

FIXED POINT THEOREM USING COMPATIBILITY OF TYPE (A) AND WEAK COMPATIBILITY IN Menger SPACE

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ABSTRACT

In this paper, the concept of compatibility of type (A) and weak compatibility in Menger space has been applied to prove a common fixed point theorem for six self maps which generalizes the result of Pant et. al. [8].

Keywords and Phrases: Menger space, Common fixed points, Compatible maps, Compatible maps of type (A) and Weak compatibility.

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1. INTRODUCTION

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [6]. It is a probabilistic generalization in which we assign to any two points x and y , a distribution function $F_{x,y}$. Schweizer and Sklar [10] studied this concept and gave some fundamental results on this space. Sehgal and Bharucha-Reid [11] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed-point theory in Menger space.

Recently, Jungck and Rhoades [5] termed a pair of self maps to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. Sessa [12] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weak commuting maps in metric spaces. Jungck [4] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [7].

Cho, Murthy and Stojakovic [1] proposed the concept of compatible maps of type (A) in Menger space and gave some fixed point theorems. Recently, using the concept of compatible mappings of type (A), Jain et. al. [2, 3] proved some interesting fixed point theorems in Menger space. In the sequel, Patel and Patel [9] proved a common fixed point theorem for four compatible maps of type (A) in Menger space by taking a new inequality.

In this paper a fixed point theorem for six self maps has been proved using the concept of weak compatibility and compatibility of type (A) which generalizes the result of Pant et. al. [8]. We also cited an example.

2. PRELIMINARIES

Definition 2.1. [7] A mapping $f : \mathbb{R} \rightarrow \mathbb{R}^+$ is called a distribution if it is non-decreasing left continuous with

$$\inf \{ f(t) \mid t \in \mathbb{R} \} = 0 \quad \text{and} \quad \sup \{ f(t) \mid t \in \mathbb{R} \} = 1.$$

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0 & , \quad t \leq 0 \\ 1 & , \quad t > 0 \end{cases}.$$

Definition 2.2. [2] A mapping $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a t -norm if it satisfies the following conditions :

$$(t-1) \quad t(a, 1) = a, \quad t(0, 0) = 0 ;$$

$$(t-2) \quad t(a, b) = t(b, a) ;$$

$$(t-3) \quad t(c, d) \geq t(a, b) ; \quad \text{for } c \geq a, d \geq b,$$

$$(t-4) \quad t(t(a, b), c) = t(a, t(b, c)) \text{ for all } a, b, c, d \in [0, 1].$$

Definition 2.3. [2] A probabilistic metric space (PM-space) is an ordered pair (X, \mathbf{f}) consisting of a non empty set X and a function $\mathbf{f} : X \times X \rightarrow L$, where L is the collection of all distribution functions and the value of \mathbf{f} at $(u, v) \in X \times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ assumed to satisfy the following conditions:

$$(PM-1) \quad F_{u,v}(x) = 1, \text{ for all } x > 0, \text{ if and only if } u = v;$$

$$(PM-2) \quad F_{u,v}(0) = 0;$$

(PM-3) $F_{u,v} = F_{v,u}$;

(PM-4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x+y) = 1$,
for all $u,v,w \in X$ and $x, y > 0$.

Definition 2.4. [2] A Menger space is a triplet (X, \mathbf{f}, t) where (X, \mathbf{f}) is a PM-space and t is a t -norm such that the inequality

(PM-5) $F_{u,w}(x+y) \geq t\{F_{u,v}(x), F_{v,w}(y)\}$, for all $u, v, w \in X, x, y \geq 0$.

Definition 2.5. [11] A sequence $\{x_n\}$ in a Menger space (X, \mathbf{f}, t) is said to be convergent and converges to a point x in X if and only if for each $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$$F_{x_n, x}(\varepsilon) > 1 - \lambda \quad \text{for all } n \geq M(\varepsilon, \lambda).$$

Further the sequence $\{x_n\}$ is said to be Cauchy sequence if for $\varepsilon > 0$ and $\lambda > 0$, there is an integer $M(\varepsilon, \lambda)$ such that

$$F_{x_n, x_m}(\varepsilon) > 1 - \lambda \quad \text{for all } m, n \geq M(\varepsilon, \lambda).$$

A Menger PM-space (X, \mathbf{f}, t) is said to be complete if every Cauchy sequence in X converges to a point in X .
A complete metric space can be treated as a complete Menger space in the following way :

Proposition 2.1. [2] If (X, d) is a metric space then the metric d induces mappings $\mathbf{f} : X \times X \rightarrow L$, defined by

$F_{p,q}(x) = H(x - d(p, q))$, $p, q \in X$, where

$$H(k) = 0, \quad \text{for } k \leq 0 \quad \text{and} \quad H(k) = 1, \quad \text{for } k > 0.$$

Further if, $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is defined by $t(a, b) = \min\{a, b\}$. Then (X, \mathbf{f}, t) is a Menger space. It is complete if (X, d) is complete.

