

INTEGRO - DIFFERENTIAL EQUATION AND AN ELASTIC PLATE WITH A CURVILINEAR HOLE IN S- PLANE

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ABSTRACT

Complex variable method has been applied to obtain an exact expression for Goursat functions for the stretched infinite plate weakened by a hole having arbitrary shape . The inner of the hole is free from stresses and it is conformally mapped on the arc of the right half -plane (s-plane) , $\text{Re } s > 0$.

The interesting cases when the infinite plate weakened by a crescent hole or a cut having the shape of a circular arc and hypotrochoidal with three round corners , are included as special cases. Many applications are considered and the program of Maple 7 is used to compute the stress components.

Keywords: *Complex variable method , curvilinear hole , conformal mapping , integro - differential equation.*

MSC (2000): 74B10 , 30C 20

1. INTRODUCTION AND BASIC EQUATIONS

The boundary value problems for the perforated infinite plates have been discussed by several authors , see Muskhelishvili [1] , England [2] , Abdou and Khamis [3] and Abdou and Asseri [4,5] .

It is known that , see Muskhelishvili [1] , the first fundamental problem, in the plane theory of elasticity is equivalent to finding two analytic functions $\phi_1(z)$ and $\psi_1(z)$ of one complex argument $z = x + iy$, $i = \sqrt{-1}$ satisfying the boundary condition

$$\phi_1(t) + t\overline{\phi_1'(t)} + \overline{\psi_1'(t)} = f(t) , \quad (1.1)$$

where $f(t)$ is a given function of stresses and t denotes the affix of a point in the boundary .

Also , the two complex functions $\phi_1(z)$ and $\psi_1(z)$ take the forms

$$\phi_1(z) = \frac{P}{4} z + \phi(z) \quad , \quad \psi_1(z) = \frac{P}{4} e^{-2i\theta} z + \psi(z). \quad (1.2)$$

Here , the physical meaning of the problem is : we have an infinite plate stretched at infinity by the application of a uniform tensile stress of intensity P , making an angle θ with the x- axis . The plate weakened by a curvilinear hole C which is free from stress . The two functions $\phi(z)$ and $\psi(z)$ of Eq.(1.2) are single valued analytic functions , called Goursat functions , within the region of the plate and bounded at infinity .

The components of stresses , in view of Goursat functions, take the form , see Hetnarski[6]

$$\begin{aligned} \sigma_{xx} &= 4\text{Re}\{\phi'(z) - M(z, \bar{z})\} , \quad \sigma_{yy} = 4\text{Re}\{\phi'(z) + M(z, \bar{z})\}, \\ \sigma_{xy} &= \text{Re}\{iM(z, \bar{z}) - \overline{M(z, \bar{z})}\}; \quad M(z, \bar{z}) = \bar{z} \phi''(z) + \psi'(z). \end{aligned} \quad (1.3)$$

Assume the conformal mapping

$$z = cw(s) = s \frac{(s+1)^3 + m(s+1)(s-1)^2 + l(s-1)^3}{(1-n)(s^2-1)(s+a)} , \quad (c > 0, |n| < 1, a = \frac{1+n}{1-n}). \quad (1.4)$$

where m, n, l are real parameters subject to the condition $w(\infty)$ is bounded and $w'(s)$ does not vanish within the right half - plane $\text{Re } s \geq 0$. The conformal mapping (1.4) conforms the inner of the curvilinear hole on the domain of the right half - plane $\text{Re } s \geq 0$.

In the present paper , the complex variable method and the conformal mapping (1.4) are used to obtain exact and closed expression for the two Goursat functions of Eq.(1.1) and (1.2), in the form of integro - differential equation , for the stretched infinite plate weakened by an arbitrary curvilinear hole which the edge is free from stresses .

The interesting cases of an infinite plate weakened by an elliptic hole , a crescent - like hole or a cut having the shape of a circular arc , and the hypotrochoidal hole with three rounded are considered here .Also , components of stress , in each case, are determined and computed using Maple 7 .

2. METHOD OF SOLUTION

The expression $\frac{w(i\tau)}{w'(i\tau)}$ can be assumed in the form

$$\frac{w(i\tau)}{w'(i\tau)} = \overline{\alpha(i\tau)} + \beta(i\tau) , \tag{2.1}$$

where

$$\alpha(s) = \frac{h}{a+s} , \quad a = \frac{1+n}{1-n} , \tag{2.2}$$

$$h = 4a^2n(n^3 + nm + l)J_0^{-1} \quad \text{and} \quad J_0 = l n^2 - 2l n^3 - mn - 2n^2 + 1 ,$$

while

$$\beta(s) = \frac{1}{s-a} \left[\frac{H(s)}{E(s)} + K \right] , \tag{2.3}$$

$$H(s) = (1-n)(s+a)^2(s^2-1) [l (s+1)^3 + m(s-1)(s+1)^2 + (s-1)^3] ,$$

and

$$E(s) = 2 [-(s+1)^4 + 2n(s+1)^3(s-1) + m(s+1)^2(s-1)^2 + 2l(s-1)^3(s+1) - nl(s-1)^4] .$$

The function $\beta(s)$ is regular within the right half - plane except at infinity.

