NUMERICAL SOLUTION OF FREDHOLM-VOLTERRA INTEGRAL EQUATION IN TWO DIMENSONAL SPACE BY USING DISCRETE ADOMIAN DECOMPOSITION METHOD

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ABSTRACT
In this paper, discrete Adomian decomposition method (DADM) use to solve two dimensional Fredholm-Volterra integral equations. This method arises when the quadrature rules are used to approximate the integrals which cannot be computed analytically. Finally, some concrete examples are given to illustrate the validity of the method.

Keywords: *Two dimensional Fredholm-Volterra integral equations; Quadrature rule; Adomian decomposition method.*

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1. INTRODUCTION

Many problems of mathematical physics, engineering and contact problems in the theory of elasticity lead to nonlinear integral equations (see[1, 2, 3]). Their solution can obtained analytically, using the theory developed by Mushkelishvili [4]. The books edited by Green [5], Tricomi [6] and Hochstadt [7] contain many different methods to solve the integral equations analytically. Since closed form solution of a large class of nonlinear integral equations is generally not available, much attention has been focused on numerical treatment. The book edited by Golberg [8] contain extensive literature survey on both approximate analytically and purely numerical techniques. The interested reader should consult the fine expositions by Atkinson [9] and Delves and Mohamed [10] for numerical methods. Several numerical methods for approximating the solution of linear and nonlinear integral equations and specially Hammerstein integral equations are known [11, 12, 13]. The classical method of successive approximation for Fredholm- Hammerstein type integral equations was introduced in [6]. Brunner in [14] applied a collocation type method and Ordokhani in [12] applied rationalized Haar function to nonlinear Volterra-Fredholm Hammerstein integral equations. A collocation type method was developed in [11]. Yousefi and Razzaghi in [15] applied Legendre Wavelets to special type of nonlinear Fredholm - Volterra integral equations of the form

$$x(s) = y(s) + \lambda_1 \int_0^s k_1(s,t)\, F(x(t))dt + \lambda_2 \int_0^1 k_2(s,t)\, G(x(t))dt \qquad ,0 \le s,t \le 1 \tag{1}$$

where $y(s)$, $k_1(s,t)$ *and* $k_2(s,t)$ are assumed to be in $L^2(R)$ on the interval $0 \le s,t \le 1$.

Yalcinbas in [16] used Taylor polynomials for solving Eq. (1) with $F(u) = u^p$ and $G(u) = u^q$. In [17], Abdou using orthogonal polynomial method of type Legendre polynomial to solve Fredholm – Volterra integral equation. Also, Yusufoglu and Erbas presented the method based on interpolation in solving linear Volterra-Fredholm integral equations [18]. In [19] Maleknejad and Sohrabi presented a method to solve nonlinear Volterra-Fredholm Hammerestein integral equations in terms of Legendre polynomials.

The topic of Adomian decomposition method (ADM), introduced by Adomian [20, 21], has been rapidly growing in recent years. ADM possesses a great potential in solving different kind of nonlinear functional equations. Application of Adomian decomposition method to different types of integral equations was discussed by many authors for example [22, 23]. El-Kalla in [24] consider the two dimensional nonlinear Fredholm-Volterra integral equations of the form

$$u(x,t) = f(x,t) + \lambda_1 \int_0^t k_1(t,\tau)\, f_1(u(x,\tau))d\tau + \lambda_2 \int_a^b k_2(x,s)\, f_2(u(t,s))ds \tag{2}$$

and he introduced new formula to estimate the error for series solution of equation (2). In this paper we consider the two dimensional nonlinear Fredhom – Volterra integral equation(2). We assume $f(x,t)$ is bounded $\forall\ x \in [a,b]$, $t \in [0,T]$, the kernel of Volterra term is bounded such that

$|k_1(t,\tau)| \le M_1 \;\; \forall\; 0 \le \tau \le t \le T < \infty$ and the kernel of the Fredholm term is bounded such that $|k_2(x,s)| \le M_2 \;\; \forall\; a \le x \le s \le b.$ The nonlinear terms $f_1(u)$ *and* $f_2(u)$ are Lipschitzian with $|f_1(u) - f_1(z)| \le L_1|u-z|$ and $|f_2(u) - f_2(w)| \le L_2|u-w|$. We apply a discrete version of the Adomian decomposition method to solve the equation(2). Discrete Adomian decomposition method introduced by Behiry *et al.* in [25] and applied to nonlinear Fredholm integral equations. This method arises when the quadrature rules are used to approximate the definite integrals which cannot be computed analytically. The discrete Adomian decomposition method (DADM) gives the numerical solution at noodes used in the quadrature rules.