The space (X, \mathbf{f}, t) so obtained is called the *induced Menger space*.

Definition 2.6. [5] Self mappings A and S of a Menger space (X, \mathbf{f}, t) are said to be weak compatible if they commute at their coincidence points i.e. $Ax = Sx$ for $x \in X$ implies $ASx = SAX$.

Definition 2.7. [7] Self mappings A and S of a Menger space (X, \mathbf{f}, t) are said to be compatible if $F_{ASx_n, SAx_n}(x) \rightarrow 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \rightarrow u$ for some u in X , as $n \rightarrow \infty$.

Definition 2.8. [1] Self maps S and T of a Menger space (X, \mathbf{f}, t) are said to be compatible of type (A) if $F_{STx_n, TTx_n}(x) \rightarrow 1$ and $F_{TSx_n, SSx_n}(x) \rightarrow 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \rightarrow u$ for some u in X , as $n \rightarrow \infty$.

Definition 2.9. [8] Self maps S and T of a Menger space (X, \mathbf{f}, t) are said to be *semi-compatible* if $F_{STx_n, Tu}(x) \rightarrow 1$ for all $x > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \rightarrow u$ for some u in X , as $n \rightarrow \infty$.

Proposition 2.2. [2] Let S and T be compatible maps type of (A), self maps of a Menger space X , let $Sx_n, Tx_n \rightarrow u$ for some u in X . Then

(i) $TSx_n \rightarrow Su$ if S is continuous.

(ii) $STu = TSu$ and $Su = Tu$ if S and T are continuous.

Proposition 2.3. Self mappings S and T of a Menger space (X, \mathbf{f}, t) are compatible of type (A) then they are weakly compatible.

Proof. Let $Su = Tu$ and $\{x_n\} = u$ is a constant sequence, $n = 1, 2, 3, \dots$ such that

$$Sx_n \rightarrow Su, Tx_n \rightarrow Tu (Su) \text{ as } n \rightarrow \infty.$$

Now, by compatibility of type (A),

$$F_{STu, TTu}(t) = F_{STx_n, TTx_n}(t) \rightarrow 1 \text{ for } t > 0,$$

then $STu = TTu$.

Similarly, $F_{TSu, SSu}(t) = F_{TSx_n, SSx_n}(t) \rightarrow 1 \text{ for } t > 0,$

then $TSu = SSu$.

$$\begin{aligned} \text{Now, } Su = Tu &\Rightarrow SSu = STu \\ &\Rightarrow TSu = TTu. \end{aligned}$$

Then we get $STu = TSu$.

Hence, the pair (S, T) is weak compatible.

But the converse is not true as seen from the following example.

Example 2.1. Let (X, d) be a metric space where $X \in \mathbb{R}^+$ and (X, \mathbf{f}, t) be the induced Menger space with $F_{p,q}(\varepsilon) = H(\varepsilon - d(p, q)), \forall p, q \in X$ and $\forall \varepsilon > 0$. Define self maps S and T as follows :

$$S(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \quad \text{and} \quad T(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} .$$

Take $x_n = n$. Now,

$$Sx_n = \frac{1}{n^2} = 0 \text{ as } n \rightarrow \infty \text{ and } Tx_n = \frac{1}{n} = 0 \text{ as } n \rightarrow \infty.$$

Further, $F_{STx_n, TTx_n}(\varepsilon) = H(\varepsilon - d(STx_n, TTx_n)) = H(\varepsilon - |n^2 - n|)$

Therefore, $\lim_{n \rightarrow \infty} F_{STx_n, TTx_n}(\varepsilon) \neq 1$.

Similarly, $\lim_{n \rightarrow \infty} F_{TSx_n, SSx_n}(\varepsilon) = H(\varepsilon - d(TSx_n, SSx_n)) = H(\varepsilon - |n^2 - n^4|)$.

Therefore, $\lim_{n \rightarrow \infty} F_{TSx_n, SSx_n}(\varepsilon) \neq 1$.