In view of (2.1) , the boundary condition (1.1) takes the form

$$\phi(i\tau) = \alpha(i\tau)\overline{\phi'(i\tau)} + \overline{\Psi_*(i\tau)} = f(i\tau) , \quad \phi(s) = \phi(w(s)) , \tag{2.4}$$

where

$$f(i\tau) = \frac{-P}{2} [w(i\tau) - e^{2i\theta} \overline{w(i\tau)}] , \tag{2.5}$$

and

$$\Psi_*(s) = \psi(w(s)) + \beta(s)\phi'(s) . \tag{2.6}$$

Here, we assume that $\phi(\infty) = \Psi(\infty) = 0$.

Multiplying both sides of (2.4) by $\frac{d\tau}{2\pi(s-i\tau)}$, and integrating with respect to τ from $-\infty$ to ∞ , we

have

$$\phi(s) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\alpha(\tau)\overline{\phi'(\tau)}}{(s-i\tau)} d\tau = F(s) , \tag{2.7}$$

where

$$F(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{(s-i\tau)} d\tau . \tag{2.8}$$

The formula (2.7) represents an integro - differential equation with Cauchy kernel. The free term $F(s)$, after using (2.5) in (2.8) , takes the form

$$F(s) = P \left[\frac{-2l}{(1-n)^3(1+s)(s+a)} + \frac{m+l}{(1-n)^2(s+a)} + \frac{(2-n)l}{(1-n)^2(1+s)} - \frac{e^{2i\theta}}{1+s} \right] . \tag{2.9}$$

To obtain the general solution of Eq.(2.7) , we assume

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\alpha(\tau)\overline{\phi'(\tau)}}{(s-i\tau)} d\tau = \frac{chpb}{s+a} , \tag{2.10}$$

where b is a complex constant to be determined.

Using (2.10) in (2.7) , then differentiating $\phi(s)$ and inserting the result of $\overline{\phi'}$ in (2.10) again , the complex constant b takes the form

$$b = (1+n)^2 \left(\frac{\cos 2\theta}{4a^2h} - i \frac{\sin 2\theta}{4a^2h} \right) + \frac{nl(n^2-2) - (m+n^2)}{(1-n)^2(4a^2-h)} , (i = \sqrt{-1}). \tag{2.11}$$

Substituting from (2.11) into (2.10) , then using the result in (2.7), the function $\phi(s)$ is completely determined in the form

$$\phi(s) = \frac{cp}{(1-n)^2} \left[\frac{J_1 + iJ_2 - 2l(1-n)^{-1}(1+s)^{-1}}{(s+a)} + \frac{(2-n)l}{1+s} - \frac{(1-n)^2 e^{2i\theta}}{1-n} \right] , \tag{2.12}$$

where

$$J_1 = \frac{(m+n^2)J_0 - n(n^3 + nm + l)[(1-n^2)^2 \cos 2\theta + nl(n^2 - 2)]}{(J_0 - n^4 - mn^2 - nl)} ,$$

$$J_2 = \frac{n(n^3 + nm + l)(1-n^2)^2 \sin 2\theta}{J_0 - n^4 - mn^2 - nl} , \tag{2.13}$$

and J_0 is given by the third formula of Eq.(2.2).

Hence, the integro - differential equation (2.7) is completely solved by the first function of Goursat $\phi(s)$ of Eq.(2.12). To obtain the second Goursat function $\psi(s)$, we use the boundary condition to obtain

$$\psi(s) = cp \left[\frac{2le^{-2i\theta}}{(1-n)^3(1+s)(s+a)} - \frac{(m+l)e^{-2i\theta}}{(1-n)^2(s+a)} + \frac{l(n+2)e^{-2i\theta}}{(1-n)^2(s+a)} + \frac{1}{1+s} \right]$$

$$+ \frac{cph}{4} \left[\frac{(s+3a)(J_1 + iJ_2)}{(1+n)^2(s+a)^2} + \frac{\sum_{j=0}^3 A_j s^j}{(1-n)^3(1+n)^2(1+s)^2(s+a)^2} - \frac{(1-n)^2(s+2+a)e^{2i\theta}}{(1+s)^2} \right] , \tag{2.14}$$

where

$$\begin{aligned}
 A_0 &= n^6 - n^5 - 7n^4 - n^3 + 14n^2 + 2n - 8, \\
 A_1 &= n^6 + 7n^5 + 3n^4 - 21n^3 + 2n^2 + 18n - 8, \\
 A_2 &= -n^6 - 3n^5 + 11n^4 + n^3 - 18n^2 + 10n, \\
 &\text{and} \\
 A_3 &= n^6 - 3n^5 + n^4 + 5n^3 - 6n^2 + 2n.
 \end{aligned}
 \tag{2.15}$$

3. SPECIAL CASES

Now , we are in a position to consider several interesting special cases (i) For $l=0$, we have the mapping function

$$z = c \frac{(1+s)^2 + m(s-1)^2}{s^2 - 1 - n(s-1)^2}, \quad c > 0, |n| < 1,
 \tag{3.1}$$

the corresponding formulas of $\phi(s)$ and $\psi(s)$ become

$$\phi(s) = \frac{cp}{(1-n)^2} \left(\frac{(m+n^2)J_3}{s+a} + \frac{(1-n)^2 e^{2i\theta}}{1+s} \right),
 \tag{3.2}$$

and

$$\psi(s) = \frac{cp}{1+s} \left(\frac{m+n_2 e^{2i\theta}}{(1-n)^2 (s+a)} \right) + cpk \left(\frac{m+n^2}{4(1-n)^2 (s+a)^2} - \frac{(s+a+2)e^{2i\theta}}{(1+a)(s+1)^2} \right),
 \tag{3.3}$$