The structure of this paper is as follows:

In section 2 we solve equation (2) by standard Adomian decomposition method. In section 3 the existence and uniqueness of the solution of equation (2)are assumed. In section 4 we solve the two-dimensional nonlinear Fredholm-Volterra integral equation by discrete Adomian decomposition method. Then in section 5 we apply the proposed method in some examples, showing the accuracy and efficiency of the method.

2. ADOMIAN DECOMPOSITION METHOD

To set up the Adomian method, consider $u(x,t)$ in the series form

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) \tag{3}$$

where the components $u_i(x,t)$ will be determined recurrently. The method defines the nonlinear functions $f_1(u)$ *and* $f_2(u)$ as the series

$$f_1[u] = \sum_{n=0}^{\infty} A_n[u_0(t),...,n_n(t)]$$

$$f_2[u] = \sum_{n=0}^{\infty} B_n[u_0(t),...,n_n(t)] \tag{4}$$

where the traditional formulas of A_n and B_n are given by

$$A_n[u_0(t),...,u_m(t)] = \frac{1}{n!}\,\frac{d^n}{d\alpha^n}\left[f_1\left\{\sum_{i=0}^{\infty}\alpha^i \; u_i\right\}\right]_{\alpha=0}$$

$$B_n[u_0(t),...,u_m(t)] = \frac{1}{n!}\,\frac{d^n}{d\alpha^n}\left[f_2\left\{\sum_{i=0}^{\infty}\alpha^i \; u_i\right\}\right]_{\alpha=0} \tag{5}$$

where α is a formal parameter.

Substituting equation (3) and (4) into (2) to get

$$\sum_{i=0}^{\infty} u_i(x,t) = f(x,t) + \lambda_1\int_0^t k_1(t,\tau)\sum_{i=0}^{\infty} A_i \; d\tau + \lambda_2\int_a^b k_2(x,s)\sum_{i=0}^{\infty} B_i \; ds \;\; , i \ge 0$$

The components $u_i(x,t),\; i \ge 0$ are computed using the following recursive relations

$$u_0(t) = f(x,t) \tag{6}$$

$$u_{i+1}(x,t) = \lambda_1\int_0^t k_1(t,\tau)\,A_i \; d\tau + \lambda_2\int_a^b k_2(x,s)\,B_i \; ds \;\; , i \ge 0 \tag{7}$$

It is noticed that the computation of each component $u_i(x,t),\; i \ge 0$ requires the computation of integrals in equation(7). If the evaluation of integrals analytically is possible, the ADM can be applied in a simple manner.

3. THE EXISTENCE AND UNIQUENESS OF THE SOLUTION

The convergence analysis of the Adomian series solution was studied for different problems and by many authors. Cherruault in [26] proved the convergence of the Adomian method for differential and operator equations and to specific type of equations in [27]. In the convergence analysis, Adomian's polynomials play a very important role, El-Kalla in [27] suggested an alternative approach for proving the convergence. This approach depends mainly on the accelerated Adomian polynomial. The formula of El-Kalla [28] is used directly to prove the convergence of the series solution to a class of nonlinear two dimensional integral equations and he proved the following theorems:

Uniqueness Theorem (1)

Theorem (1)

Problem (2) has a unique solution whenever

$$0<\alpha<1 \ , \ \alpha=\alpha_1+\alpha_2 \ , \ \alpha_1=|\lambda_1| L_1 \ M_1 \ T \ , \ \alpha_2=|\lambda_2| L_2 \ M_2 \ (b-a)$$

Convergence Theorem (2)

Theorem (2)

The series solution (3) of problem (2) using Adomian decomposition method converges if $0<\alpha<1$ and $f(x,t)$ bounded on the interval $J=[a,b]\times[0,T]$.

