_M \lim_{n\to\infty} (Sx_{2n}, S\,f\,x_{2n}, \frac{t}{2}),$$

$$\lim_{n\to\infty} N(f\,f\,x_{2n}, f\,x_{2n}, t) \leq H_N \lim_{n\to\infty} N(Sx_{2n}, S\,f\,x_{2n}, \frac{t}{2})$$

This along with the continuity of f and S imply that

$$H_M(fA, Sz, t) \geq M(A, Sz, \frac{t}{2}), \tag{2.13}$$

$$H_N(f\,A, Sz, t) \leq N(A, Sz, \frac{t}{2}) \tag{2.14}$$

And
$$M(f\,z, z, t) \geq H_M(A, Sz, t) \tag{2.15}$$
$$N(f\,z, z, t) \leq H_N(A, Sz, t) \tag{2.16}$$

Now

$$M(f\,z, Sz, t) \geq M(f\,z, f\,g\,x_{2n+1}, \frac{t}{2}) * M(f\,g\,x_{2n+1}, Sz, \frac{t}{2})$$

$$\geq M(f\,z, f\,g\,x_{2n+1}, \frac{t}{2}) * H_M(f\,Sx_{2n}, Sz, \frac{t}{2})$$

$$\geq M(f\,z, f\,g\,x_{2n+1}, \frac{t}{2}) * H_M(f\,Sx_{2n}, S\,f\,x_{2n}, \frac{t}{4}) * H_M(S\,f\,x_{2n}, Sz, \frac{t}{4})$$

$$N(f\,z, Sz, t) \leq N(f\,z, f\,g\,x_{2n+1}, \frac{t}{2}) \lozenge N(f\,g\,x_{2n+1}, Sz, \frac{t}{2})$$

$$\leq N(f\,z, f\,g\,x_{2n+1}, \frac{t}{2}) \lozenge H_N(f\,Sx_{2n}, Sz, \frac{t}{2})$$

$$\leq N(f\,z, f\,g\,x_{2n+1}, \frac{t}{2}) \lozenge H_N(f\,Sx_{2n}, S\,f\,x_{2n}, \frac{t}{4}) \lozenge H_N(S\,f\,x_{2n}, Sz, \frac{t}{4})$$

and letting $n \to \infty$ we have

$$M(f z, Sz, t) \geq \lim_{n\to\infty} H_M(f Sx_{2n},S f x_{2n}, \frac{t}{4}) \geq H_M (A, Sz, \frac{t}{4}).$$

$$N(f z, Sz, t) \leq \lim_{n\to\infty} N(f Sx_{2n},S f x_{2n}, \frac{t}{4}) \leq H_N(A, Sz, \frac{t}{4}).$$

Now using (2.1) we have

$F(H_M (Sz, T x_{2n+1}, kt),M(f z, g x_{2n+1}, t),M(f z,Sz, t),M(g x_{2n+1},T x_{2n+1}, 2t),$
$M(f z, T x_{2n+1}, 2t), M(g x_{2n+1},Sz, t)) \geq 0$

$F(H_M (Sz, T x_{2n+1},k t),M(f z, g x_{2n+1}, t),M(f z, Sz, t),M(g x_{2n+1},T x_{2n+1}, t),$
$M(f z, fx_{2n+2}, t) *M(fx_{2n+2}, T x_{2n+1}, t), M(g x_{2n+1},Sz, t)) \geq 0$ (2.15)

Now using (2.2) we have

$F(H_N(Sz, T x_{2n+1}, kt),N(f z, g x_{2n+1}, t),N(f z, Sz, t),N(g x_{2n+1},T x_{2n+1}, 2t),$
$N(f z, T x_{2n+1}, 2t), N(g x_{2n+1},Sz, t)) \leq 0$

$F(H_N(Sz, T x_{2n+1}, kt), N(f z, g x_{2n+1}, t),N(f z, Sz, t), N(g x_{2n+1},T x_{2n+1}, t),$
$N(f z, fx_{2n+2},t)\Diamond N(fx_{2n+2}, T x_{2n+1}, t) , N(g x_{2n+1},Sz, t)) \leq 0$ (2.17)

Letting $n\to\infty$ in(2.15) and (2.16)we obtain

$F(M(Sz, A, kt),M(f z, z, t),M(f z, Sz, t),M(z, A, t),$
$M(f z, z, t)*M(z, A, t),M(z, Sz, t)) \geq 0$ (2.18)

$F(N(Sz, A, t),N(f z, z, t),N(f z, Sz, t),N(z, A, t),$
$N(f z, z, t)\Diamond N(z, A, t),N(z, Sz, t)) \leq 0$ (2.19)

and so

$F(M(Sz, A, t), M(Sz, A, t), M(Sz, A, t), 1, M(Sz, A, t)*1, M(Sz, A, t)) \geq 0$ $F(N(Sz, A ,t), N(Sz, A, t), N(Sz, A, t), 0 , N(Sz, A, t) \Diamond 0, N(Sz, A, t)) \leq 0$

which is a contradiction to F_a and F_b. Thus $M(Sz, A, t) = 1$, $N(Sz, A, t) = 0$ and so, from (2.13), (2.14), (2.15) and (2.16) we have $z = f z \in Sz$. Thus we say

$M(Sz, A, kt) \geq M(Sz, A, t)$ and $N(Sz, A, kt) \leq N(Sz, A, t)$. Similarly we have $z = g z \in Tz$. Thus z is a common fixed point of this four mappings.

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