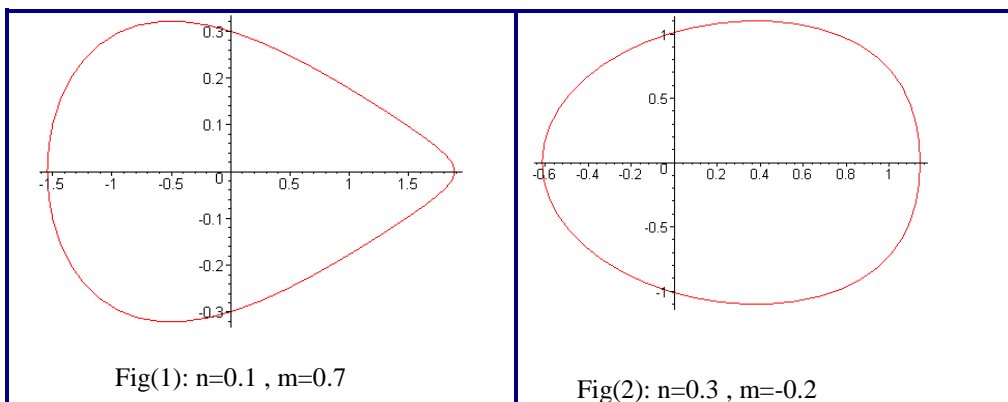
where

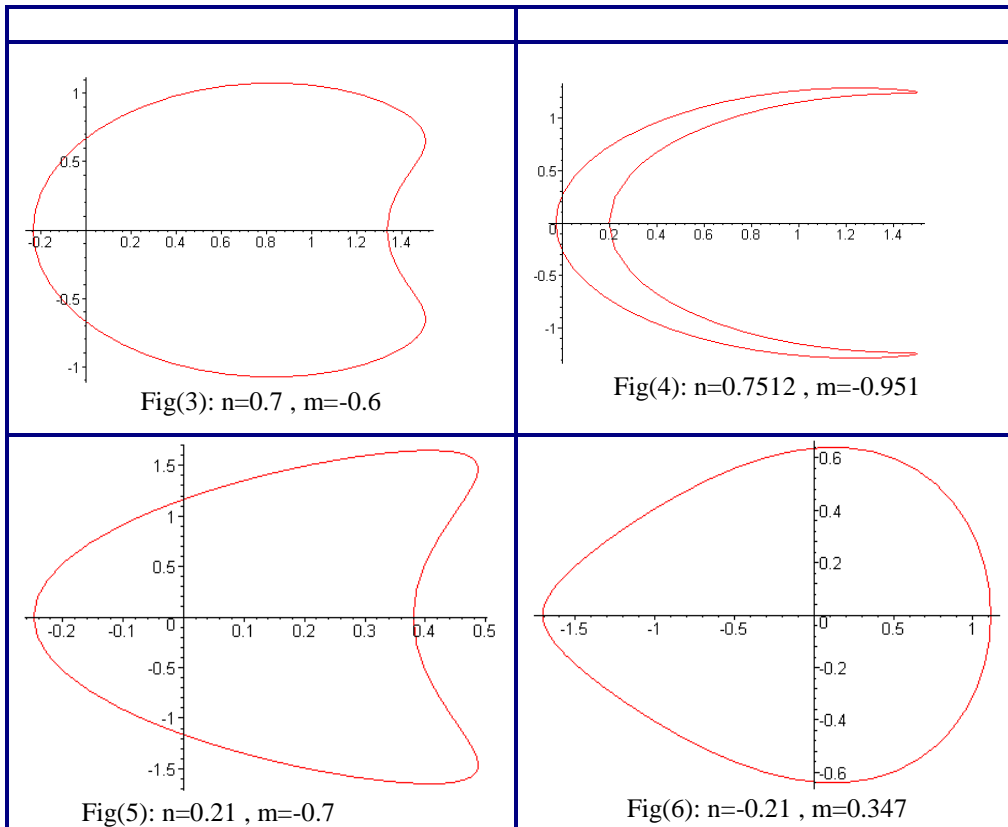
$$J_3 = \frac{(m+2)n^2 - 1 + n^2(n^2 - 1)^2 \cos 2\theta}{n^4 - 1 + n^2(1+m)} + in^2 \sin 2\theta, \quad k = \frac{4n^2 a^2 (m+n^2)}{1 - (m-2)n^2}.
 \tag{3.4}$$

Also, when $s = \frac{\zeta + 1}{\zeta - 1}, l = 0$, we have the mapping function

$$z = c \frac{\zeta + m\zeta^{-1}}{1 - n\zeta^{-1}}, \quad (c > 0, |n| < 1).
 \tag{3.5}$$

The mapping function (3.5) maps the curvilinear hole C in z- plane onto the domain of outside unit circle γ in ζ - plane, under the condition that $w'(\zeta)$ does not vanish or become infinite outside the unit circle γ . The following graphs clear up the shape of the rational mapping of Eq.(3.5), see Figs(1-6)