Error Estimate (3)

Theorem (3)

The maximum absolute truncation error of the series solution (3) to problem (2) is estimated to be

$$\max_{\forall x,t\in J}\left|u(x,t)-\sum_{i=0}^{m}u_i(x,t)\right|\le\frac{\alpha^m}{1-\alpha}\left(\frac{\varphi_1\,\alpha_1}{L_1}+\frac{\varphi_2\,\alpha_2}{L_2}\right) \quad (8).$$

where

$$\varphi_1=\max_{\forall x,t\in J}|f_1(u_0)| \quad and \quad \varphi_2=\max_{\forall x,t\in J}|f_2(u_0)|$$

4. DISCRETE ADOMIAN DECOMPOSITION METHOD

In the cases where the evaluation of integrals in (7) is analytically impossible, the ADM cannot be applied, so we consider any numerical integration scheme be given by the following formula

$$\int_a^b g(s)\ ds\approx\sum_{j=0}^{n}w_{n,j}\,g(s_{n,j})$$

where $g(s)$ is continuous function $s_{n,j}=a+jh$ are the nodes of the quadrature rule, $h=\frac{b-a}{n}$ and $w_{n,j}$, $n=0,1,2,3,...$ are the weights functions.

In order to use the quadrature rule for equation (7), let $\tau=t\quad v$, we get

$$u_{i+1}(x,t)=\lambda_1\ t\int_0^1 k_1(t,t\,v)\,A_i\ dv+\lambda_2\int_a^b k_2(x,s)\,B_i\ \ ds\ \ ,\ i\ge 0 \quad (9)$$

Now by applying quadrature integration formula, equation(9) may be approximated as

$$u_{i+1}(x,t_i)=\lambda_1\sum_{r=0}^{n}w_{1r}\ \ k_1(t,v_i)\,A_{i,r}\ +\lambda_2\sum_{r=0}^{n}w_{2r}\ \ k_2(x,s_i)\,B_{i,r}\qquad ,i\ge 0 \quad (10)$$

The approximate solution of equation(2) using DADM can be obtained by summing the approximate values to the components $u_i(x,t),\ i\ge 0$ represented by equations (6) and (10) at the nodes $s_{n,i}$, $i=0,1,2,3,...,n$.

The solution $u_i(x_{n,i},t_{n,i}),\ i\ge 0$ at these nodes using DADM of equation (2) can be written as

$$u(x_{n,i},t_{n,i})=\sum_{i=0}^{\infty}u_i(x_{n,i},t_{n,i}) \tag{11}$$

In practice all the terms of the series in (11) cannot be determined and the solution will be approximated by series of the form

$$\varphi(x_{n,i},t_{n,i})=\sum_{i=0}^{n}u_i(x_{n,i},t_{n,i})$$

5. ILLUSTRATIVE EXAMPLES

In this section, we applied presented method in this paper for solving integral equation (2) and solved some examples. In these examples we used Simpson's rule to approximate the integrals. The computations associated with the example were performed using Mathematica 7.0 software.

Example 1

$$u(x,t)=f(x,t)-\int_0^t k_1(t,\tau)\;u(x,\tau)d\tau-\int_0^1 k_2(x,s)\;u(t,s)ds \qquad ,\ 0\le x,t\le 1$$

where

$$k_1(t,\tau)=-4t^2+\tau t \quad , \quad k_2(x,s)=5x+2s^2-5 \quad ,$$

$$f(x,t)=-10+xt-2t+\tfrac{5}{2}x(6+t)-\tfrac{1}{6}t^3(36+10xt)$$

with the exact solution $u(x,t)=xt+3$

Table (1) shows the computed error $|e_m^n|=|u_{exact}-u_{appr.}|$, where m is the number of components $u_0,u_1,,...,u_m$, and n is the number of the nodes of the quadrature rule.