Hence, the pair (S, T) is not compatible of type (A).

Also, 0 and 1 are coincidence points of S and T, since

$$ST(1) = T(1) = 1 = S(1) = TS(1) \quad \text{and}$$

$$ST(0) = T(0) = 1 = S(0) = TS(0).$$

Hence, the pair (S, T) is weak compatible.

Remark 2.1. In view of above example, it follows that the concept of weak-compatibility is more general than that of compatible maps of type (A).

Now, we give an example of pair of self maps (I, L) which are compatible of type (A) but not-semi-compatible.

Example 2.2. Let (X, d) be a metric space where $X = [0, 1]$ and (X, \mathbf{F}, t) be the induced Menger space with $F_{x,y} = \frac{t}{t + d(x,y)}$ for all $t > 0$.

Define self maps I and L as follows :

$$I(x) = x \text{ for all } x \in X \quad \text{and} \quad L(x) = \begin{cases} x, & \text{if } 0 \leq x < \frac{1}{2} \\ 1, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Taking $x_n = \frac{1}{2} - \frac{1}{n}$, we get $Ix_n = x_n = \frac{1}{2} - \frac{1}{n}$ and $Lx_n = \frac{1}{2} - \frac{1}{n}$.

Thus, $Lx_n \rightarrow \frac{1}{2}$ as $n \rightarrow \infty$ and $Ix_n \rightarrow \frac{1}{2}$, as $n \rightarrow \infty$.

Hence, $x = \frac{1}{2}$

Since $Lx_n = \frac{1}{2} - \frac{1}{n}$

Therefore, $ILx_n = I\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2} - \frac{1}{n}$

and $LLx_n = L\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2} - \frac{1}{n}$.

Consider $\lim_{n \rightarrow \infty} F_{ILx_n, LLx_n}(t) = \lim_{n \rightarrow \infty} F_{\frac{1}{2} - \frac{1}{n}, \frac{1}{2} - \frac{1}{n}}(t) = 1$ for $t > 0$.

Also, $Lx_n = L\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2} - \frac{1}{n}$ and $Ix_n = I\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{1}{2} - \frac{1}{n}$.

Consider $\lim_{n \rightarrow \infty} F_{Lx_n, Ix_n}(t) = \lim_{n \rightarrow \infty} F_{\frac{1}{2} - \frac{1}{n}, \frac{1}{2} - \frac{1}{n}}(t) = 1$ for $t > 0$.

Therefore, by definition, (I, L) is compatible mapping of type (A).

Now, $\lim_{n \rightarrow \infty} F_{ILx_n, Lx_n}(t) = \lim_{n \rightarrow \infty} F_{\frac{1}{2} - \frac{1}{n}, \frac{1}{2} - \frac{1}{n}}(t) < 1$ for $t > 0$.

Therefore, (I, L) is not semi-compatible mapping. Thus the pair (I, L) of self maps is compatible of type (A) but not semi-compatible.

Remark 2.2. In view of above example, it follows that the concept of compatible maps of type (A) is more general than that of semi-compatible maps.

Lemma 2.1. [14] Let $\{x_n\}$ be a sequence in a Menger space (X, F, t) with continuous t-norms t and $t(a, a) \geq a$. If there exists a constant $k \in (0, 1)$ such that $F_{x_n, x_{n+1}}(kt) \geq F_{x_{n-1}, x_n}(t)$ for all $t \geq 0$ and $n = 1, 2, 3, \dots$, then $\{x_n\}$ is a Cauchy sequence in X .

Lemma 2.3. [14] Let (X, F, t) be a Menger space. If there exists a constant $k \in (0, 1)$ such that $F_{x, y}(kt) \geq F_{x, y}(t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

A class of implicit relation. Let Φ be the set of all real continuous functions $\phi : (R^+)^4 \rightarrow R$, non-decreasing in the first argument with the property :

- a. For $u, v \geq 0$, $\phi(u, v, v, u) \geq 0$ or $\phi(u, v, u, v) \geq 0$ implies that $u \geq v$.
- b. $\phi(v, v, 1, 1) \geq 0$ implies that $u \geq 1$.

Example 2.3. Define $\phi(t_1, t_2, t_3, t_4) = 18t_1 - 16t_2 + 8t_3 - 10t_4$. Then $\phi \in \Phi$.