The corresponding Goursat functions in ζ - plane after excluding the constant term , in this case , become

$$\phi(\zeta) = \frac{cp}{2} \left[e^{2i\theta} \zeta^{-1} + (m + n^2) \left(\frac{1}{2} - J_3 \right) (\zeta - n)^{-1} \right], \tag{3.6}$$

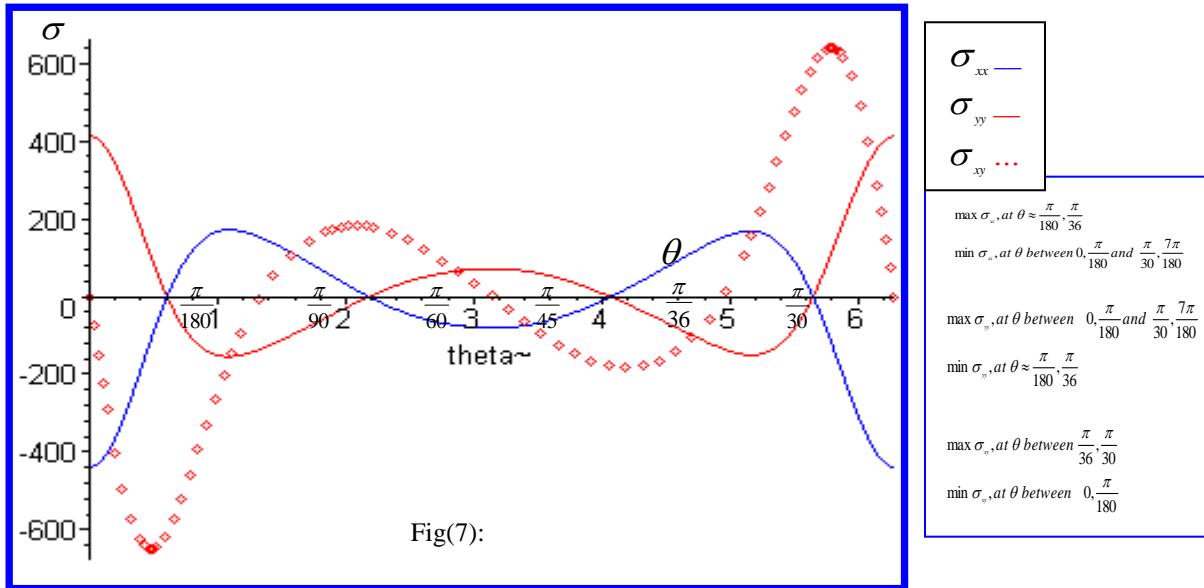
and

$$\begin{aligned} \psi(\zeta) = & \frac{-cp}{4} + \frac{cphn^2\zeta}{2(1-n\zeta)} \left[e^{2i\theta} + (m + n^2) \left(\frac{1}{2} - J_3 \right) (1 - n^2)^{-2} \right] \\ & + c \frac{w(\zeta^{-1})}{w'(\zeta)} p \left[e^{2i\theta} \zeta^{-2} + (m + n^2) \left(\frac{1}{2} - J_3 \right) (\zeta - n)^{-2} - \frac{1}{2} \right], \end{aligned} \tag{3.7}$$

where J_3 is given by (3.4) and

$$h = \frac{(m + n^2)(1 - n^2)^2}{1 - (m + 2)n^2}. \tag{3.8}$$

As an application for the conformal mapping (3.1) and Goursat functions we assume $n=0.25, m=9, c=2, p=0.25$, $0 \leq \theta \leq 2\pi$, the stress components , σ_{xx} , σ_{yy} and σ_{xy} , in this case, are calculated by computer and illustrated in Fig.7;



Fig(7):

The results of (3.6) , (3.7) agree with Abdou and Khar- Eldin [7].

(ii) For $m=l=0$, we have the mapping function

$$z = c \frac{(1+s)^2}{(1+s)\{(s-1) - n(s+1)\}} \tag{3.9}$$

Here, the inner edge of the infinite plate is the inverse of an elliptic limaçon

. In this case , the Goursat functions (2.12)and (2.14) become

$$\phi(s) = \frac{cp}{(1-n)^2} \left[\frac{J_4 + in^2 \sin 2\theta}{(s+a)} - \frac{(1-n)^2 e^{2i\theta}}{s+1} \right], \quad a = \frac{1+n}{1-n}, \tag{3.10}$$

$$\begin{aligned} \psi(s) = & \frac{cpk_1}{4} \left[(s+3a) \frac{J_4 + in^2 \sin 2\theta}{(1+n)^2 (s+a)^2} - \frac{(1-n)^2 (s+2+a)e^{2i\theta}}{(1+s)^2} \right] \\ & + cp \left[-\frac{n^2 e^{-2i\theta}}{(1-n)^2 (s+a)} + \frac{1}{1+s} \right], \end{aligned} \tag{3.11}$$