t	x	$\lvert e_5^8\rvert$
0.125	0.00	1.17911E-4
	0.125	1.31059E-3
	0.250	6.8627E-3
	0.375	3.17028E-2
	0.500	1.27255E-1
0.250	0.00	1.111361E-4
	0.125	1.23778E-3
	0.250	6.50074E-3
	0.375	2.82015E-2
	0.500	1.13418E-1
0.375	0.00	1.055E-4
	0.125	1.117264E-3
	0.250	6.15905E-3
	0.375	2.67301E-2
	0.500	1.07631E-1
0.500	0.00	1.0022E-4
	0.125	1.11140E-3
	0.250	5.85161E-3
	0.375	2.5479E-2
	0.500	1.02453E-1

Table(1)

Example 2

$$u(x,t) = f(x,t) + \frac{1}{10}\int_0^t (t-\tau)\ u^2(x,\tau)d\tau + \frac{1}{2}\int_0^1 (x^2 - s)\ u^2(t,s)ds$$

$$f(x,t) = \frac{t}{120}[t(1-2x^2) + 120x(1-x) - t^3x^2(1-x)^2]$$

With the exact solution $u(x,t) = xt(1-x)$.

Table (2) shows the computed error $|e_m^n| = |u_{exact} - u_{appr.}|$, where m is the number of components $u_0, u_1, ,..., u_m$, and n is the nodes of the quadrature rule.

t	x	$\lvert e_8 \rvert$	t	x	$\lvert e_8 \rvert$
0.125	0.00	0.0	0.625	0.00	0.0
	0.125	1.22932E-7		0.125	2.79622E-8
	0.250	4.90712E-7		0.250	1.12427E-7
	0.375	1.10179E-6		0.375	2.54232E-7
	0.500	1.9546E-6		0.500	4.54145E-7
	0.625	3.04722E-6		0.625	7.12787E-7
	0.750	4.37711E-6		0.750	1.03051E-6
	0.875	5.95338E-6		0.875	1.40721E-6
	1.00	7.74075E-6		1.00	1.84213E-6
0.250	0.00	0.0	0.750	0.00	0.0
	0.125	1.11061E-7		0.125	1.55655E-8
	0.250	4.43431E-7		0.250	6.09579E-8
	0.375	9.95886E-7		0.375	1.34263E-7
	0.500	1.76717E-6		0.500	2.33668E-7
	0.625	2.75592E-6		0.625	3.57545E-7
	0.750	3.96055E-6		0.750	5.04573E-7
	0.875	5.37903E-6		0.875	6.73899E-7
	1.00	7.00861E-6		1.00	8.6534E-7
0.375	0.00	0.0	0.875	0.00	0.0
	0.125	9.12754E-8		0.125	6.70073E-8
	0.250	3.64623E-7		0.250	2.65863E-7
	0.375	8.19321E-7		0.375	5.93362E-7
	0.500	1.45462E-6		0.500	1.04640E-6
	0.625	2.2697E-6		0.625	1.62207E-6
	0.750	3.26351E-6		0.750	2.31774E-6
	0.875	4.43464E-6		0.875	3.13118E-6
	1.00	5.78099E-6		1.00	4.06076E-6
0.500	0.00	0.0	1.00	0.00	0.0
	0.125	6-35759E-8		0.125	1.26363E-7
	0.250	2.54288E-7		0.250	5.02284E-7
	0.375	5.72099E-7		0.375	1.12303E-6
	0.500	1.01693E-6		0.500	1.98393E-6
	0.625	1.58861E-6		0.625	3.08039E-6
	0.750	2.28672E-6		0.750	4.40798E-6
	0.875	3.11046E-6		0.875	5.96246E-6
	1.00	4.05886E-6		1.00	7.73989E-6

Table(2)

6. CONCLUSION

Nonlinear integral equations in two dimensional space are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. For this purpose, the presented method can be proposed. We approximate the nonlinear parts by Adomian polynomials. A quadrature rule is used to compute the integrals.

As shown by numerical examples, the accuracy of this method is reasonable and increasing m (the number of approximation terms) reduces the error.

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