3. MAIN RESULT

Theorem 3.1. Let A, B, L, M, S and T be self mappings on a complete Menger space (X, F, t) with $t(a, a) \geq a$, for some $a \in [0, 1]$, satisfying :

- (3.1.1) $L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$;
- (3.1.2) $ST(X)$ and $AB(X)$ are complete subspace of X ;
- (3.1.3) either AB or L is continuous;
- (3.1.4) (L, AB) is compatible maps of type (A) and (M, ST) is weak compatible;
- (3.1.5) for some $\phi \in \Phi$, there exists $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$,

$$\phi(F_{Lx, My}(kt), F_{ABx, STy}(t), F_{Lx, ABx}(t), F_{My, STy}(kt)) \geq 0$$

then A, B, L, M, S and T have a unique common fixed point in X .

Proof. Let $x_0 \in X$. From condition (3.1.1) $\exists x_1, x_2 \in X$ such that

$$Lx_0 = STx_1 = y_0 \quad \text{and} \quad Mx_1 = ABx_2 = y_1.$$

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Lx_{2n} = STx_{2n+1} = y_{2n} \quad \text{and} \quad Mx_{2n+1} = ABx_{2n+2} = y_{2n+1}$$

For $n = 0, 1, 2, \dots$

Step 1. Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{Lx_{2n}, Mx_{2n+1}}(kt), F_{ABx_{2n}, STx_{2n+1}}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mx_{2n+1}, STx_{2n+1}}(kt)) \geq 0.$$

Letting $n \rightarrow \infty$, we get

$$\phi(F_{y_{2n}, y_{2n+1}}(kt), F_{y_{2n-1}, y_{2n}}(t), F_{y_{2n}, y_{2n-1}}(t), F_{y_{2n+1}, y_{2n}}(kt)) \geq 0.$$

Using (a), we get

$$F_{y_{2n}, y_{2n+1}}(kt) \geq F_{y_{2n-1}, y_{2n}}(t).$$

Therefore, for all n even or odd, we have

$$F_{y_n, y_{n+1}}(kt) \geq F_{y_{n-1}, y_n}(t).$$

Therefore, by lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X , which is complete.

Hence $\{y_n\} \rightarrow z \in X$. Also its subsequences converges as follows :

$$\{Lx_{2n}\} \rightarrow z, \quad \{ABx_{2n}\} \rightarrow z, \quad \{Mx_{2n+1}\} \rightarrow z \quad \text{and} \quad \{STx_{2n+1}\} \rightarrow z.$$

Case I. When AB is continuous.

As AB is continuous, $(AB)^2x_{2n} \rightarrow ABz$ and $(AB)Lx_{2n} \rightarrow ABz$.

As (L, AB) is compatible pair of type (A), so by proposition (2.2), $L(AB)x_{2n} \rightarrow ABz$.

Step 2. Putting $x = ABx_{2n}$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{LABx_{2n}, Mx_{2n+1}}(kt), F_{ABABx_{2n}, STx_{2n+1}}(t), F_{LABx_{2n}, ABABx_{2n}}(t), F_{Mx_{2n+1}, STx_{2n+1}}(kt)) \geq 0$$

Letting $n \rightarrow \infty$, we get

$$\phi(F_{ABz, z}(kt), F_{ABz, z}(t), F_{ABz, ABz}(t), F_{z, z}(kt)) \geq 0$$

$$\phi(F_{ABz, z}(kt), F_{ABz, z}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{ABz, z}(kt), F_{ABz, z}(t), 1, 1) \geq 0.$$

Using (b), we get

$$F_{ABz, z}(t) = 1, \text{ for all } t > 0,$$

i.e. $ABz = z$.

Step 3. Putting $x = z$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{Lz, Mx_{2n+1}}(kt), F_{ABz, STx_{2n+1}}(t), F_{Lz, ABz}(t), F_{Mx_{2n+1}, STx_{2n+1}}(kt)) \geq 0.$$

Letting $n \rightarrow \infty$, we get

$$\phi(F_{Lz, z}(kt), F_{ABz, z}(t), F_{Lz, ABz}(t), F_{z, z}(kt)) \geq 0$$

$$\phi(F_{Lz, z}(kt), 1, F_{Lz, z}(t), 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{Lz, z}(kt), 1, F_{Lz, z}(t), 1) \geq 0.$$

Using (a), we get

$$F_{z, Lz}(kt) = 1, \text{ for all } t > 0,$$

i.e. $z = Lz$.

Thus, we have $z = Lz = ABz$.