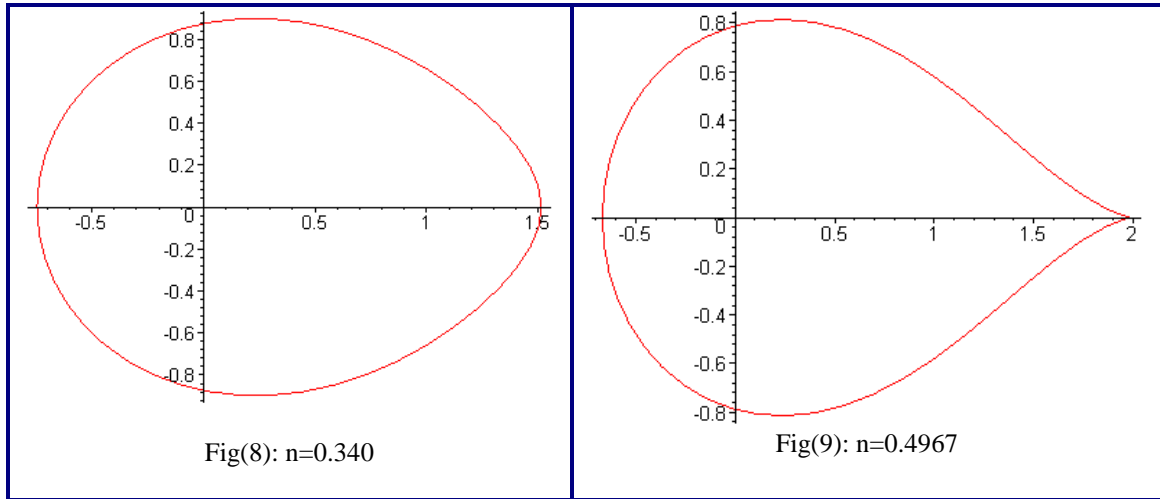
where

$$J_4 = \frac{n^2(1-2n^2) - n^4(1-n^2)^2 \cos 2\theta}{(1-2n^2 - n^4)}, \quad k_1 = \frac{4a^2 n^2}{1-2n^2}.$$

Also, when $s = \frac{\zeta+1}{\zeta-1}$, we have the transformation mapping function

$$z = c \frac{\zeta}{1-n\zeta^{-1}} \tag{3.12}$$

The following graphs clear up the shape of the rational mapping on the domain outside a unit circle γ in Eq.(3.12) , see Figs(8-9)



The corresponding complex functions of equations (2.12) and (2.14), after using (3.12) and excluding the constant term ,become

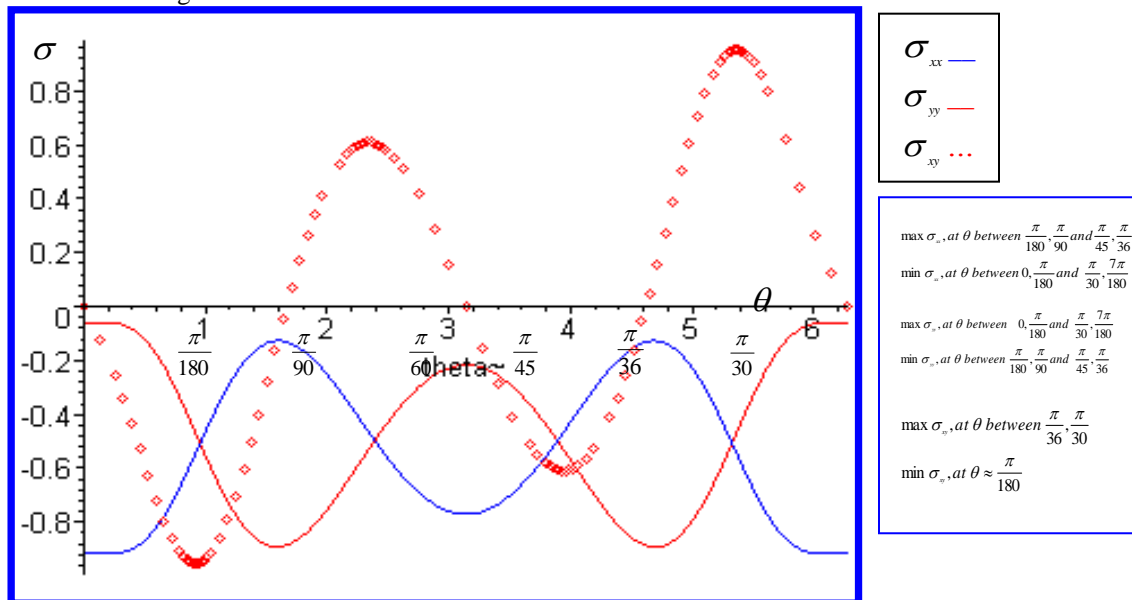
$$\phi(\zeta) = \frac{cpe^{2i\theta}}{2\zeta} + \frac{cph_1}{4(n-\zeta)} \left[1 + \frac{\nu}{1-\nu} - \frac{2n^2 \cos 2\theta}{1+\nu} + \frac{2in^2 \sin 2\theta}{1+\nu} \right], \quad (3.13)$$

$$\psi(\zeta) = -\frac{cp}{4} - \frac{w(\zeta^{-1})}{w'(\zeta)} \phi_*(\zeta) + \frac{\zeta}{1-n\zeta} (h_1 \phi_*(n^{-1})); \quad (3.14)$$