Step 4. Putting $x = Bz$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{LBz, Mx_{2n+1}}(kt), F_{ABBz, STx_{2n+1}}(t), F_{LBz, ABBz}(t), F_{Mx_{2n+1}, STx_{2n}}(kt)) \geq 0.$$

Letting $n \rightarrow \infty$, we get

$$\begin{aligned} \phi(F_{Bz, z}(kt), F_{Bz, z}(t), F_{Bz, Bz}(t), F_{z, z}(kt)) &\geq 0 \\ \phi(F_{Bz, z}(kt), F_{Bz, z}(t), 1, 1) &\geq 0. \end{aligned}$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{Bz, z}(t), F_{Bz, z}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{Bz, z}(t) = 1, \text{ for all } t > 0,$$

i.e. $z = Bz$.

Since $z = ABz$, we also have

$$z = Az.$$

Therefore, $z = Az = Bz = Lz$.

Step 5. As $L(X) \subseteq ST(X)$, there exists $v \in X$ such that

$$z = Lz = STv.$$

Putting $x = x_{2n}$ and $y = v$ in (3.1.5), we get

$$\phi(F_{Lx_{2n}, Mv}(kt), F_{ABx_{2n}, STv}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mv, STv}(kt)) \geq 0.$$

Letting $n \rightarrow \infty$, we get

$$\phi(F_{z, Mv}(kt), F_{z, STv}(t), F_{z, z}(t), F_{Mv, z}(kt)) \geq 0$$

$$\phi(F_{z, Mv}(kt), 1, 1, F_{z, Mv}(kt)) \geq 0$$

Using (a), we have

$$F_{z, Mv}(kt) \geq 1, \text{ for all } t > 0.$$

Hence, $F_{z, Mv}(t) = 1$.

Thus, $z = Mv$.

Therefore, $z = Mv = STv$.

As (M, ST) weakly compatible, we have

$$STMv = MSTv. \quad \text{Thus, } STz = Mz.$$

Step 6. Putting $x = x_{2n}$ and $y = z$ in (3.1.5), we get

$$\phi(F_{Lx_{2n}, Mz}(kt), F_{ABx_{2n}, STz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mz, STz}(kt)) \geq 0$$

Letting $n \rightarrow \infty$, we get

$$\phi(F_{z, Mz}(kt), F_{z, Mz}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, Mz}(t), F_{z, Mz}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{z, Mz}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{z, Mz}(t) = 1$, we have

$$z = Mz = STz.$$

Step 7. Putting $x = x_{2n}$ and $y = Tz$ in (3.1.5) and using Step 5, we get

$$\phi(F_{Lx_{2n}, MTz}(kt), F_{ABx_{2n}, STTz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{MTz, STTz}(kt)) \geq 0.$$

Letting $n \rightarrow \infty$, we get

$$\phi(F_{Lz, Tz}(kt), F_{z, Tz}(t), F_{z, z}(t), F_{Tz, Tz}(kt)) \geq 0$$

$$\phi(F_{z, Tz}(kt), F_{z, Tz}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, Tz}(t), F_{z, Tz}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{z, Tz}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{z, Tz}(t) = 1$, we have

$$z = Tz.$$

Since $Tz = STz$, we also have $z = Sz$.

Hence

$$Az = Bz = Lz = Mz = Tz = Sz = z.$$

Hence, the six self maps have a common fixed point in this case.

Case II. When L is continuous.

As L is continuous, $L^2x_{2n} \rightarrow Lz$ and $L(AB)x_{2n} \rightarrow Lz$.

As (L, AB) is compatible map of type (A), so by proposition (2.2),

$$L(AB)x_{2n} \rightarrow ABz.$$

By uniqueness of limit in Menger space, we have

$$Lz = ABz.$$

Step 8. Putting $x = z$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{Lz, Mx_{2n+1}}(kt), F_{ABz, STx_{2n+1}}(t), F_{Lz, ABz}(t), F_{Mx_{2n+1}, STx_{2n+1}}(kt)) \square \square 0.$$

Letting $n \rightarrow \infty$, we get

$$\phi(F_{Lz, z}(kt), F_{Lz, z}(t), F_{Lz, Lz}(t), F_{z, z}(kt)) \geq 0$$

$$\phi(F_{Lz, z}(kt), F_{Lz, z}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{Lz, z}(t), F_{Lz, z}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{z, Lz}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{z, Lz}(t) = 1$

$$\Rightarrow z = Lz.$$

Therefore,

$$z = Lz = ABz.$$

Step 9. Putting $x = Bz$ and $y = x_{2n+1}$ in (3.1.5), we get

$$\phi(F_{LBz, Mx_{2n+1}}(kt), F_{ABBz, STx_{2n+1}}(t), F_{LBz, ABBz}(t), F_{Mx_{2n+1}, STx_{2n+1}}(kt)) \square \square 0.$$

Letting $n \rightarrow \infty$, we get

$$\phi(F_{Bz, z}(kt), F_{Bz, z}(t), F_{Bz, Bz}(t), F_{z, z}(kt)) \geq 0$$

$$\phi(F_{Bz, z}(kt), F_{Bz, z}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{Bz, z}(t), F_{Bz, z}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{Bz, z}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{Bz, z}(t) = 1$

$$\Rightarrow z = Bz.$$

Since $z = ABz$, we also have $z = Az$.

Therefore, $z = Az = Bz = Lz$.

Step 10. As $L(X) \subseteq ST(X)$, there exists $v \in X$ such that

$$z = Lz = STv.$$

Putting $x = x_{2n}$ and $y = v$ in (3.1.5), we get

$$\phi(F_{Lx_{2n}, Mv}(kt), F_{ABx_{2n}, STv}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mv, STv}(kt)) \geq 0.$$

Letting $n \rightarrow \infty$, we get

$$\phi(F_{z, Mv}(kt), F_{z, STv}(t), F_{z, z}(t), F_{Mv, z}(kt)) \geq 0$$

$$\phi(F_{z, Mv}(kt), 1, 1, F_{z, Mv}(kt)) \geq 0$$

Using (a), we have

$$F_{z, Mv}(kt) \geq 1, \text{ for all } t > 0.$$

Hence, $F_{z, Mv}(t) = 1$.

Thus, $z = Mv$.

Therefore, $z = Mv = STv$.

As (M, ST) weakly compatible, we have

$$STMv = MSTv.$$

Thus, $STz = Mz$.

Step 11. Putting $x = x_{2n}$ and $y = z$ in (3.1.5), we get

$$\phi(F_{Lx_{2n}, Mz}(kt), F_{ABx_{2n}, STz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{Mz, STz}(kt)) \geq 0$$

Letting $n \rightarrow \infty$, we get

$$\phi(F_{z, Mz}(kt), F_{z, Mz}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, Mz}(t), F_{z, Mz}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{z, Mz}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{z, Mz}(t) = 1$, we have

$$z = Mz = STz.$$

Step 12. Putting $x = x_{2n}$ and $y = Tz$ in (3.1.5) and using Step 5, we get

$$\phi(F_{Lx_{2n}, MTz}(kt), F_{ABx_{2n}, STTz}(t), F_{Lx_{2n}, ABx_{2n}}(t), F_{MTz, STTz}(kt)) \geq 0.$$

Letting $n \rightarrow \infty$, we get

$$\phi(F_{Lz, Tz}(kt), F_{z, Tz}(t), F_{z, z}(t), F_{Tz, Tz}(kt)) \geq 0$$

$$\phi(F_{z, Tz}(kt), F_{z, Tz}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, Tz}(t), F_{z, Tz}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{z, Tz}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{z, Tz}(t) = 1$, we have

$$z = Tz.$$

Since $Tz = STz$, we also have $z = Sz$.

Hence $Az = Bz = Lz = Mz = Tz = Sz = z$.

Hence, the six self maps have a common fixed point in this case also.

Uniqueness. Let w be another common fixed point of A, B, L, M, S and T ; then $w = Aw = Bw = Lw = Mw = Sw = Tw$.

Putting $x = z$ and $y = w$ in (3.1.5), we get

$$\phi(F_{Lz, Mw}(kt), F_{ABz, STw}(t), F_{Lz, ABz}(t), F_{Mw, STw}(kt)) \geq 0$$

$$\phi(F_{z, w}(kt), F_{z, w}(t), F_{z, z}(t), F_{w, w}(kt)) \geq 0$$

$$\phi(F_{z, w}(kt), F_{z, w}(t), 1, 1) \geq 0.$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(F_{z, w}(t), F_{z, w}(t), 1, 1) \geq 0.$$

Using (b), we have

$$F_{z, w}(t) \geq 1, \text{ for all } t > 0.$$

Thus, $F_{z, w}(t) = 1$,

i.e., $z = w$.

Therefore, z is a unique common fixed point of A, B, L, M, S & T .

This completes the proof.

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