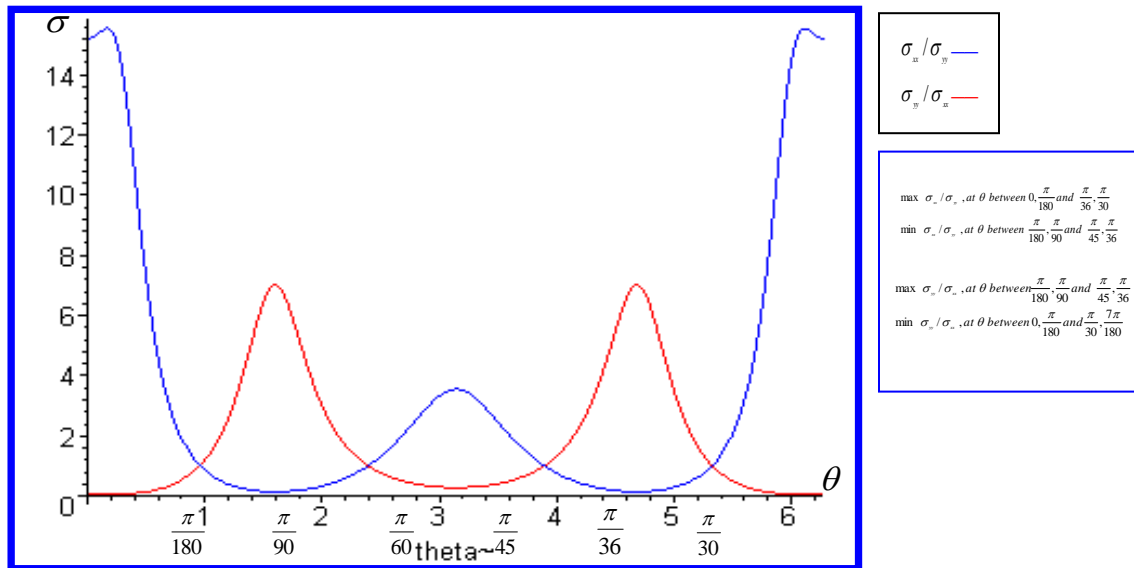
where

$$h_1 = \frac{n^2(1-n^2)^2}{1-2n^2}, \quad \nu = \frac{n^4}{1-2n^2} \quad \text{and} \quad \phi_*(\zeta) = \phi'(\zeta) + \frac{cp}{4}.$$

Also ,when $n=0.25, c=2, p=0.25, 0 \leq \theta \leq 2\pi$ the stress components σ_{xx}, σ_{yy} and σ_{xy} , for the Goursat functions, (3.10), (3.11) in s-plane and the corresponding functions (3.13), (3.14) in ζ - plane are calculated and illustrated in Figs. 10-11



Fig(10):



Fig(11):

(iii) For $n=0$, we have the transform mapping function

$$\frac{z}{c} = \frac{s+1}{s-1} + m \left(\frac{s-1}{s+1} \right) + l \left(\frac{s-1}{s+1} \right)^2. \tag{3.15}$$

The mapping (3.15) represents a circle when $m=l=0$, an elliptic form when $l=0$ and a triangle when $m=0$. In this case, the Goursat functions become

$$\phi(s) = \left(\frac{-2l}{(1+s)^2} + \frac{m+2l}{1+s} - \frac{e^{2i\theta}}{1+s} \right), \tag{3.16}$$

and

$$\psi(s) = cp \left(\frac{-2le^{-2i\theta}}{(1+s)^2} + \frac{m+2l}{1+s} e^{-2i\theta} - \frac{1}{1+s} \right). \tag{3.17}$$

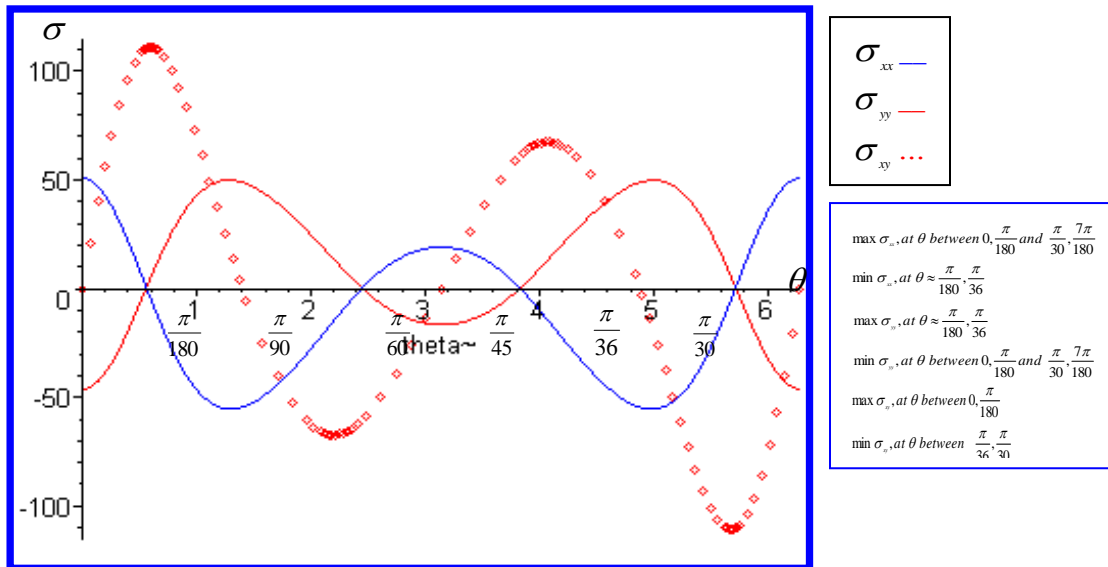
Also, when $s = \frac{\zeta + 1}{\zeta - 1}$, we have the mapping function $z = c(\zeta + m\zeta^{-1} + l\zeta^{-1})$,

and the two corresponding complex functions, after neglecting the constant term, become

$$\phi(\zeta) = \left(\frac{cpe^{2i\theta}}{2\zeta} - \frac{clp}{\zeta^2} - \frac{cmp}{4\zeta} \right), \tag{3.18}$$

$$\psi(\zeta) = -\frac{cp}{4} - \frac{w(\zeta^{-1})}{w'(\zeta)} \phi(\zeta) + \frac{cp\zeta}{4} (m + l\zeta). \tag{3.19}$$

Hence, for $m=9, l=0.25, c=2, p=0.25$, $0 \leq \theta \leq 2\pi$ the stress components σ_{xx} , σ_{yy} and σ_{xy} are illustrated in Figs. 12

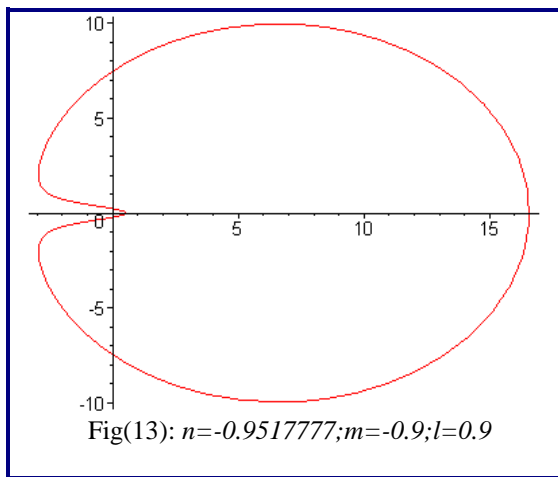


Fig(12):

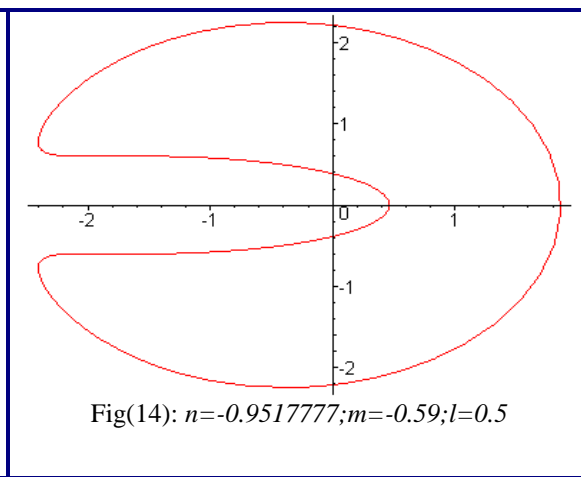
The general agreement between s- plane and ζ - plane can be obtain ,when $s = \frac{\zeta + 1}{\zeta - 1}$ in (1.4) , whose mapping function become

$$z = c \frac{\zeta + m\zeta^{-1} + l\zeta^{-2}}{1 - n\zeta^{-1}} \quad (c > 0, |n| < 1). \tag{3.20}$$

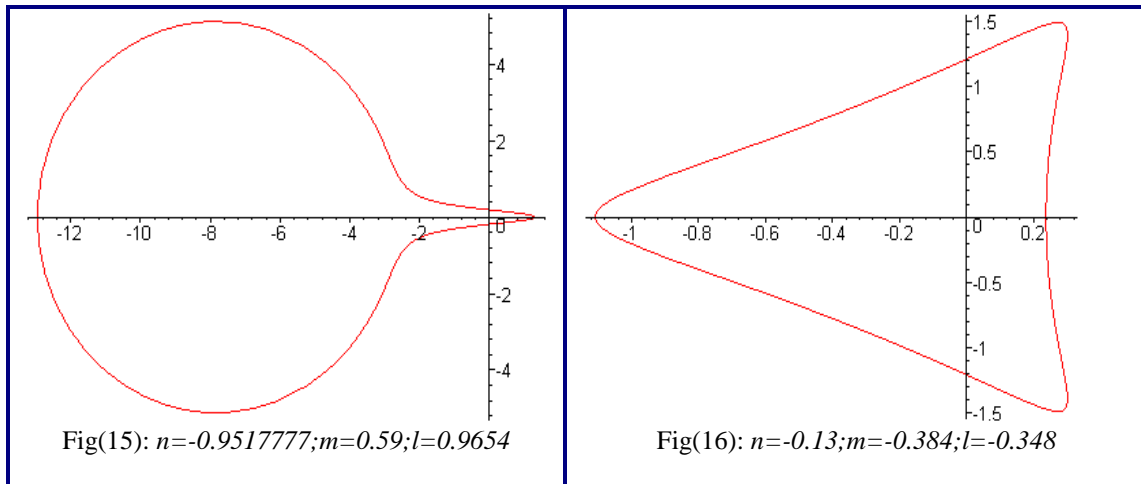
The following graphs clear up the shape of the curvilinear hole by using the rational mapping of Eq(3.20) on the domain outside a unit circle γ , see Figs(13-16)



Fig(13): $n=-0.9517777; m=-0.9; l=0.9$



Fig(14): $n=-0.9517777; m=-0.59; l=0.5$



Also, the two complex potential functions $\phi(z)$ and $\psi(z)$ become

$$\phi(\zeta) = \frac{cpe^{2i\theta}}{2\zeta} + \frac{cpl}{\zeta(n-\zeta)} + \left[\frac{h_1}{4} + \frac{h_1\nu_1}{4(1-\nu_1)} - \frac{n(1+nh_1)\cos 2\theta}{2(1-\nu_1)} + \frac{in(1+nh_1)}{2(1+\nu_1)}\sin 2\theta + \frac{nm(2-n^2)\nu_1}{4(1-\nu_1)} \right], \tag{3.21}$$

and

$$\psi(s) = -\frac{cp}{4} - \frac{w(\zeta^{-1})}{w'(\zeta)}\phi_*(\zeta) + \frac{\zeta}{1-n\zeta}(h_1\phi_*(n^{-1}) + l B_*(\zeta)). \tag{3.22}$$

where

$$h_1 = \frac{(1-n^2)^2(m+n^2) + l(nm + 2ln^2 + n^3 - ln^4)}{(1-2n^2 + n^2m - 2n^3l + n^5l)}, \tag{3.23}$$

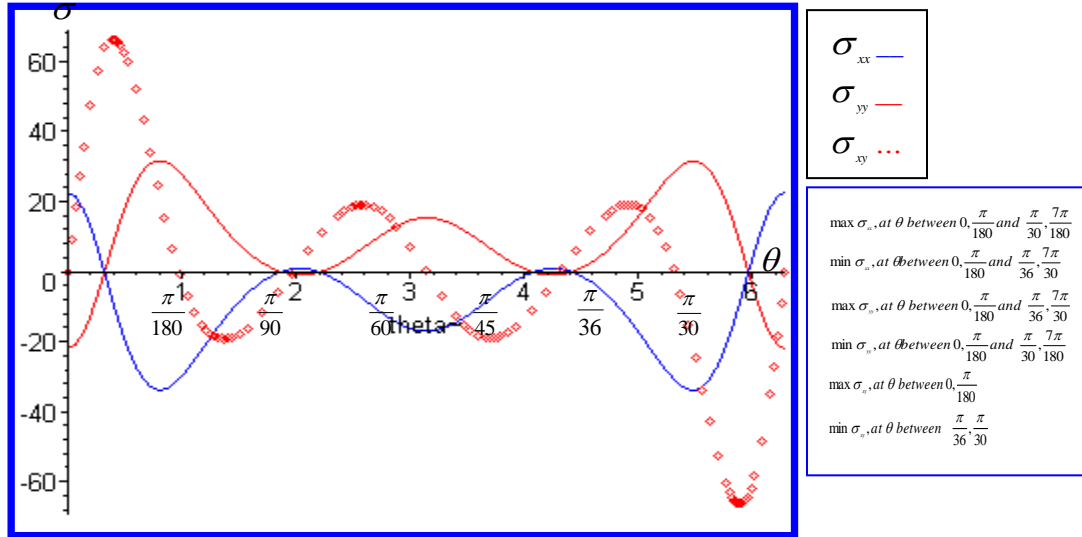
$$B_*(\zeta) = \frac{cp\zeta}{4} + n^{-1}\phi'(n^{-1}),$$

and

$$\nu_1 = \frac{n(1+nh_1)}{(1-n^2)^2}. \tag{3.24}$$

The results of Eqs. (3.21) and (3.22), agree with the work of Abdou [8].

And for $n=0.25, c=2, p=0.25, m=9, l=0.25, 0 \leq \theta \leq 2\pi$ the stress components σ_{xx}, σ_{yy} and σ_{xy} are computing and illustrated in Fig. 17



Fig(17):

4. CONCLUSIONS

1- Formula (1.1) represents, in two dimensional problems, in the theory of elasticity , the first fundamental problems in the complex plane , z- plane .

2- In the two dimensional problems, the transformation mapping functions help us to transform the difficult region into one of simpler shape , which gives the solutions without difficulty .

3- the transformation mapping $z = cw(s)$, $c > 0, s = \sigma + i\tau$, transforms the domain of right half - plane .

While the mapping $z = cw(\zeta), |\zeta| > 1$ transforms the domain of the infinite plate with a curvilinear hole onto the domain outside the unit circle.

4- The transformation $s = \frac{\zeta + 1}{\zeta - 1}$, $s = \sigma + i\tau$, transforms the domain of right half - plane onto the domain

outside the unit circle .The inverse case can be obtained ,if we take $\zeta = \frac{1 + s}{s - 1}$.

5- The interesting cases when the shape of the hole is an ellipse , a triangle , hypotrochoidal , a crescent or a cut having the shape of a circular arc are included as special ones of this work.

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